Exchanged Crossed Cube: A Novel Interconnection Network for Parallel Computation

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Abstract—The topology of interconnection networks plays a key role in the performance of parallel computing systems. A new interconnection network called exchanged crossed cube (ECQ) is proposed and analyzed in this paper. We prove that ECQ has the better properties than other variations of the basic hypercube in terms of the smaller diameter, fewer links, and lower cost factor, which indicates the reduced communication overhead, lower hardware cost, and more balanced consideration among performance and cost. Furthermore, it maintains several attractive advantages including recursive structure, high partitionability, and strong connectivity. Furthermore, the optimal routing and broadcasting algorithms are proposed for this new network topology.

Index Terms—Interconnection networks, hypercube, exchanged crossed cube, interprocessor communication, parallel computation

1 Introduction

Significant progress has been made in the past decades in developing massively parallel computing architectures. It is well known that the interconnection network plays an important role in large-scale parallel systems [3], [12], [14] because there is usually a need for one processor to communicate with other processors when a collection of processors execute a program in parallel to solve problems. Much of the computation power is wasted if the processors spend a considerable amount of time in communication, such as routing and broadcasting. Thus, it is necessary for the processors to communicate efficiently with one another, and such efficient interprocessor communication requires the support from a carefully designed interconnection network. Among all the topologies proposed in the current literature, the hypercube (denoted by HQ for short) has received much attention due to its many attractive properties, including regularity, symmetry, small diameter, strong connectivity, recursive construction, partitionability, and relatively small link complexity [8], [13], [16], [17].

Variations of this fundamental topology have been proposed in the literature to further enhance some of its features [1], [9], [10], [19], [20], [21]. Among all the features of a network topology, the diameter (i.e., the maximum of the shortest distances between all pairs of nodes, which has appreciable influence on communication latency) and the hardware cost are two of the most important factors in determining its performance and cost. Thus, a number of approaches have been suggested to improve the performance of HQ by reducing its diameter. The crossed cube, denoted by CQ, is an excellent example of such a topology [6]. A CQ is derived from an HQ by changing the way of connection of some HQ links. The diameter of a CQ is almost half of that of its corresponding HQ. Specifically, the diameter of an n-dimensional CQ is $\lceil (n+1)/2 \rceil$ and the diameter of an n-dimensional HQ is n. However, the CQ makes no improvement in the hardware cost compared to the HQ. An n-dimensional HQ is composed of 2^n nodes and has n links per node. The number of links of an HQ, which is directly related to the hardware cost, grows more drastically than the number of processors. This leads to rapid increase of the hardware cost as an HQ scales up. The exchanged hypercube, denoted by EH, is an excellent topology with the lower hardware cost [11]. An EH is based on link removal from an HQ, which makes the network more cost-effective as it scales up. Unfortunately, the availability of rich connectivity in the EH is reduced. The EH offers major reduction in the hardware cost compared to the HQ, but no improvement over the diameter of the HQ. The demand for reduction of the diameter of the HQ as well as its hardware cost motivates our investigation in proposing a new interconnection network.

The objective of this paper is to present a network topology with not only a smaller diameter but also the lower hardware cost. To this end, a new interconnection network, called exchanged crossed cube, denoted by ECQ, is proposed for large-scale parallel computation. This interconnection topology retains most of the topological features of the EH, and at the same time combines many attractive features of the CQ. In particular, the ECQ has the following desirable properties:

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- The diameter of an ECQ is almost the same as that of a CQ, but much smaller than that of an EH. Furthermore, the mean distance among the vertices is smaller than that of an EH.
- The hardware cost of an ECQ is almost the same as that of an EH, but much lower than that of a CQ.
- An interconnection network with a large diameter has very low message passing bandwidth but a network with a high node degree is very expensive. The cost factor (i.e., the product of the diameter and the node degree [2]) of the ECQ is better than that of the CQ and EH, which indicates that the ECQ is able to offer a better combined consideration between the cost and performance of a parallel computing system.

The aforementioned three main advantages of the ECQ reveal that the ECQ is able to achieve the improved performance and scalability compared with all the hypercube variations proposed previously in the current literature.

The binomial tree [15] is one of the most frequently used spanning tree structures for parallel applications in various systems, especially in the HQ. It has several desirable properties and is applicable in solving many problems, for example, computing prefix sums. Load balancing can also be achieved in this tree structure [5], [7]. The exchanged tree (ET) proposed in [11] is a good method for constructing the spanning tree of an EH. However, by taking the wrong link and the wrong degree of an EH, the exchanged tree is not a spanning tree of an EH. Further explanation and analysis can be found in Section 4.2. To overcome this deficiency, we propose the improved exchanged tree and prove that it is a spanning tree of an EH and an ECQ. Furthermore, it offers an efficient way for broadcasting communication in the EH and ECQ.

The organization of this paper is as follows: Section 2 presents the construction of the ECQ. Section 3 discusses its various topological properties and compares the ECQ with the HQ and its variants. In Section 4, efficient routing and broadcasting algorithms are proposed. Finally, the work is summarized and concluded in Section 5.

2 Exchanged Crossed Cube

An undirected graph is often adopted to model an interconnection network, in which vertices correspond to the processing elements and edges correspond to the bidirectional links. Let $G = (V, E)$ be an undirected graph, where V and E denote the vertex set and the edge set of G, respectively. The graphs considered here are labeled graphs and the used labels are binary strings. The notation G^x denotes the labeled graph obtained by prefixing every vertex label in Graph G with x.

Before constructing an ECQ, we give the following definitions.

Definition 1. *Two binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair related, denoted by $x \sim y$, if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. The case that x and y are not pair related is denoted by $x \not\sim y$.*

Definition 2. *An n-dimensional crossed cube, denoted by CQ_n, is a labeled graph defined inductively as follows: CQ_1 is K_2 (i.e., the complete graph on two vertices with labels 0 and 1). For $n > 1$, CQ_n contains CQ^0_{n-1} and CQ^1_{n-1} joined according to the following rule: vertex $u = 0u_{n-2}\cdots u_0$ from CQ^0_{n-1} and vertex $v = 1v_{n-2}\cdots v_0$ from CQ^1_{n-1} are adjacent in CQ_n if and only if 1) $u_{n-2} = v_{n-2}$ if n is even, and 2) $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$ for $0 \le i \le \lfloor (n-1)/2 \rfloor$.*

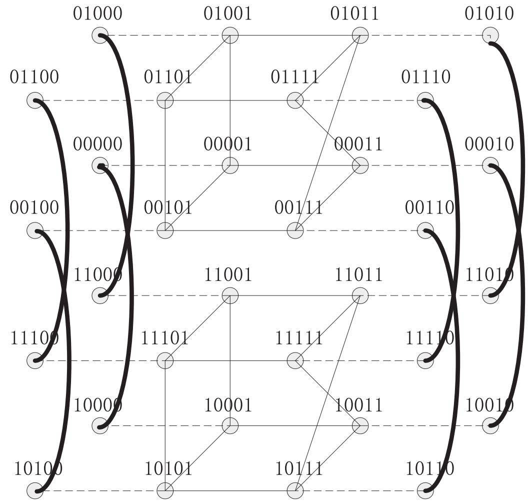

Fig. 1. An exchanged crossed cube ECQ(1, 3).

The procedure of constructing an ECQ is presented as follow.

An ECQ is defined as an undirected graph $\mathrm{ECQ}(s,t) = G(V,E)$, where $s \ge 1$ and $t \ge 1$. V is the set of vertices, i.e., $V = \{a_{s-1}\dots a_0 b_{t-1}\dots b_0 c \mid a_i, b_j, c \in \{0,1\}, i \in [0,s), j \in [0,t)\}$. E is the set of edges, i.e., $E = \{(v_1, v_2) \mid (v_1,\ v_2) \in V \times V\}$, which consists of three types of edges, i.e., E_1, E_2, and E_3, as described below:

1. E_1: $v_1[0] \ne v_2[0]$, $v_1 \oplus v_2 = 1$, where $\oplus$ denotes the exclusive-OR operator.
2. E_2: $v_1[0] = v_2[0] = 0$, $v_1[t:1] = v_2[t:1]$, where $v[x:y]$ denotes the bit pattern of v between dimensions y and x inclusive, $v_1[s+t:t+1]$ denoted by $a = a_{s-1}\cdots a_0$ and $v_2[s+t:t+1]$ denoted by $a' = a'_{s-1}\cdots a'_0$ are joined according to the following rule: For all $s \ge 1$, if and only if there exists an $l (1 \le l \le s)$ with $a_{s-1}\cdots a_l = a'_{s-1}\cdots a'_l$; $a_{l-1} \ne a'_{l-1}$, $a_{l-2} = a'_{l-2}$ if l is even; $a_{2i+1}a_{2i} \sim a'_{2i+1}a'_{2i}$ for $0 \le i < \lfloor (l-1)/2 \rfloor$.
3. E_3: $v_1[0] = v_2[0] = 1$, $v_1[s+t:t+1] = v_2[s+t:t+1]$, $v_1[t:1]$ denoted by $b = b_{t-1}\cdots b_0$ and $v_2[t:1]$ denoted by $b' = b'_{t-1}\cdots b'_0$ are joined according to the following rule: For all $t \ge 1$, if and only if there exists an $l (1 \le l \le t)$ with $b_{t-1}\cdots b_l = b'_{t-1}\cdots b'_l$; $b_{l-1} \ne b'_{l-1}$, $b_{l-2} = b'_{l-2}$ if l is even; $b_{2i+1}b_{2i} \sim b'_{2i+1}b'_{2i}$ for $0 \le i < \lfloor (l-1)/2 \rfloor$.

The above definition shows that an $\mathrm{ECQ}(s,t)$ has three disjoint sets of edges. Fig. 1 shows an example of an ECQ with $s = 1$ and $t = 3$, where the dashed links, bold links and solid links correspond to E_1, E_2 and E_3, respectively.

Let u and v be two distinct vertices in CQ_n. The ith double bit of vertex u is defined as a 2-bit string $u_{2i+1}u_{2i}$ for $0 \le i \le \lfloor n/2 \rfloor - 1$, and simply a single bit u_{2i} for $i = \lfloor n/2 \rfloor$ when n is odd. For $k < n$, the k-prefix of u, denoted by $p_k(u)$, is defined as $u_{n-1}u_{n-2}\dots u_{n-k}$. Bit l is called *the most significant* *different bit* between u and v if $p_{n-l-1}(u) = p_{n-l-1}(v)$ and $u_l \neq v_l$. $i^* = \lfloor l/2 \rfloor$ is called the *most significant different double bit*. We define a function ρ on u and v as follows:

$$\rho_j(u,v) = 0, \text{ for all } j \geq i^*+1;$$
$$\rho_{i^*}(u,v) = \begin{cases} 2, & \text{if } u_{2i^*+1}u_{2i^*} = \bar{v}_{2i^*+1}\bar{v}_{2i^*}; \\ 1, & \text{otherwise.} \end{cases}$$

Subsequently, we recursively define $\rho_j(u,v)$ for $j \leq i^*-1$ using the notion of distance-preserving pair related (abbreviated as *d.-p. pair related*), which is motivated from the concept of pair related.

Definition 3. *$u_{2j+1}u_{2j}$ and $v_{2j+1}v_{2j}$, for $j \leq i^*-1$, are* distance-preserving pair related *if one of the following conditions holds:*

1. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in \{(01,01),(11,11)\}$ *and*
$$\sum_{k=j+1}^{\lfloor (n-1)/2 \rfloor} \rho_k(u,v) \quad \text{is even;}$$
2. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in \{(01,11),(11,01)\}$ *and*
$$\sum_{k=j+1}^{\lfloor (n-1)/2 \rfloor} \rho_k(u,v) \quad \text{is odd;}$$
3. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in \{(00,00),(10,10)\}$.

We write $u_{2i+1}u_{2i} \overset{d.-p.}{\sim} v_{2i+1}v_{2i}$ if $u_{2i+1}u_{2i}$ and $v_{2i+1}v_{2i}$ are *d.-p. pair related*, and $u_{2i+1}u_{2i} \overset{d.-p.}{\nsim} v_{2i+1}v_{2i}$ otherwise.

Then, $\rho_j(u,v)$ for $j \leq i^*-1$ is recursively defined as follows:

$$\rho_j(u,v) = \begin{cases} 1, & \text{if } u_{2j+1}u_{2j} \overset{d.-p.}{\sim} v_{2j+1}v_{2j}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the *pair related distance* between u and v, denoted by $\rho(u,v)$, is defined as $\rho(u,v) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \rho_j(u,v)$.

3 Topological Properties of Exchanged Crossed Cube

This section first presents the important topological properties of ECQ and then compares its performance with the HQ, CQ, and EH.

3.1 Topological Properties

The fundamental properties of ECQ, including the total number of nodes and edges, degree, expandability, isomorphism, decomposition, diameter, and cost factor are investigated in this section.

3.1.1 Nodes

The number of nodes in an ECQ is given by the following theorem:

Theorem 1. *The total number of nodes in* $\mathrm{ECQ}(s,t)$ *is* $N(s,t) = 2^{s+t+1}$.

3.1.2 Edges

The edge set of $\mathrm{ECQ}(s,t)$, denoted by $E(s,t)$, is composed of three types, i.e., E_1, E_2, and E_3. The definition of ECQ reveals that 1) the number of edges in E_1 is 2^{s+t}; 2) the number of edges in E_3 is $2^t \times s2^{s-1} = s2^{s+t-1}$; and 3) the number of edges in E_2 is $2^s \times t2^{t-1} = t2^{s+t-1}$. So, we have the following theorems on the number of edges.

Theorem 2. *The number of edges in* $\mathrm{ECQ}(s,t)$ *is* $E(s,t) = (s+t+2)2^{s+t-1}$.

Theorem 3. *The number of edges in* $\mathrm{ECQ}(s,t)$ *is about half of that in* CQ_{s+t+1}.

Proof. An $(s+t+1)$-dimensional crossed cube, denoted by CQ_{s+t+1}, has $(s+t+1)2^{s+t}$ edges, and an $\mathrm{ECQ}(s,t)$ has $(s+t+2)2^{s+t-1}$ edges. Hence, we have

$$\frac{(s+t+2)2^{s+t-1}}{(s+t+1)2^{s+t}} = \frac{s+t+2}{2(s+t+1)} = \frac{1}{2} + \frac{1}{2(s+t+1)},$$

which approaches 1/2 as $s \to +\infty$ and/or $t \to +\infty$. □

3.1.3 Degree

The degree of a node in a graph is defined as the total number of edges connected to the node. The degree of a network is defined as the largest degree of all the vertices in its graph representation.

Theorem 4. *The degree of nodes whose bit addresses end in 0 (0-ending nodes) of an* $\mathrm{ECQ}(s,t)$ *is* $s+1$*, while the degree of nodes whose bit addresses end in 1 (1-ending nodes) is* $t+1$*. So, the degree of an* $\mathrm{ECQ}(s,t)$ *is* $\max\{s+1, t+1\}$.

3.1.4 Expandability

We use $IE = \Delta T/T$ to measure the *incremental expandability* of an interconnection network, where ΔT represents the smallest change in the number of network components (nodes and edges) needed to increase the existing number of components T while retaining its topological characteristics. Let IE_{node} and IE_{edge} denote the node expandability and the edge expandability, respectively.

Theorem 5. *For* $\mathrm{ECQ}(s,t)$, $IE_{\text{node}}(s,t) = 1$ *and* $IE_{\text{edge}}(s,t) \to 1$*, asymptotically.*

Proof. Since an $\mathrm{ECQ}(s,t)$ has 2^{s+t+1} nodes and $(s+t+2)2^{s+t-1}$ edges, we have

$$IE_{\text{node}}(s,t) = \frac{\Delta T_{\text{node}}(s,t)}{T_{\text{node}}(s,t)} = \frac{2^{s+t+1+1} - 2^{s+t+1}}{2^{s+t+1}} = 1,$$

and

$$\begin{aligned} IE_{\text{edge}}(s,t) &= \frac{\Delta T_{\text{edge}}(s,t)}{T_{\text{edge}}(s,t)} \\ &= \frac{(s+t+3)2^{s+t} - (s+t+2)2^{s+t-1}}{(s+t+2)2^{s+t-1}} \\ &= \frac{(s+t+4)2^{s+t-1}}{(s+t+2)2^{s+t-1}} \\ &= 1 + \frac{2}{s+t+2}, \end{aligned}$$

which approaches 1 as $s \to +\infty$ and/or $t \to +\infty$.

TABLE 1
Vertex Distance in an ECQ

No.	$a_{s-1}\cdots a_0 = a'_{s-1}\cdots a'_0$	$b_{t-1}\cdots b_0 = b'_{t-1}\cdots b'_0$	c	c'	Distance
1	Yes	Yes	0	1	1
2	Yes	Yes	1	0	1
3	Yes	No	0	0	$\rho(b,b')+2$
4	Yes	No	0	1	$\rho(b,b')+1$
5	Yes	No	1	0	$\rho(b,b')+1$
6	Yes	No	1	1	$\rho(b,b')$
7	No	Yes	0	0	$\rho(a,a')$
8	No	Yes	0	1	$\rho(a,a')+1$
9	No	Yes	1	0	$\rho(a,a')+1$
10	No	Yes	1	1	$\rho(a,a')+2$
11	No	No	0	0	$\rho(a,a')+\rho(b,b')+2$
12	No	No	0	1	$\rho(a,a')+\rho(b,b')+1$
13	No	No	1	0	$\rho(a,a')+\rho(b,b')+1$
14	No	No	1	1	$\rho(a,a')+\rho(b,b')+2$

Notice that the IE_{node} of an n-dimensional CQ_n is 1 and the IE_{edge} is 2 asymptotically. This means that a CQ_n has the much higher hardware cost when it is expanded into a CQ_{n+1}. □

3.1.5 Isomorphism

Two graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are isomorphic if and only if 1) there is a bijection (one-to-one correspondence) f from V_1 to V_2, and 2) there is a bijection g from E_1 to E_2 that maps each edge (u, v) to $(f(u), f(v))$ [18].

Theorem 6. *$ECQ(s,t)$ and $ECQ(t,s)$ are isomorphic.*

Proof. According to the definition of ECQ, we have $ECQ(s,t) = (V_1, E_1)$, where

$$V_1 = \{a_{s-1}\dots a_0 b_{t-1}\dots b_0 c \mid a_i, b_j, c \in \{0,1\}, i \in [0,s), j \in [0,t)\},$$

and $ECQ(t,s) = (V_2, E_2)$, where

$$V_2 = \{a_{t-1}\dots a_0 b_{s-1}\dots b_0 c \mid a_i, b_j, c \in \{0,1\}, i \in [0,t), j \in [0,s)\}.$$

Considering an arbitrary vertex $u = a_{s-1}\dots a_0 b_{t-1}\dots b_0 c$ in $ECQ(s,t)$, we can find a vertex $v = a'_{t-1}\dots a'_0 b'_{s-1}\dots b'_0 c'$ in $ECQ(t,s)$, obtained by the following bijection f:

$$a_{s-1} = b'_{s-1}, a_{s-2} = b'_{s-2}, \dots, a_0 = b'_0,$$
$$b_{t-1} = a'_{t-1}, b_{t-2} = a'_{t-2}, \dots, b_0 = a'_0, c = \bar{c'}.$$

Similarly, we can find that every vertex v' in $ECQ(t,s)$ has a corresponding vertex u' in $ECQ(s,t)$. An $ECQ(s,t)$ has the same number of vertices as an $ECQ(t,s)$. Hence, f is a bijection from V_1 to V_2.

Based on the definition of ECQ, if (u,v) is an arbitrary edge in $ECQ(s,t)$, then $(f(u), f(v))$ is an edge in $ECQ(t,s)$. Similarly, if (u', v') is an arbitrary edge in $ECQ(t,s)$, then $(f(u'), f(v'))$ is also an edge in $ECQ(s,t)$. Hence, there is a bijection g from E_1 to E_2 that maps each edge (u,v) to $(f(u), f(v))$.

Based on the above analysis and the definition of isomorphic groups, we can conclude that $ECQ(s,t)$ and $ECQ(t,s)$ are isomorphic. □

3.1.6 Decomposition

The following theorems give the partitionability of the ECQ:

Theorem 7. *An $ECQ(s,t)$ can be decomposed into two copies of $ECQ(s-1,t)$ or $ECQ(s,t-1)$.*

According to the definition of ECQ and EH, $ECQ(s,t)$ can be decomposed into $ECQ(s-1,t)$ or $ECQ(s,t-1)$ using the same way as $EH(s,t)$ is decomposed into $EH(s-1,t)$ or $EH(s,t-1)$. The decomposition of EH was reported in [11]. Following the same method, Theorems 7 can be proven straightforwardly.

Theorem 8. *An $ECQ(s,t)$ can be decomposed into 2^s topological networks of CQ_t and 2^t topological networks of CQ_s.*

Proof. According to the definition of ECQ, an $ECQ(s,t)$ has one set of nodes $V = \{a_{s-1}\dots a_0 b_{t-1}\dots b_0 c \mid a_i, b_j, c \in \{0,1\}, i \in [0,s), j \in [0,t)\}$ and three disjoint sets of edges E_1, E_2, and E_3. Furthermore, based on the definition of CQ, E_2 and E_3 can compose the edge set of a CQ_s and a CQ_t, respectively. $V_1 = \{a_{s-1}\dots a_0 | a_i \in \{0,1\}, i \in [0,s)\}$ is the node set of CQ_s and $V_2 = \{b_{t-1}\dots b_0 | b_j \in \{0,1\}, j \in [0,t)\}$ is the node set of CQ_t. There are 2^s V_2 and 2^t V_1 in V. The set of V_1 and E_2 can compose a CQ_s while the set of V_2 and E_3 can compose a CQ_t. Thus, an $ECQ(s,t)$ can be decomposed into 2^s topological networks of CQ_t and 2^t topological networks of CQ_s. □

3.1.7 Diameter

The distance between two distinct vertices u and v, denoted by $d(u,v)$, is the length of a shortest path from u to v. The diameter of $G = (V,E)$, denoted by D_G, is defined to be $\max\{d(u,v) : u, v \in V\}$. In a network system, the diameter represents the worst-case communication delay between two processors in the network.

Theorem 9. *The diameter of an $ECQ(s,t)$ is $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 2$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x.*

Proof. Let $u = a_{s-1}\cdots a_0 b_{t-1}\cdots b_0 c$ and $v = a'_{s-1}\cdots a'_0 b'_{t-1}\cdots b'_0 c'$ be the source vertex and the destination vertex, commonly abbreviated as $u = abc$ and $v = a'b'c'$, where $a = a_{s-1}\cdots a_0$, $b = b_{t-1}\cdots b_0$, $a' = a'_{s-1}\cdots a'_0$, $b' = b'_{t-1}\cdots b'_0$. The results in Table 1 list the results of the distance between u and v under various cases with

TABLE 2
Comparison of the Cost Factors of Various Topologies

Dimension n	HQ	CQ	EH	ECQ
$s=1,\ t=1,\ n=3$	9	6	8	8
$s=2,\ t=1,\ n=4$	16	12	12.5	12.5
$s=2,\ t=2,\ n=5$	25	15	18	18
$s=5,\ t=5,\ n=11$	121	66	72	48
$s=6,\ t=5,\ n=12$	132	84	84.5	58.5
$s=10,\ t=10,\ n=21$	441	231	242	154
$s=11,\ t=10,\ n=22$	484	264	264.5	161
$s=20,\ t=20,\ n=41$	1681	861	882	504
$s=50,\ t=50,\ n=101$	10201	5151	5202	2754

different values of a and a', b and b', c and c', respectively. From Theorems 4 and 8, and the results in Table 1, we can see that the maximum distance in an $\mathrm{ECQ}(s,t)$ is $\max\{\rho(a,a')\}+\max\{\rho(b,b')\}+2$, where 2 is due to the fact that routing has to use dimension 0 twice: $1\rightarrow 0$ and $0\rightarrow 1$. The diameter of CQ_n is $\lceil\frac{n+1}{2}\rceil$. $\max\{\rho(a,a')\}$ and $\max\{\rho(b,b')\}$ are the length of the diameter of CQ_s and CQ_t. Hence, the diameter of an $\mathrm{ECQ}(s,t)$ is given by

$$D_{\mathrm{ECQ}(s,t)} = \max\{\rho(a,a')\}+\max\{\rho(b,b')\}+2 = \left\lceil\frac{s+1}{2}\right\rceil + \left\lceil\frac{t+1}{2}\right\rceil + 2.$$

□

3.1.8 Cost Factor

The cost factor is defined as the product of the diameter and the node degree, which is a good criterion to measure the performance and the hardware cost a multiprocessor system [2]. For an $\mathrm{ECQ}(s,t)$, the diameter is given by Theorem 9. In general, the average node degree, denoted by $\bar{d}(s,t)$, is given by

$$\bar{d}(s,t) = \frac{(s+t+2)2^{s+t}}{2^{s+t+1}} = \frac{s+t+2}{2}.$$

Thus, the cost factor of an $\mathrm{ECQ}(s,t)$, denoted by $\phi(s,t)$, is given by

$$\phi(s,t) = D_{\mathrm{ECQ}(s,t)} \times \bar{d}(s,t) = \left(\left\lceil\frac{s+1}{2}\right\rceil + \left\lceil\frac{t+1}{2}\right\rceil + 2\right) \times \left(\frac{s+t+2}{2}\right).$$

Table 2 compares the cost factors of the HQ, CQ, EH, and ECQ for the same dimension $n (n = s+t+1)$. While the cost factor of ECQ can be calculated using the above equation and the cost factor of HQ, CQ, and EH is n^2, $\lceil\frac{n}{2}\rceil \times n$, and $(s+t+2)\times(\frac{s+t+2}{2})$, respectively. The cost factor of a network measures not only the cost of the processors, but also the cost of the communications. In general, for a desirable interconnection structure, the cost factor should be as small as possible. Fig. 2 depicts the curves of the cost factor versus the number of nodes for different topologies.

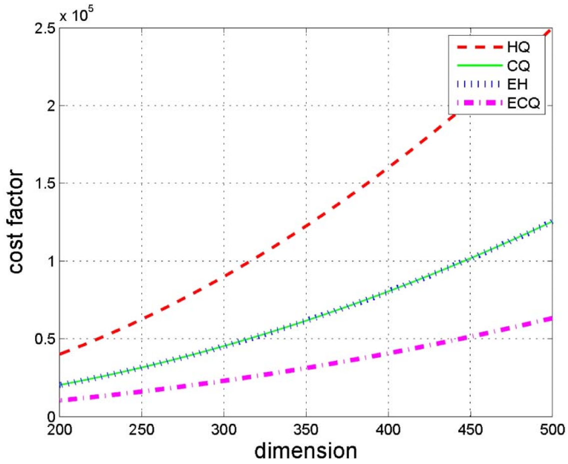

Fig. 2. Comparison of the cost factors.

3.1.9 Connectivity

In this section, the proposed ECQ is proven to be Hamilton-connected.

Theorem 10. $\mathrm{ECQ}(s,t)$ *is Hamiltonian with a closed cycle encompassing all nodes only once.*

Proof. $\mathrm{ECQ}(s,t)$ can be proven to be Hamilton-connected through an inductive method. First of all, it is obvious that $\mathrm{ECQ}(1,2)$ and $\mathrm{ECQ}(2,2)$ are Hamiltonian. According to Theorem 7, $\mathrm{ECQ}(3,2)$ can be decomposed into two $\mathrm{ECQ}(2,2)$ and thus is Hamiltonian. In the same way, we can inductively obtain that $\mathrm{ECQ}(4,2)$, $\mathrm{ECQ}(5,2),\ldots,$ $\mathrm{ECQ}(s,2)$, $\mathrm{ECQ}(t,2)$ are all Hamilton connected. According to Theorem 6, $\mathrm{ECQ}(t,2)$ is isomorphic to $\mathrm{ECQ}(2,t)$, and thus, $\mathrm{ECQ}(2,t)$ is also Hamilton connected. According to Theorem 7, $\mathrm{ECQ}(3,t)$ can be decomposed into two $\mathrm{ECQ}(2,t)$. As a result, $\mathrm{ECQ}(3,t)$ is Hamiltonian. In the same way, we can prove that $\mathrm{CQ}(4,t)$, $\mathrm{ECQ}(5,t),\ldots,$ $\mathrm{ECQ}(s,t)$ are all Hamilton connected. □

3.2 Performance Evaluation

Table 3 presents a straightforward comparison among the HQ, CQ, EH, and ECQ in terms of the following properties: the total number of links, node degree, diameter, expandability, and decomposition, which have significant impact on the performance of a parallel computing system. In general, for a desirable interconnection structure, the total

TABLE 3
Comparison of Various Networks ($n = s+t+1$)

Network	Links	Node Degree	Diameter	Expandability	Decomposition
HQ_n	$n2^{n-1}$	n	n	Sub-optimal	Complex
CQ_n	$n2^{n-1}$	n	$\lceil\frac{n+1}{2}\rceil$	Sub-optimal	Complex
$\mathrm{EH}(s,t)$	$(n+1)2^{n-2}$	$s+1/t+1$	$s+t+1$	Optimal	Simple
$\mathrm{ECQ}(s,t)$	$(n+1)2^{n-2}$	$s+1/t+1$	$\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+2$	Optimal	Simple

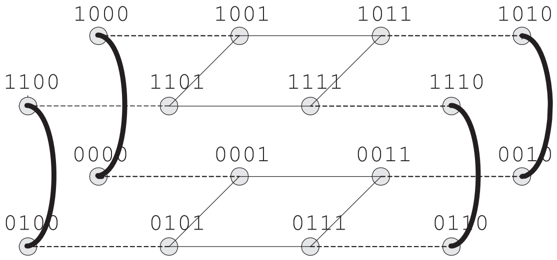

Fig. 3. The exchanged hypercube $\mathrm{EH}(1, 2)$.

number of links, the number of links per node, and the diameter should all be as small as possible. A network with a large number of links or a large node degree tends to increase the hardware cost. A smaller diameter means the lower communication overheads. Table 3 shows that these networks have their own advantages and disadvantages. For example, the diameter of the CQ is the smallest, i.e., $\lceil \frac{n+1}{2} \rceil$; but it requires much more links than the EH and ECQ. Among these networks, we find that the ECQ offers the best balance among the properties listed in Table 3. Such balanced performance and cost make the ECQ suitable for large-scale parallel computing systems.

4 Communication in Exchanged Crossed Cube

In large-scale parallel and distributed systems, communication is an important problem, which considers how the processors can exchange messages efficiently and reliably. Two most important communication primitives are routing and broadcasting. This section will present the routing and broadcasting algorithms we develop for the ECQ.

4.1 Routing

An optimal routing algorithm is to find the shortest path between a source and destination pair, where the source sends a message to the destination. Suppose the source is $u = a_{s-1} \cdots a_0 b_{t-1} \cdots b_0 c$ and the destination is $v = a'_{s-1} \cdots a'_0 b'_{t-1} \cdots b'_0 c'$. According to the definition of the ECQ and Table 1, we can divide the path from u to v into three parts, namely, $\rho(a, a')$, $\rho(b, b')$, and $\rho(c, c')$, respectively. There are four cases to be considered:

- *Case 1:* $a = a'$ and $b = b'$. There are two subcases:
  - Case 1.1: $c = c'$. The source and the destination are identical.
  - Case 1.2: $c \neq c'$. It is obvious that the communication between u and v can take place on the 0-dimension edge.
- *Case 2:* $a = a'$ and $b \neq b'$. There are four subcases:
  - Case 2.1: $c = c' = 1$. From the definition of the ECQ and Theorem 8, it is easy to find that u and v are in the same CQ_t. Routing in the same CQ can be done by the routing algorithm developed in [4], which is better than the lookup table approach discussed in [1]. The number of hops is bounded by $\rho(b, b')$.
  - Case 2.2: $c = 0, c' = 1$. A message can be sent to u's neighbor $u_1 = a_{s-1} \cdots a_0 b_{t-1} \cdots b_0 1$ via the 0-dimension edge, which results in u_1 and v in the same CQ_t.
  - Case 2.3: $c = 1, c' = 0$. We must find the vertex $v_1 = a'_{s-1} \cdots a'_0 b'_{t-1} \cdots b'_0 1$, which is a neighbor of v. This makes that v_1 and u are in the same CQ_t. So, a message can be sent from u to v_1, and then from v_1 to v via the 0-dimension edge. The number of hops is now bounded by $\rho(b, b') + 1$.
  - Case 2.4: $c = c' = 0$. We cannot find an intermediate vertex like u_1 or v_1 to transmit a message in one CQ. However, from Cases 2.2 and 2.3, we can find the neighbor u_1 of the source u and the neighbor v_1 of the destination v, such that u_1 and v_1 are in the same CQ_t. So, a message can be transmitted in this way: $u \rightarrow u_1 \rightarrow v_1 \rightarrow v$. Thus, the number of hops is bounded by $\rho(b, b') + 2$.
- *Case 3:* $a \neq a'$ and $b = b'$. $\mathrm{ECQ}(s, t)$ and $\mathrm{ECQ}(t, s)$ are isomorphic according to Theorem 6. So, the routing procedure is similar to that of Case 2. However, the distance $\rho(b, b')$ in Case 2 should be changed to $\rho(a, a')$.
- *Case 4:* $a \neq a'$ and $b \neq b'$. There must be a processor $u_3 = a'_{s-1} \cdots a'_0 b_{t-1} \cdots b_0 c$ in the same CQ_s with u. We first transmit the message from u to u_3 to make u_3 and v satisfying Case 2. Then, the problem is reduced to that deserved in Case 2. The number of hops in the first step is $\rho(a, a')$. So, the total number of hops between u and v is obtained by adding $\rho(a, a')$ to the number of hops in each subcase of Case 2. However, it is bounded by $\rho(a, a') + \rho(b, b') + 2$.

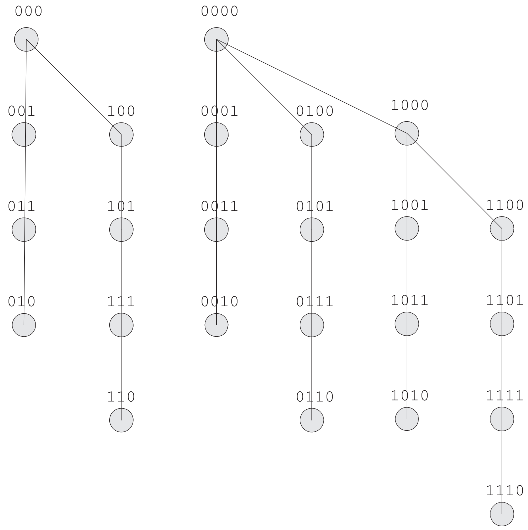

Fig. 4. The exchanged tree $ET(1, 1)$ and $ET(2, 1)$.

4.2 Broadcasting

Broadcasting is a communication pattern in a network where a data set is to be copied from one node to all other nodes. The exchanged tree proposed in [6] provides a very good method for constructing a spanning tree for broadcasting. However, there is a serious problem in the ET, which is derived by taking the wrong link and the wrong degree. Thus, the ET is not a spanning tree of the EH. The $\mathrm{EH}(1, 2)$ shown in Fig. 3 and $\mathrm{ET}(2, 1)$ shown in Fig. 4 are

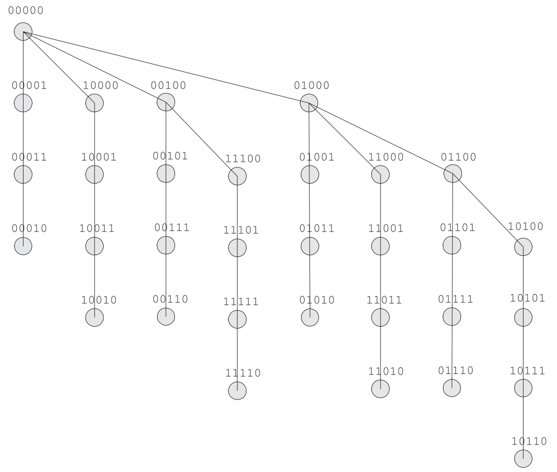

Fig. 5. The improved exchanged tree $\mathrm{IET}(1,3)$.

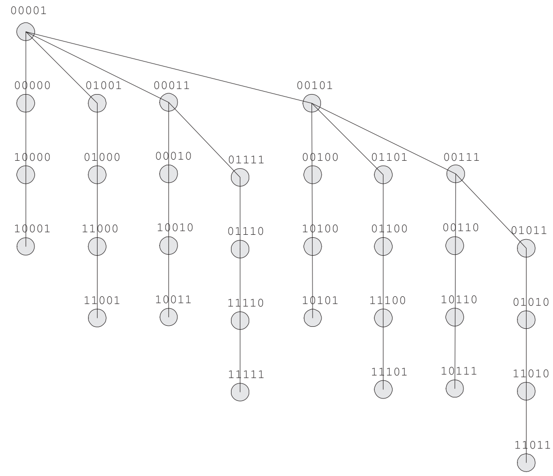

Fig. 6. The improved exchanged tree $\mathrm{IET}(3,1)$.

obtained by the construction procedure proposed in [6]. Obviously, link $(0000, 0100)$ in $\mathrm{ET}(2,1)$ is not an edge of vertices 0000 and 0100 in $\mathrm{EH}(1,2)$. Thus, $\mathrm{ET}(2,1)$ is not a spanning tree of $\mathrm{EH}(1,2)$. To overcome this defection, necessary modifications to ET should be made as follows:

The improved ET is a spanning tree of an EH and an ECQ that provides an effective and efficient way for broadcasting in an EH and an ECQ. The construction of the improved exchanged tree (denoted by $\mathrm{IET}(s,t)$ for short) is divided into two cases.

Case 1. When $s \le t$, $\mathrm{IET}(s,t)$ is constructed by the sequence:

$$IET(1,1) \to IET(1,2) \to IET(1,3) \to \cdots \to IET(1,t) \to IET(2,t) \to \cdots \to IET(s,t).$$

If $\mathrm{IET}(1,t)$ is given, $\mathrm{IET}(1,t+1)$ is constructed as follows: Let G_1 and G_2 be two $\mathrm{IET}(1,t)$s. Relabel G_1 and G_2 by inserting a 0 and a 1 between the most significant first and second bits of the original vertex addresses, respectively. Thus, there are two mappings:

$$a_0 b_{t-1} b_{t-2} \dots b_0 c \to a_0 0 b_{t-1} b_{t-2} \dots b_0 c;$$
$$a_0 b_{t-1} b_{t-2} \dots b_0 c \to a_0 1 b_{t-1} b_{t-2} \dots b_0 c.$$

Then, make the root of G_2 the rightmost child of G_1's root. If $\mathrm{IET}(s,t)$ is given, $\mathrm{IET}(s+1,t)$ is constructed as follows: Let G_3 and G_4 be two $\mathrm{IET}(s,t)$s. Relabel G_3 and G_4 by adding a 0 and a 1 to the leftmost bit of the original vertex addresses, respectively. Thus, there also are two mappings:

$$a_0 b_{t-1} b_{t-2} \dots b_0 c \to 0 a_0 b_{t-1} b_{t-2} \dots b_0 c;$$
$$a_0 b_{t-1} b_{t-2} \dots b_0 c \to 1 a_0 b_{t-1} b_{t-2} \dots b_0 c.$$

Then, make the root of G_4 the rightmost child of G_3's root. So, an $\mathrm{IET}(s,t)$ is constructed when $s \le t$.

Case 2. When $s > t$, Theorem 6 reveals that $\mathrm{ECQ}(s,t)$ is isomorphic to $\mathrm{ECQ}(t,s)$. So, $\mathrm{IET}(s,t)$ can be constructed in a similar way. Based on the method of Case 1, we can construct the $\mathrm{IET}(t,s)$, where $t < s$. Then, for an arbitrary vertex u in $\mathrm{IET}(t,s)$, denoted by $u = a_{t-1} \dots a_0 b_{s-1} \dots b_0 c$, we can find the vertex $v = a'_{t-1} \dots a'_0 b'_{s-1} \dots b'_0 c'$ in $\mathrm{IET}(s,t)$, satisfying the mapping function f:

$$a_{t-1} = b'_{t-1}, a_{t-2} = b'_{t-2}, \dots, a_0 = b'_0,$$
$$b_{s-1} = a'_{s-1}, b_{s-2} = a'_{s-2}, \dots, b_0 = a'_0, c = \bar{c'}.$$

Because f is a bijection, an $\mathrm{IET}(s,t)$ is constructed when $s > t$.

According to Cases 1 and 2, an $\mathrm{IET}(s,t)$ with arbitrary integers s and t can be constructed.

The improved exchanged tree inherits all the properties of the exchanged tree. An extra but very important property of $\mathrm{IET}(s,t)$ is presented below.

Theorem 11. $\mathrm{IET}(s,t)$ *is a spanning tree of* $\mathrm{ECQ}(t,s)$.

Proof. From the construction procedure for $\mathrm{IET}(s,t)$, all vertices in $\mathrm{ECQ}(t,s)$ are used in $\mathrm{IET}(s,t)$. By the construction in Case 1 and the fact that 0^{s+t+1} and 10^{s+t} are neighbors, it is known that each edge in $\mathrm{IET}(s,t)$ is also in $\mathrm{ECQ}(t,s)$ when $s \le t$. Then, based on the bijection f and the construction in Case 2, it is known that each edge in $\mathrm{IET}(s,t)$ is also in $\mathrm{ECQ}(t,s)$ when $s > t$. So, $\mathrm{IET}(s,t)$ is a subset of $\mathrm{ECQ}(t,s)$. Because $\mathrm{IET}(s,t)$ has the structure of a tree, it is a spanning tree of $\mathrm{ECQ}(t,s)$. □

Theorem 11 provides a good way of broadcasting in $\mathrm{ECQ}(s,t)$. A message can be broadcasted from one node to all other nodes by the $\mathrm{IET}(t,s)$. Figs. 5 and 6 show the spanning tree of $\mathrm{ECQ}(3,1)$ and $\mathrm{ECQ}(1,3)$, respectively. From Figs. 5, 6, and 7, we can know how to construct $\mathrm{IET}(3,1)$ which is a spanning tree of $\mathrm{ECQ}(1,3)$.

5 Conclusions

This paper presents a new interconnection topology called exchanged crossed cube. A recursive procedure for constructing an ECQ is proposed. This new topology has many desirable properties such as regularity, expandability, isomorphism, and decomposition. The diameter and node degree of an ECQ have low values, and hence, the cost factor of the ECQ is less than that of other topologies such as the hypercube, crossed cube, and exchanged hypercube. An optimal routing algorithm that guarantees the shortest path and a broadcasting algorithm are developed. The attractive properties of the ECQ make it applicable to large-scale parallel computing systems very well.

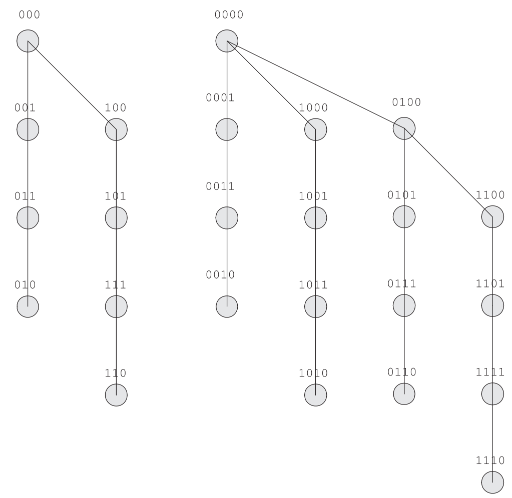

Fig. 7. The improved exchanged tree $\mathrm{IET}(1,1)$ and $\mathrm{IET}(1,2)$.

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