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Optimal number of annuli for maximizing the lifetime of sensor networks

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HIGHLIGHTS

- Maximize the lifetime of a wireless sensor network by optimal network design.
- Represent the network lifetime as a function of the number of annuli.
- Find an expression of the optimal number of annuli for an arbitrary sensor density function.

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ABSTRACT

The most effective way to maximize the lifetime of a wireless sensor network (WSN) is to allocate initial energy to sensors such that they exhaust their energy at the same time. The lifetime of a WSN as well as an optimal initial energy allocation are determined by a network design. The main contribution of the paper is to show that the lifetime of a WSN can be maximized by an optimal network design. We represent the network lifetime as a function of the number m of annuli and show that m has significant impact on network lifetime. We prove that if the energy consumed by data transmission is proportional to $d^\alpha + c$, where d is the distance of data transmission and α and c are some constants, then for a circular area of interest with radius R, the optimal number of annuli that maximizes the network lifetime is $m = R((\alpha - 1)/c)^{1/\alpha}$ for an arbitrary sensor density function.

1. Introduction

Wireless sensor networks (WSNs) provide pervasive instrumentation that enables us to observe and interact with the physical and social world and to realize the vision of an embedded Internet. WSNs consisting of mass-produced intelligent sensors have been widely used in environmental and habitat monitoring, climate control, surveillance, intelligent alarms, structural monitoring, ecophysiology, equipment maintenance, medical diagnostics, disaster management, emergence response, asset tracking, healthcare, and manufacturing process flow [9,10].

Due to severe energy constraint in sensors, the lifetime of a WSN has gained substantial research attention [13]. Energy consumption in WSNs contains two components, namely, the energy required for data sensing and the energy used for data transmission. Research in lifetime maximization of WSNs has been focused on the first component only [5,6,11,25], and the second component only [7,12,17,18,22], and both components [1,2,29]. We believe that the lifetime maximization problem of WSNs should be studied by taking both components of energy consumption into consideration [8].

Several methods have been proposed to increase the lifetime of a WSN, including redundant sensors [28], nonuniform sensor distributions [26], and aggregation and forwarding nodes for data transmission [15,27]. All these methods are based on the observation that sensors consume their battery power at different speeds. In particular, sensors close to a base station consume energy much faster than sensors far away from the base station [16,21]. Therefore, the most effective way to maximize the lifetime of a WSN is to allocate initial energy to sensors such that they exhaust their energy at the same time [1,20,23,24].

We find that the lifetime of a WSN as well as an optimal initial energy allocation are determined by a network design. Network lifetime maximization is a two-stage process, namely, optimal network design and optimal energy allocation. In reality, a WSN design includes the locations, sensing ranges, communication ranges, and data generation rates of all sensors, energy consumption for both data sensing and data transmission, as well as a routing algorithm for data transmission to a base station (i.e., a sink). All these factors have impact on sensor and network lifetime as well as optimal energy allocation [20].

The main contribution of the paper is to show that the lifetime of a WSN can be maximized by an optimal network design. By proper modeling and simplification, we represent the network

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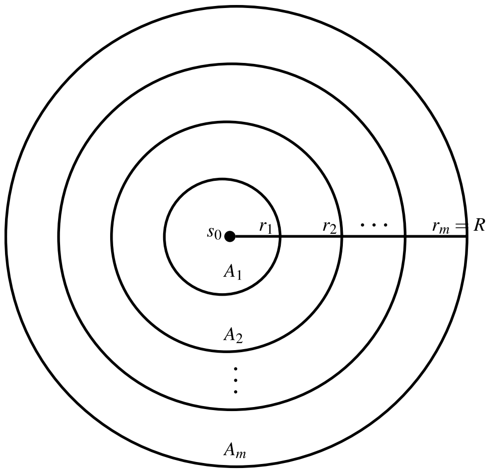

Fig. 1. A circular area of radius R with m annuli.

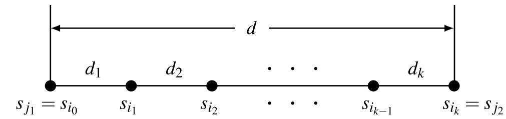

Fig. 2. A data transmission path.

lifetime obtained by optimal energy allocation as a function of the number m of annuli and show that m has a significant impact on network lifetime. We prove that if the energy consumed by data transmission is proportional to $d^\alpha + c$, where d is the distance of data transmission and α and c are some constants, then for a circular area of interest with radius R, the optimal number of annuli that maximizes the network lifetime is

$$m = R\left(\frac{\alpha-1}{c}\right)^{1/\alpha},$$

for an arbitrary sensor density function. (Notice that for real applications, m should be rounded to the nearest integer, i.e., either $\lfloor m\rfloor$ or $\lceil m\rceil$; however, we will eliminate such notations for clarity of presentation.)

The organization of the paper is as follows. In Section 2, we present the network model used in our study. In Section 3, we develop analytical forms of network lifetime and optimal energy allocation. In Section 4, we derive the optimal number of annuli for a uniform distribution. In Section 5, we extend our results in Section 3 to arbitrary sensor density functions. In Section 6, we derive the optimal number of annuli for a nonuniform distribution. In Section 7, we demonstrate numerical examples. In Section 8, we prove our general result. We conclude the paper in Section 9.

2. The network model

Let us consider a circular area of interest A which has radius R meters (see Fig. 1). Assume that A is divided into m annuli (also called coronae) $A_1, A_2, \ldots, A_m$ by m circles with radii $r_1, r_2, \ldots, r_m$ centered at a sink, where $0 < r_1 < r_2 < \cdots < r_m = R$ [23]. For convenience, we assume that there is A_0 with width $r_0 = 0$ which contains a sink. All sensors report sensory data to the sink. For a fixed R, the number m of annuli as well as the sequence of values $(r_1, r_2, \ldots, r_{m-1})$ is called a *network design* or a *network configuration*, which has a significant impact on energy consumption and network lifetime.

Annulus A_j has width $r_j - r_{j-1}$, where $1 \le j \le m$. In this paper, we consider the case when all annuli have identical width r, i.e., $r_j - r_{j-1} = r$ for all $1 \le j \le m$. In other words, we have $r_j = jr$, where $r = R/m$.

Assume that there are N sensors $s_1, s_2, \ldots, s_N$ uniformly distributed in A (later, we will consider nonuniform sensor distributions). We use s_0 to represent a sink. All sensors in A_j are designed in such a way that they have the same transmission range $r_j - r_{j-1}$. All sensors also have certain sensing range. It is assumed that N is sufficiently large such that a WSN is connected. Furthermore, it is assumed that the sensing range is sufficiently large such that A is well covered. Let N_j be the number of sensors in A_j. Then, we have

$$N_j = \left(\frac{\pi r_j^2 - \pi r_{j-1}^2}{\pi R^2}\right)N = \left(\frac{r_j^2 - r_{j-1}^2}{R^2}\right)N = \left(\frac{2j-1}{m^2}\right)N,$$

and $N = N_1 + N_2 + \cdots + N_m$.

The amount of energy consumed by a sensor to sense and receive data in one unit of time is p mJ/s.

The amount of energy needed to transmit one bit over distance d meters is $(a_1 d^\alpha + a_2)$ pJ, where a_1 is the energy required to run a transmitter amplifier, a_2 is the energy used to activate a transmitter circuitry, and $2 \le \alpha \le 6$ is a constant [14]. The above expression has significant implication in minimizing energy cost of data transmission in WSNs. Consider a sensor s_{j_1} which sends a bit to another sensor s_{j_2} along a path $(s_{i_0}, s_{i_1}, s_{i_2}, \ldots, s_{i_k})$ with k hops, where $i_0 = j_1$ and $i_k = j_2$ (see Fig. 2). For simplicity, we assume that the $(k+1)$ sensors are on the same line, such that the distance between s_{j_1} and s_{j_2} is d, and the distance between $s_{i_{l-1}}$ and s_{i_l} is d_l, for all $1 \le l \le k$, with $d_1 + d_2 + \cdots + d_k = d$. Then, the energy consumed by the above data transmission is a function of $d_1, d_2, \ldots, d_k$,

$$E(d_1, d_2, \ldots, d_k) = \sum_{l=1}^{k}(a_1 d_l^\alpha + a_2).$$

It has been known that due to the convexity of d^α, the above function is minimized when $d_1 = d_2 = \cdots = d_k = d/k$ [3,23]. Hence, $E(d_1, d_2, \ldots, d_k)$ becomes a function of k,

$$E(k) = k\left(a_1\left(\frac{d}{k}\right)^\alpha + a_2\right) = \frac{a_1 d^\alpha}{k^{\alpha-1}} + a_2 k.$$

It is clear that to minimize $E(k)$, the first term prefers multiple hops of short distance, while the second term prefers a single hop of long distance. The function $E(k)$ is minimized when

$$\frac{dE(k)}{dk} = -\frac{a_1(\alpha-1)d^\alpha}{k^\alpha} + a_2 = 0,$$

that is,

$$k = \left(\frac{a_1}{a_2}(\alpha-1)\right)^{1/\alpha} d,$$

which gives

$$\begin{aligned} E(k) &= \left(\frac{a_1}{a_2}(\alpha-1)\right)^{1/\alpha}\left(\frac{\alpha}{\alpha-1}\right)a_2 d \\ &= a_1^{1/\alpha} a_2^{1-1/\alpha}\left(\frac{\alpha}{(\alpha-1)^{1-1/\alpha}}\right)d. \end{aligned}$$

Such a phenomenon inspires the optimal network configuration problem solved in this paper.

Assume that each datum has size b bytes $= 8b$ bits. Then, the amount of energy needed to transmit one datum over distance d meters is

$$\begin{aligned} q &= 8b(a_1 d^\alpha + a_2)\ \text{pJ} = 8a_1 b\left(d^\alpha + \frac{a_2}{a_1}\right)\ \text{pJ} \\ &= \frac{8a_1 b}{10^6}(d^\alpha + c)\ \text{mJ} = a(d^\alpha + c)\ \text{mJ}, \end{aligned}$$

where $a = a_1 b/125{,}000$ mJ/m^α and $c = a_2/a_1$ m^α. For instance, when $b = 25$ bytes, $a_1 = 10$ pJ/bit/m^α, $a_2 = 50$ nJ/bit $= 50{,}000$ pJ/bit, we have $a = 0.002$ mJ/m^α, $c = 5000$ m^α, and $q = 0.002d^\alpha + 10$ mJ. Based on the above discussion, we know that the amount of energy consumed by a sensor in A_j to transmit a datum is

$$q_j = a((r_j - r_{j-1})^\alpha + c) = a(r^\alpha + c) \text{ mJ},$$

for all $1 \le j \le m$.

When a datum is transmitted from a sensor s_j in A_j to the sink s_0, the datum is sent along a path $(s_j, s_{j-1}, s_{j-2}, \ldots, s_1, s_0)$ from s_j to s_0, where $s_i \in A_i$ for all $j \ge i \ge 0$. Assume that each sensor senses and transmits μ data to a sink per second. This implies that sensors in A_j contribute $N_j\mu$ data transmissions per second to all A_i's, where $1 \le i \le j$. It is also assumed that all sensors in A_j are treated equally such that they all perform the same amount of data transmission. Since there are $(N_j + N_{j+1} + \cdots + N_m)\mu$ data transmissions to be performed by N_j sensors in A_j per second, a sensor in A_j performs β_j data transmissions in one unit of time, where

$$\beta_j = \frac{1}{N_j}(N_j + N_{j+1} + \cdots + N_m)\mu = \left(\frac{r_m^2 - r_{j-1}^2}{r_j^2 - r_{j-1}^2}\right)\mu = \left(\frac{m^2 - (j-1)^2}{2j-1}\right)\mu,$$

for all $1 \le j \le m$.

A sensor in A_j is equipped with E_j amount of initial energy. Let E denote the total energy budget, i.e.,

$$E = \sum_{j=1}^{m} N_j E_j.$$

Once a sensor is deployed, E_j is not renewable or replenishable. The network lifetime is determined by the *initial energy allocation* $(E_1, E_2, \ldots, E_m)$, which is determined by a network design, i.e., $(r_1, r_2, \ldots, r_{m-1})$.

3. Network lifetime and optimal energy allocation

The energy consumed by a sensor in A_j in one unit of time is $p + \beta_j q_j$, which implies that the lifetime of a sensor in A_j is

$$L_j = \frac{E_j}{p + \beta_j q_j}.$$

A reasonable definition of *network lifetime* L is

$$L = \min(L_1, L_2, \ldots, L_m),$$

since when all sensors in A_j run out of battery power, a WSN becomes disconnected and in-operational. It is clear that L is maximized if and only if $L_1 = L_2 = \cdots = L_m$, that is, all sensors die at the same time; otherwise, we can allocate energy from sensors which work longer to sensors which die sooner so that the network lifetime is increased.

To have an identical lifetime L for all the sensors, i.e.,

$$\frac{E_j}{p + \beta_j q_j} = L,$$

we need

$$E_j = L(p + \beta_j q_j),$$

for all $1 \le j \le m$. Since $N_1 E_1 + N_2 E_2 + \cdots + N_m E_m = E$, i.e.,

$$\sum_{j=1}^{m} N_j L(p + \beta_j q_j) = E,$$

we get

$$L = \frac{E}{\sum_{j=1}^{m} N_j(p + \beta_j q_j)},$$

and

$$E_j = \frac{(p + \beta_j q_j)E}{\sum_{j=1}^{m} N_j(p + \beta_j q_j)},$$

for all $1 \le j \le m$. An initial energy allocation $(E_1, E_2, \ldots, E_m)$ that results in $L_1 = L_2 = \cdots = L_m = L$ is called an *optimal energy allocation*.

Notice that for a uniform distribution of sensors, we have

$$\begin{aligned}
\sum_{j=1}^{m} N_j(p + \beta_j q_j) &= \sum_{j=1}^{m} N_j p + \sum_{j=1}^{m} N_j \beta_j q_j \\
&= Np + \sum_{j=1}^{m} (N_j + N_{j+1} + \cdots + N_m)\mu q_j \\
&= Np + \sum_{j=1}^{m} \left(\frac{r_m^2 - r_{j-1}^2}{R^2}\right) N\mu a((r_j - r_{j-1})^\alpha + c) \\
&= Np + \frac{N\mu a}{R^2} \sum_{j=1}^{m} (r_m^2 - r_{j-1}^2)((r_j - r_{j-1})^\alpha + c) \\
&= N\left(p + \frac{\mu a}{R^2} \sum_{j=1}^{m} (r_m^2 - r_{j-1}^2)((r_j - r_{j-1})^\alpha + c)\right).
\end{aligned}$$

Hence, the network lifetime is

$$L = \frac{E}{N\left(p + \frac{\mu a}{R^2} \sum_{j=1}^{m} (r_m^2 - r_{j-1}^2)((r_j - r_{j-1})^\alpha + c)\right)}.$$

Since

$$\begin{aligned}
p + \beta_j q_j &= p + \frac{1}{N_j}(N_j + N_{j+1} + \cdots + N_m) \\
&\quad \times \mu a((r_j - r_{j-1})^\alpha + c) \\
&= p + \left(\frac{r_m^2 - r_{j-1}^2}{r_j^2 - r_{j-1}^2}\right)\mu a((r_j - r_{j-1})^\alpha + c),
\end{aligned}$$

we obtain the optimal energy allocation

$$E_j = \frac{E}{N} \cdot \frac{p + \left(\frac{r_m^2 - r_{j-1}^2}{r_j^2 - r_{j-1}^2}\right)\mu a((r_j - r_{j-1})^\alpha + c)}{p + \frac{\mu a}{R^2} \sum_{j=1}^{m} (r_m^2 - r_{j-1}^2)((r_j - r_{j-1})^\alpha + c)},$$

for all $1 \le j \le m$.

It is clear that the network lifetime L is a function of $r_1, r_2, \ldots, r_{m-1}$. To maximize the network lifetime, we need to minimize the following function:

$$F(r_1, r_2, \ldots, r_{m-1}) = \frac{1}{R^2} \sum_{j=1}^{m} (r_m^2 - r_{j-1}^2)((r_j - r_{j-1})^\alpha + c),$$

where $r_0 = 0$ and $r_m = R$. The above function gives the average number of m^α taken by a single data transmission. It is clear that $F(r_1, r_2, \ldots, r_{m-1})$ is a quantity determined by a network design $(r_1, r_2, \ldots, r_{m-1})$ and $F(r_1, r_2, \ldots, r_{m-1})$ determines the energy expenditure of data transmission.

When all annuli have identical width r, i.e., $r_1 = r_2 = \cdots = r_m = r$, an optimal network design is actually an optimal choice of m, the number of annuli. This is the focus of our investigation in this paper.

4. Optimal number of annuli—uniform distributions

When all annuli have identical width $r = R/m$, the function $F(r_1, r_2, \ldots, r_{m-1})$ becomes a function of m:

$$F(m) = \frac{1}{R^2}\sum_{j=1}^{m}(m^2-(j-1)^2)r^2(r^\alpha+c)$$
$$= \sum_{j=1}^{m}\left(1-\left(\frac{j-1}{m}\right)^2\right)\left(\left(\frac{R}{m}\right)^\alpha+c\right)$$
$$= \left(m-\frac{1}{m^2}\sum_{j=1}^{m}(j-1)^2\right)\left(\left(\frac{R}{m}\right)^\alpha+c\right)$$
$$= \left(m-\frac{1}{m^2}\cdot\frac{m(m-1)(2m-1)}{6}\right)\left(\left(\frac{R}{m}\right)^\alpha+c\right)$$
$$= \left(\frac{4m^2+3m-1}{6m}\right)\left(\frac{cm^\alpha+R^\alpha}{m^\alpha}\right)$$
$$= \frac{1}{6}\left(\frac{4\,cm^{\alpha+2}+3\,cm^{\alpha+1}-cm^\alpha+4R^\alpha m^2+3R^\alpha m-R^\alpha}{m^{\alpha+1}}\right)$$
$$= \frac{1}{6}\left(4\,cm+3c-\frac{c}{m}+\frac{4R^\alpha}{m^{\alpha-1}}+\frac{3R^\alpha}{m^\alpha}-\frac{R^\alpha}{m^{\alpha+1}}\right).$$

To minimize $F(m)$, we consider

$$\frac{dF(m)}{dm} = \frac{1}{6}\left(4c+\frac{c}{m^2}-\frac{4(\alpha-1)R^\alpha}{m^\alpha}-\frac{3\alpha R^\alpha}{m^{\alpha+1}}+\frac{(\alpha+1)R^\alpha}{m^{\alpha+2}}\right)$$
$$= \frac{1}{6}\left(\frac{4\,cm^{\alpha+2}+cm^\alpha-4(\alpha-1)R^\alpha m^2-3\alpha R^\alpha m+(\alpha+1)R^\alpha}{m^{\alpha+2}}\right).$$

To satisfy the condition $dF(m)/dm = 0$, we need to solve the equation

$$4cm^{\alpha+2}+cm^\alpha-4(\alpha-1)R^\alpha m^2-3\alpha R^\alpha m+(\alpha+1)R^\alpha = 0.$$

Although there is no closed-form solution to the above equation, it can be solved numerically by using the bisection method [4, p. 22].

Notice that by considering only the dominant terms in the above equation, we get

$$4cm^{\alpha+2}-4(\alpha-1)R^\alpha m^2 = 0,$$

that is,

$$cm^\alpha-(\alpha-1)R^\alpha = 0.$$

The last equation implies that

$$m \approx R\left(\frac{\alpha-1}{c}\right)^{1/\alpha},$$

which can be used as an approximate solution to the equation of m.

By using the above closed-form approximation of m and the fact that

$$F(m) \approx \frac{2m}{3}\left(\left(\frac{R}{m}\right)^\alpha+c\right),$$

we get network lifetime

$$L = \frac{E}{N(p+\mu aF(m))}$$
$$\approx \frac{E}{N\left(p+\mu a\cdot\frac{2m}{3}\left(\left(\frac{R}{m}\right)^\alpha+c\right)\right)}$$
$$\approx \frac{E}{N\left(p+\mu a\cdot\frac{2}{3}R\left(\frac{\alpha-1}{c}\right)^{1/\alpha}\left(\frac{c}{\alpha-1}+c\right)\right)}$$
$$= \frac{E}{N\left(p+\frac{2}{3}\alpha\mu aR\left(\frac{c}{\alpha-1}\right)^{1-1/\alpha}\right)}.$$

Table 1
Optimal number of annuli.

α	Optimal m	Numerical solution	Closed-form approximation
2.0	3	3.29258	2.82843
2.5	8	8.21365	7.79612
3.0	15	15.07804	14.73613
3.5	23	23.08025	22.79705
4.0	32	31.54171	31.30169
4.5	40	40.01296	39.80528
5.0	48	48.22777	48.04498
5.5	56	56.04305	55.87994
6.0	63	63.39276	63.24555

Notice that the function

$$g(\alpha) = \alpha\left(\frac{c}{\alpha-1}\right)^{1-1/\alpha}$$

is an increasing function of α, since $dg(\alpha)/d\alpha > 0$ for all $\alpha \geq 2$. The last equation for L means that for a given energy budget E, the lifetime of a WSN is a decreasing function of seven parameters: N, p, α, μ, R, a, and c. In other words, the network lifetime is reduced if there are more sensors, more energy consumption for sensing and receiving data, increased value of α, increased sensor reporting rate, larger area of interest, more energy consumption for transmitter amplification and activation.

In Table 1, we show the optimal value of m for $\alpha = 2.0, 2.5, 3.0, \ldots, 6.0$, with $c = 5000$ and $R = 200$. We also give the numerical solution to the equation of m and our closed-form approximate solution. It is clear that the optimal value of m is the numerical solution rounded to the nearest integer. It is also observed that the closed-form approximate solution is very accurate.

Finally, the optimal energy allocation is

$$E_j = \frac{E}{N(p+\mu aF(m))}\left(p+\left(\frac{m^2-(j-1)^2}{2j-1}\right)\right.$$
$$\left.\times\ \mu a\left(\left(\frac{R}{m}\right)^\alpha+c\right)\right),$$

for all $1 \leq j \leq m$.

5. Nonuniform sensor distributions

All our studies in previous sections can be easily extended to any sensor distributions.

Let $f(r)$ be any *sensor density function* (or *sensor distribution function*) in a circular area of interest A with radius R, where $0 \leq r \leq R$. In other words, the number of sensors in a small area z with distance r to the sink is $f(r)z$. The function $f(r)$ should satisfy

$$\int_0^R 2\pi rf(r)dr = N.$$

The number of sensors in A_j is

$$N_j = \int_{r_{j-1}}^{r_j} 2\pi rf(r)dr,$$

for all $1 \leq j \leq m$. For instance, for a uniform distribution, we have

$$f(r) = \frac{N}{\pi R^2},$$

and

$$\int_0^R 2\pi rf(r)dr = \frac{N}{R^2}\int_0^R 2rdr = \left(\frac{N}{R^2}\right)r^2\Big|_0^R = N,$$

and

$$N_j = \frac{N}{R^2}\int_{r_{j-1}}^{r_j} 2rdr = \left(\frac{N}{R^2}\right)r^2\Big|_{r_{j-1}}^{r_j} = \left(\frac{r_j^2-r_{j-1}^2}{R^2}\right)N,$$

for all $1 \leq j \leq m$.

To extend the results in Section 3, we notice that

$$N_j + N_{j+1} + \cdots + N_m = \int_{r_{j-1}}^{r_m} 2\pi r f(r) dr.$$

Hence, the network lifetime is

$$L = \frac{E}{Np + \sum_{j=1}^{m} (N_j + N_{j+1} + \cdots + N_m)\mu q_j} = \frac{E}{Np + \mu a \sum_{j=1}^{m} \left(\int_{r_{j-1}}^{r_m} 2\pi r f(r) dr\right)((r_j - r_{j-1})^\alpha + c)}.$$

Since

$$p + \beta_j q_j = p + \frac{1}{N_j}(N_j + N_{j+1} + \cdots + N_m)\mu a((r_j - r_{j-1})^\alpha + c) = p + \frac{\int_{r_{j-1}}^{r_m} 2\pi r f(r) dr}{\int_{r_{j-1}}^{r_j} 2\pi r f(r) dr} \cdot \mu a((r_j - r_{j-1})^\alpha + c),$$

the optimal energy allocation is

$$E_j = E \cdot \frac{p + \left(\int_{r_{j-1}}^{r_m} 2\pi r f(r) dr\right)\left(\int_{r_{j-1}}^{r_j} 2\pi r f(r) dr\right)^{-1} \mu a((r_j - r_{j-1})^\alpha + c)}{Np + \mu a \sum_{j=1}^{m} \left(\int_{r_{j-1}}^{r_m} 2\pi r f(r) dr\right)((r_j - r_{j-1})^\alpha + c)},$$

for all $1 \le j \le m$.

6. Optimal number of annuli—nonuniform distributions

As an example of nonuniform sensor distribution, let us consider

$$f(r) = \left(\frac{N}{\pi \ln(1 + 1/u)}\right)\left(\frac{1}{r^2 + uR^2}\right),$$

where $u > 0$. It is easy to see that

$$\begin{aligned}\int_0^R 2\pi r f(r) dr &= \int_0^R 2\pi r \left(\frac{N}{\pi \ln(1 + 1/u)}\right)\left(\frac{1}{r^2 + uR^2}\right) dr \\ &= \frac{N}{\ln(1 + 1/u)} \int_0^R \left(\frac{1}{r^2 + uR^2}\right) d(r^2 + uR^2) \\ &= \frac{N}{\ln(1 + 1/u)} \int_{uR^2}^{(u+1)R^2} \frac{dx}{x} \quad (\text{letting } x = r^2 + uR^2) \\ &= \frac{N}{\ln(1 + 1/u)} \ln x \Big|_{uR^2}^{(u+1)R^2} \\ &= \frac{N}{\ln(1 + 1/u)} (\ln((u + 1)R^2) - \ln(uR^2)) \\ &= N.\end{aligned}$$

Notice that the ratio of the largest density (when $r = 0$) to the smallest density (when $r = R$) is $(1 + 1/u)$. Thus, the parameter u indicates uniformity of sensor distribution. For small u, sensors are more densely distributed in the area closer to the sink. As $u \to 0$, the sensor density near the sink can be arbitrarily large. One the other hand, as u increases, sensors are more evenly distributed in A. For very large u, we have $\ln(1 + 1/u) \approx 1/u$, and

$$f(r) \approx \left(\frac{N}{\pi(1/u)}\right)\left(\frac{1}{r^2 + uR^2}\right) = \frac{N}{\pi(r^2/u + R^2)} \approx \frac{N}{\pi R^2}.$$

That is, as $u \to \infty$, $f(r)$ approaches a uniform distribution.

The above $f(r)$ gives rise to

$$N_j = \frac{N}{\ln(1 + 1/u)} \ln\left(\frac{r_j^2 + uR^2}{r_{j-1}^2 + uR^2}\right) = \frac{N}{\ln(1 + 1/u)} \ln\left(\frac{j^2 + um^2}{(j - 1)^2 + um^2}\right),$$

for all $1 \le j \le m$. For very large u, we have

$$\ln\left(\frac{r_j^2 + uR^2}{r_{j-1}^2 + uR^2}\right) = \ln\left(1 + \frac{r_j^2 - r_{j-1}^2}{r_{j-1}^2 + uR^2}\right) \approx \frac{r_j^2 - r_{j-1}^2}{r_{j-1}^2 + uR^2},$$

and

$$N_j \approx \frac{N}{(1/u)} \cdot \frac{r_j^2 - r_{j-1}^2}{r_{j-1}^2 + uR^2} \approx \left(\frac{r_j^2 - r_{j-1}^2}{R^2}\right) N,$$

which is the N_j for a uniform distribution.

Since

$$N_j + N_{j+1} + \cdots + N_m = \frac{N}{\ln(1 + 1/u)} \ln\left(\prod_{i=j}^{m} \frac{r_i^2 + uR^2}{r_{i-1}^2 + uR^2}\right) = \frac{N}{\ln(1 + 1/u)} \ln\left(\frac{r_m^2 + uR^2}{r_{j-1}^2 + uR^2}\right),$$

we have

$$L = \frac{E}{N\left(p + \frac{\mu a}{\ln(1+1/u)} \sum_{j=1}^{m} \ln\left(\frac{r_m^2 + uR^2}{r_{j-1}^2 + uR^2}\right)((r_j - r_{j-1})^\alpha + c)\right)}.$$

Because

$$p + \beta_j q_j = p + \frac{1}{N_j}(N_j + N_{j+1} + \cdots + N_m)\mu a((r_j - r_{j-1})^\alpha + c) = p + \frac{\ln\left(\frac{r_m^2 + uR^2}{r_{j-1}^2 + uR^2}\right)}{\ln\left(\frac{r_j^2 + uR^2}{r_{j-1}^2 + uR^2}\right)} \cdot \mu a((r_j - r_{j-1})^\alpha + c),$$

we obtain

$$E_j = \frac{E}{N} \cdot \frac{p + \ln\left(\frac{r_m^2 + uR^2}{r_{j-1}^2 + uR^2}\right)\left(\ln\left(\frac{r_j^2 + uR^2}{r_{j-1}^2 + uR^2}\right)\right)^{-1} \mu a((r_j - r_{j-1})^\alpha + c)}{p + \frac{\mu a}{\ln(1+1/u)} \sum_{j=1}^{m} \ln\left(\frac{r_m^2 + uR^2}{r_{j-1}^2 + uR^2}\right)((r_j - r_{j-1})^\alpha + c)},$$

for all $1 \le j \le m$.

To maximize the network lifetime, we need to minimize the following function:

$$F(r_1, r_2, \ldots, r_{m-1}) = \frac{1}{\ln(1 + 1/u)} \sum_{j=1}^{m} \ln\left(\frac{r_m^2 + uR^2}{r_{j-1}^2 + uR^2}\right)((r_j - r_{j-1})^\alpha + c),$$

where $r_0 = 0$ and $r_m = R$.

When $r_j - r_{j-1} = r$ for all $1 \le j \le m$, i.e., $r_j = jr$, where $r = R/m$, the function $F(r_1, r_2, \ldots, r_{m-1})$ becomes a function of m:

$$F(m) = \frac{1}{\ln(1 + 1/u)} F_1(m),$$

where

$$\begin{aligned}
F_1(m) &= \sum_{j=1}^{m} \ln\left(\frac{R^2+uR^2}{((j-1)r)^2+uR^2}\right)(r^\alpha+c)\\
&= \sum_{j=1}^{m} \ln\left(\frac{(u+1)R^2}{((j-1)R/m)^2+uR^2}\right)\left(\left(\frac{R}{m}\right)^\alpha+c\right)\\
&= \left(\left(\frac{R}{m}\right)^\alpha+c\right)\sum_{j=1}^{m}\ln\left(\frac{(u+1)}{((j-1)/m)^2+u}\right)\\
&= \left(\left(\frac{R}{m}\right)^\alpha+c\right)\sum_{j=1}^{m}\ln\left(\frac{(u+1)m^2}{um^2+(j-1)^2}\right)\\
&= \left(\left(\frac{R}{m}\right)^\alpha+c\right)\\
&\quad\times\sum_{j=1}^{m}\left(\ln((u+1)m^2)-\ln(um^2+(j-1)^2)\right)\\
&= \left(\left(\frac{R}{m}\right)^\alpha+c\right)\\
&\quad\times\left(m\ln((u+1)m^2)-\sum_{j=1}^{m}\ln(um^2+(j-1)^2)\right).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\sum_{j=1}^{m}\ln(um^2+(j-1)^2)\approx\int_0^m \ln(um^2+x^2)dx\\
&= x\ln(um^2+x^2)\Big|_0^m-\int_0^m\left(\frac{2x^2}{um^2+x^2}\right)dx\\
&= m\ln(um^2+m^2)-2\int_0^m\left(1-\frac{um^2}{um^2+x^2}\right)dx\\
&= m\ln((u+1)m^2)-2m+2um^2\int_0^m\frac{dx}{um^2+x^2}\\
&= m\ln((u+1)m^2)-2m+2um^2\cdot\frac{1}{\sqrt{u}m}\tan^{-1}\left(\frac{x}{\sqrt{u}m}\right)\Big|_0^m\\
&= m\ln((u+1)m^2)-2m+2\sqrt{u}m\tan^{-1}\left(\frac{1}{\sqrt{u}}\right).
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
F_1(m) &\approx 2m\left(1-\sqrt{u}\tan^{-1}\left(\frac{1}{\sqrt{u}}\right)\right)\left(\left(\frac{R}{m}\right)^\alpha+c\right)\\
&= 2\left(1-\sqrt{u}\tan^{-1}\left(\frac{1}{\sqrt{u}}\right)\right)\left(cm+\frac{R^\alpha}{m^{\alpha-1}}\right).
\end{aligned}$$

To minimize $F_1(m)$, we only need to minimize

$$G(m)=cm+\frac{R^\alpha}{m^{\alpha-1}}.$$

To satisfy

$$\frac{dG(m)}{dm}=c-\frac{(\alpha-1)R^\alpha}{m^\alpha}=0,$$

we need

$$m=R\left(\frac{\alpha-1}{c}\right)^{1/\alpha},$$

which can be used as an approximate solution to m. Surprisingly, the above m is independent of u and identical to that of a uniform distribution.

The above m yields

$$\begin{aligned}
F_1(m) &= 2R\left(\frac{\alpha-1}{c}\right)^{1/\alpha}\left(1-\sqrt{u}\tan^{-1}\left(\frac{1}{\sqrt{u}}\right)\right)\left(\frac{c}{\alpha-1}+c\right)\\
&= 2\alpha R\left(\frac{c}{\alpha-1}\right)^{1-1/\alpha}\left(1-\sqrt{u}\tan^{-1}\left(\frac{1}{\sqrt{u}}\right)\right).
\end{aligned}$$

The network lifetime is

$$\begin{aligned}
L &= \frac{E}{N(p+\mu aF(m))}\\
&= \frac{E}{N\left(p+\frac{\mu a}{\ln(1+1/u)}\cdot F_1(m)\right)}\\
&= \frac{E}{N\left(p+2\alpha\mu aR\left(\frac{c}{\alpha-1}\right)^{1-1/\alpha}\left(\frac{1-\sqrt{u}\tan^{-1}(1/\sqrt{u})}{\ln(1+1/u)}\right)\right)}.
\end{aligned}$$

Let us examine the function

$$y(u)=\frac{1-\sqrt{u}\tan^{-1}(1/\sqrt{u})}{\ln(1+1/u)}.$$

It is clear that since

$$\lim_{u\to0}\tan^{-1}\left(\frac{1}{\sqrt{u}}\right)=\frac{\pi}{2},$$

and

$$\lim_{u\to0}\ln\left(1+\frac{1}{u}\right)=\infty,$$

we have

$$\lim_{u\to0}y(u)=0,$$

that is, as sensors are more and more densely distributed around a sink, the energy expended for data transmission becomes less and less significant. On the other hand, for $u>1$, we have

$$\tan^{-1}\left(\frac{1}{\sqrt{u}}\right)=\frac{1}{\sqrt{u}}-\frac{1}{3\sqrt{u}^3}+\frac{1}{5\sqrt{u}^5}-\cdots,$$

and

$$\ln\left(1+\frac{1}{u}\right)=\frac{1}{u}-\frac{1}{2u^2}+\frac{1}{3u^3}-\cdots,$$

and

$$\lim_{u\to\infty}y(u)=\lim_{u\to\infty}\frac{1/(3u)-1/(5u^2)+1/(7u^3)-\cdots}{1/u-1/(2u^2)+1/(3u^3)-\cdots}=\frac{1}{3},$$

which yields L identical to that of a uniform distribution.

Finally, the optimal energy allocation is

$$\begin{aligned}
E_j &= \frac{E}{N(p+\mu aF(m))}\\
&\quad\times\left(p+\ln\left(\frac{(u+1)m^2}{(j-1)^2+um^2}\right)\left(\ln\left(\frac{j^2+um^2}{(j-1)^2+um^2}\right)\right)^{-1}\right.\\
&\quad\left.\times\ \mu a\left(\left(\frac{R}{m}\right)^\alpha+c\right)\right),
\end{aligned}$$

for all $1\le j\le m$.

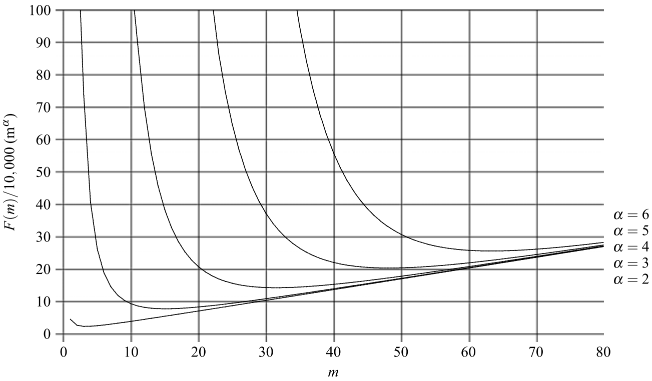

Fig. 3. $F(m)$ vs. number of annuli m (uniform distribution).

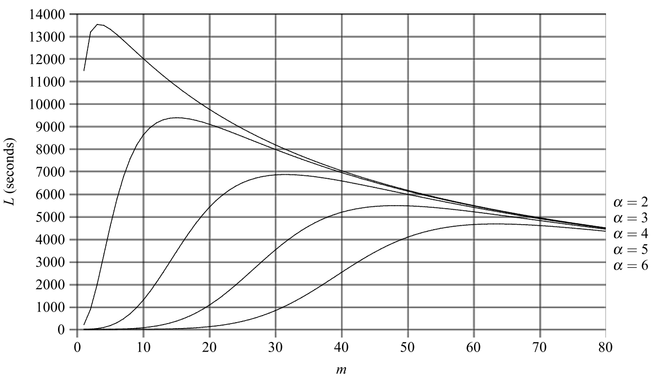

Fig. 4. Network lifetime L vs. number of annuli m (uniform distribution, varying α).

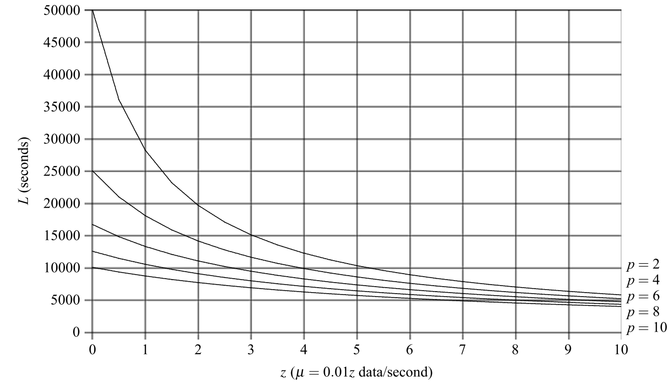

Fig. 5. Network lifetime L vs. sensor reporting rate μ (uniform distribution, varying p).

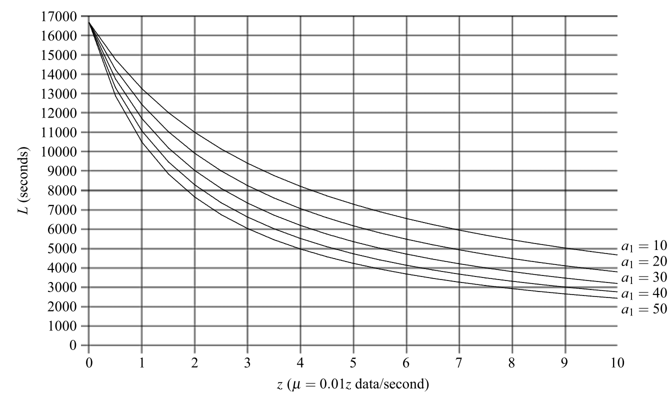

Fig. 6. Network lifetime L vs. sensor reporting rate μ (uniform distribution, varying a and c).

7. Numerical examples

To show a numerical example of optimal number of annuli for a uniform distribution of sensors, we set $c = 5000$ and $R = 200$. In Fig. 3, we display the value of $F(m)$ (actually $F(m)/10,000$) for $\alpha = 2, 3, 4, 5, 6$, where $1 \le m \le 80$. We observe that for all α, as m increases, $F(m)$ decreases rapidly. After $F(m)$ reaches its minimum value, $F(m)$ increases gradually and almost linearly as m increases. Thus, the energy used for data transmission is very sensitive to the choice of m. Furthermore, for small to moderate m, there is significant difference between the $F(m)$ values for different α values; however, as m increases, such a difference diminishes, since the impact of the term $(R/m)^\alpha$ in $F(m)$ becomes less and less important. Notice that the optimal m which minimizes $F(m)$ is given in Table 1.

In Fig. 4, we demonstrate network lifetime L as a function of m, where $1 \le m \le 80$, and show the effect of α on L. We set $p = 6, a = 0.002, c = 5000, \mu = 0.03, R = 200, E/N = 100$ Jules, and $\alpha = 2, 3, 4, 5, 6$. We observe that for all α, as m increases, L increases rapidly. After L reaches its maximum value, L decreases gradually as m increases. Thus, the network lifetime is very sensitive to the choice of m. Furthermore, for small to moderate m, there is noticeable difference between the network lifetime for different α values; however, as m increases, such a difference diminishes, since the impact of the term $(R/m)^\alpha$ in $F(m)$ becomes less and less important.

In Fig. 5, we demonstrate network lifetime L as a function of sensor reporting rate μ, where $0 \le \mu \le 0.1$, and show the effect of p on L. We set $\alpha = 3, a = 0.002, c = 5000, R = 200, m = 15$ (the optimal choice), $E/N = 100$, and $p = 2, 4, 6, 8, 10$. It is observed that when μ is small, the energy p consumed by a sensor to sense and receive data in one unit of time has a strong impact on the network lifetime. However, such an impact diminishes as μ increases, since energy consumed by data transmission gradually dominates the total energy expenditure.

In Fig. 6, we demonstrate network lifetime L as a function of sensor reporting rate μ, where $0 \le \mu \le 0.1$, and show the effect of a and c on L. We set $p = 6, \alpha = 3, R = 200, m = 15$ (the optimal choice), $E/N = 100$, and $a_1 = 10, 20, 30, 40, 50$ and $a_2 = 50,000$, which result in $a = 0.002$ and $c = 5000$, $a = 0.004$ and $c = 2500$, $a = 0.006$ and $c = 1667$, $a = 0.008$ and $c = 1250$, $a = 0.010$ and $c = 1000$. These combinations of a and c give rise to $aF(m) = 155, 204, 254, 304$, and 354, that is, increased cost for data transmission and reduced network lifetime.

In Fig. 7, we show the normalized optimal energy allocation E_j/E_1, where $1 \le j \le m$. We set $p = 6, a = 0.002, c = 5000, \mu = 0.03, R = 200$, and $\alpha = 2$ and $m = 3$, $\alpha = 3$ and $m = 15$, $\alpha = 4$ and $m = 32$, $\alpha = 5$ and $m = 48$, $\alpha = 6$ and $m = 63$. Each m is the optimal choice for the corresponding α. It is observed that an optimal energy allocation is not balanced. In particular, we have $E_1 > E_2 > \cdots > E_m$. Sensors closer to a sink receive significantly more energy than sensors far away from the sink. Such an imbalance increases as α increases.

To show a numerical example of optimal number of annuli for a nonuniform distribution of sensors, we consider the following nonuniform sensor distribution function,

$$f(r) = \left(\frac{N}{\pi \ln(1 + 1/u)}\right)\left(\frac{1}{r^2 + uR^2}\right).$$

In Fig. 8, we display the above $f(r)$, where $0 \le r \le R$, assuming that $N = 10,000$, $R = 200$, and $u = 0.125, 0.250, 0.500, 1.000, 2.000, 4.000, 8.000$. It can be seen that as u increases, $f(r)$ approaches the uniform distribution $f(r) = N/(\pi R^2) = 0.0795774$.

In Figs. 9–13, we continue to use the above nonuniform sensor density function, where $u = 0.5$, i.e., the ratio of the largest density

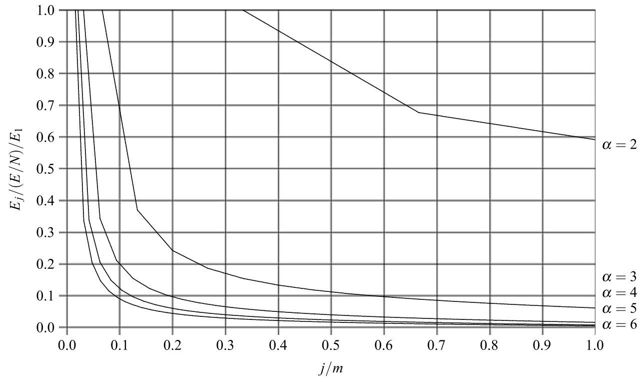

Fig. 7. Optimal energy allocation E_j vs. j (uniform distribution).

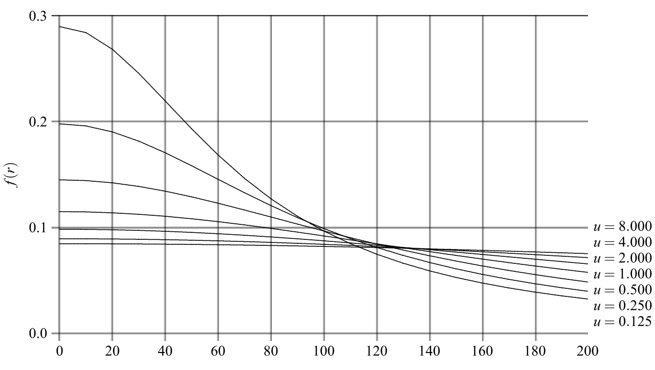

Fig. 8. Nonuniform sensor distribution functions.

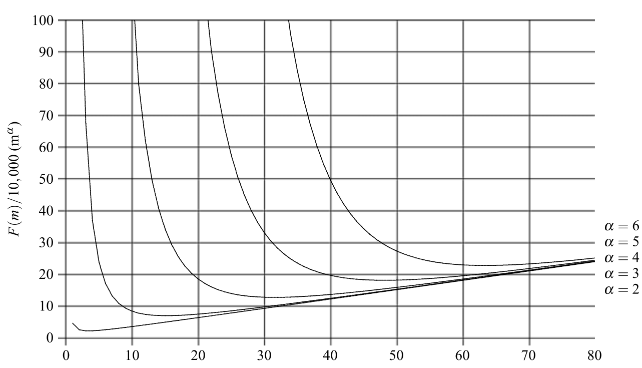

Fig. 9. $F(m)$ vs. number of annuli m (nonuniform distribution).

to the smallest density is 3. Again, we set $c = 5000$ and $R = 200$. In Fig. 9, we display the value of $F(m)$ (actually $F(m)/10,000$) for $\alpha = 2, 3, 4, 5, 6$, where $1 \le m \le 80$. As expected, the behavior of $F(m)$ is similar to and less than that of a uniform distribution in Fig. 3. Furthermore, the optimal m which minimizes $F(m)$ is identical to that in Table 1.

In Fig. 10, we demonstrate network lifetime L as a function of m, where $1 \le m \le 80$, and show the effect of α on L. We set $p = 6$, $a = 0.002$, $c = 5000$, $\mu = 0.03$, $R = 200$, $E/N = 100$, and $\alpha = 2, 3, 4, 5, 6$. As expected, the behavior of L is similar to and greater than that of a uniform distribution in Fig. 4.

In Fig. 11, we demonstrate network lifetime L as a function of sensor reporting rate μ, where $0 \le \mu \le 0.1$, and show the effect of p on L. We set $\alpha = 3$, $a = 0.002$, $c = 5000$, $R = 200$, $m = 15$ (the optimal choice), $E/N = 100$, and $p = 2, 4, 6, 8, 10$. As expected, the behavior of L is similar to and greater than that of a uniform distribution in Fig. 5.

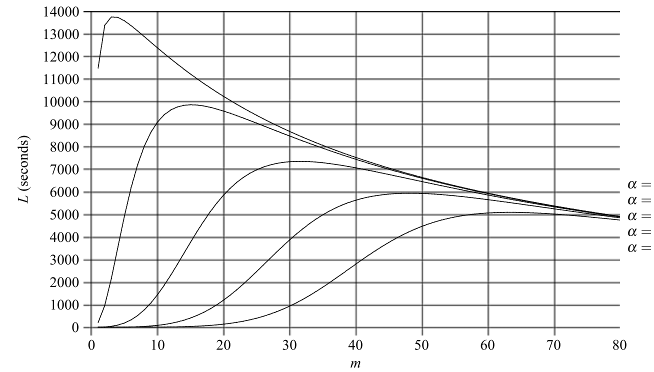

Fig. 10. Network lifetime L vs. number of annuli m (nonuniform distribution, varying α).

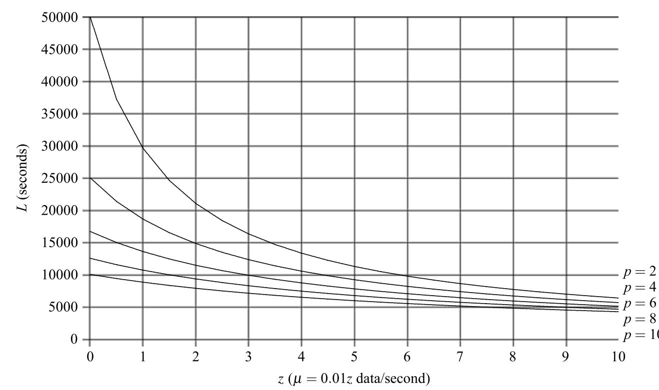

Fig. 11. Network lifetime L vs. sensor reporting rate μ (nonuniform distribution, varying p).

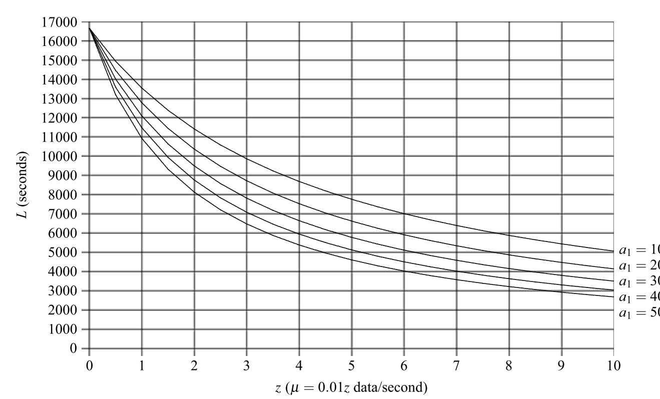

Fig. 12. Network lifetime L vs. sensor reporting rate μ (nonuniform distribution, varying a and c).

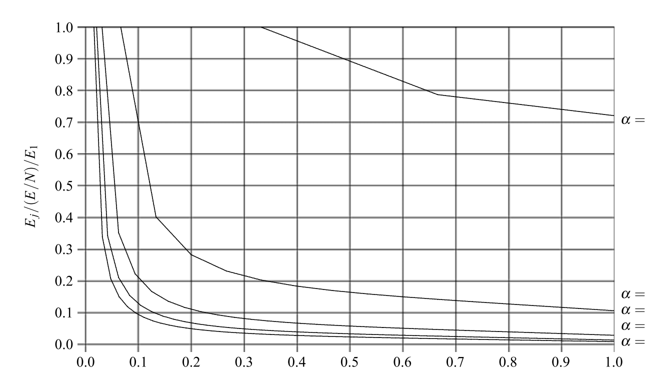

Fig. 13. Optimal energy allocation E_j vs. j (nonuniform distribution).

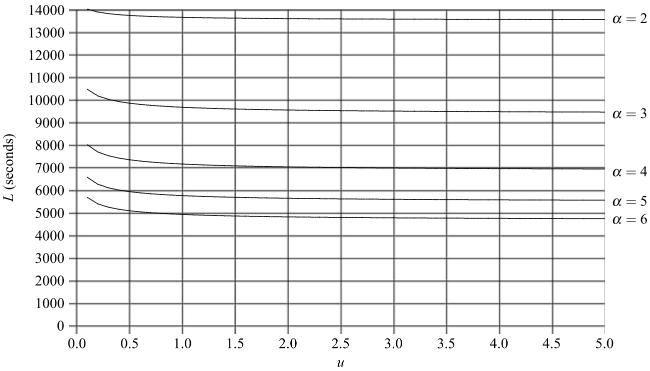

Fig. 14. Network lifetime L vs. u (nonuniform distribution, varying α).

In Fig. 12, we demonstrate network lifetime L as a function of sensor reporting rate μ, where $0 \le \mu \le 0.1$, and show the effect of a and c on L. We set $p = 6, \alpha = 3, R = 200, m = 15$ (the optimal choice), $E/N = 100$, and $a_1 = 10, 20, 30, 40, 50$ and $a_2 = 50,000$, which result in $a = 0.002$ and $c = 5000$, $a = 0.004$ and $c = 2500$, $a = 0.006$ and $c = 1667$, $a = 0.008$ and $c = 1250$, $a = 0.010$ and $c = 1000$. These combinations of a and c give rise to $aF(m) = 138, 182, 227, 271$, and 315. As expected, the behavior of L is similar to and greater than that of a uniform distribution in Fig. 6.

In Fig. 13, we show the normalized optimal energy allocation E_j/E_1, with $1 \le j \le m$. We set $p = 6$, $a = 0.002$, $c = 5000$, $\mu = 0.03$, $R = 200$, and $\alpha = 2$ and $m = 3$, $\alpha = 3$ and $m = 15$, $\alpha = 4$ and $m = 32$, $\alpha = 5$ and $m = 48$, $\alpha = 6$ and $m = 63$. Each m is the optimal choice for the corresponding α. It is observed that a nonuniform sensor distribution results in more balanced optimal energy allocation than a uniform distribution in Fig. 7.

In Fig. 14, we demonstrate network lifetime L as a function of u, where $0 < u \le 5$, and show the impact of u on L. We set $p = 6$, $a = 0.002$, $c = 5000$, $\mu = 0.03$, $R = 200$, $E/N = 100$, and $\alpha = 2$ with $m = 3$, $\alpha = 3$ with $m = 15$, $\alpha = 4$ with $m = 32$, $\alpha = 5$ with $m = 48$, $\alpha = 6$ with $m = 63$. We observe that when $u < 1$, the network lifetime can be increased noticeably by using a nonuniform sensor distribution. As u increases, the network lifetime approaches that of a uniform distribution.

8. A general result

Recall that a sensor density function $f(r)$ satisfies

$$\int_0^R 2\pi r f(r) dr = N.$$

In fact, we require the above condition to be satisfied for all $N > 0$. This means that $f(r)$ can be represented as $f(r) = Ng(r)$, where

$$\int_0^R 2\pi r g(r) dr = 1.$$

The following theorem is the main result of the paper.

Theorem 1. *The optimal number of annuli that maximizes the network lifetime is*

$$m = R\left(\frac{\alpha - 1}{c}\right)^{1/\alpha},$$

and the optimal annulus width is

$$r = \left(\frac{c}{\alpha - 1}\right)^{1/\alpha},$$

for any sensor density function.

Proof. Recall that the network lifetime is

$$L = \frac{E}{Np + \mu a \sum_{j=1}^{m} \left(\int_{r_{j-1}}^{r_m} 2\pi r f(r) dr\right) ((r_j - r_{j-1})^\alpha + c)}$$

$$= \frac{E}{N\left(p + \mu a \sum_{j=1}^{m} \left(\int_{r_{j-1}}^{r_m} 2\pi r g(r) dr\right) ((r_j - r_{j-1})^\alpha + c)\right)}$$

$$= \frac{E}{N\left(p + \mu a \left(\left(\frac{R}{m}\right)^\alpha + c\right) \sum_{j=1}^{m} \left(\int_{(j-1)R/m}^{R} 2\pi r g(r) dr\right)\right)}.$$

Notice that

$$\sum_{j=1}^{m} \left(\int_{(j-1)(R/m)}^{R} 2\pi r g(r) dr\right) \le \sum_{j=1}^{m} \left(\int_0^R 2\pi r g(r) dr\right) = m.$$

Also, we have

$$\sum_{j=1}^{m} \left(\int_{(j-1)(R/m)}^{R} 2\pi r g(r) dr\right) \ge \sum_{j=1}^{k} \left(\int_{(k-1)(R/m)}^{R} 2\pi r g(r) dr\right) = k\left(\int_{(k-1)(R/m)}^{R} 2\pi r g(r) dr\right),$$

for all $1 \le k \le m$. For convenience, let $\phi = (k-1)/m$, i.e., $k = \phi m + 1$, where $0 \le \phi \le 1$, and we treat ϕ as a continuous variable. Then, we get

$$k\left(\int_{(k-1)(R/m)}^{R} 2\pi r g(r) dr\right) = (\phi m + 1)\left(\int_{\phi R}^{R} 2\pi r g(r) dr\right) > m\left(\phi \int_{\phi R}^{R} 2\pi r g(r) dr\right).$$

Consider the function

$$h(\phi) = \phi \int_{\phi R}^{R} 2\pi r g(r) dr.$$

Because $h(0) = h(1) = 0$ and $h(\phi) \ge 0$, there is a ϕ^* which maximizes $h(\phi)$. Since

$$\sum_{j=1}^{m} \left(\int_{(j-1)(R/m)}^{R} 2\pi r g(r) dr\right) \ge m h(\phi),$$

for all $0 \le \phi \le 1$, we have

$$\sum_{j=1}^{m} \left(\int_{(j-1)(R/m)}^{R} 2\pi r g(r) dr\right) \ge m h(\phi^*).$$

The above discussion implies that

$$\sum_{j=1}^{m} \left(\int_{(j-1)(R/m)}^{R} 2\pi r g(r) dr\right) = Cm + o(m) \approx Cm,$$

for large m, where $h(\phi^*) \le C \le 1$ is some constant.

Now, the network lifetime is

$$L = \frac{E}{N\left(p + C\mu a m\left(\left(\frac{R}{m}\right)^\alpha + c\right)\right)},$$

which is maximized when

$$F(m) = m\left(\left(\frac{R}{m}\right)^\alpha + c\right)$$

is minimized. By considering $dF(m)/dm = 0$, we get

$$m = R\left(\frac{\alpha-1}{c}\right)^{1/\alpha},$$

and

$$r = \frac{R}{m} = \left(\frac{c}{\alpha-1}\right)^{1/\alpha}.$$

This proves the theorem. ■

As an example, let us consider a uniform distribution with

$$g(r) = \frac{1}{\pi R^2},$$

and

$$h(\phi) = \phi\int_{\phi R}^{R} 2\pi r g(r)dr = \phi(1-\phi^2).$$

Since

$$\frac{dh(\phi)}{d\phi} = 1 - 3\phi^2,$$

we get $\phi^* = 1/\sqrt{3}$ and $h(\phi^*) = 2/(3\sqrt{3})$ and $2/(3\sqrt{3}) \le C \le 1$. As we have already known, the actual value of C is 2/3.

As another example, let us consider a nonuniform distribution with

$$g(r) = \left(\frac{1}{\pi \ln(1+1/u)}\right)\left(\frac{1}{r^2+uR^2}\right),$$

and

$$h(\phi) = \phi\int_{\phi R}^{R} 2\pi r g(r)dr = \frac{\phi}{\ln(1+1/u)}\ln\left(\frac{u+1}{u+\phi^2}\right),$$

where $u = 0.5$. Since

$$\frac{dh(\phi)}{d\phi} = \frac{1}{\pi \ln(1+1/u)}\left(\ln\left(\frac{u+1}{u+\phi^2}\right) - \frac{2\phi^2}{u+\phi^2}\right),$$

we get $\phi^* = 0.5085156$ and $h(\phi^*) = 0.3155675$ and $0.3155675 \le C \le 1$, by solving the equation

$$y(\phi) = \ln\left(\frac{u+1}{u+\phi^2}\right) - \frac{2\phi^2}{u+\phi^2} = 0.$$

The above equation can be solved by using the bisection method, i.e., by searching ϕ^* in the range [0, 1] and noticing that $y(\phi)$ is a decreasing function of ϕ in [0, 1]. As we have already known, the actual value of C is

$$C = \frac{2(1-\sqrt{u}\tan^{-1}(1/\sqrt{u}))}{\ln(1+1/u)} = \frac{2-\sqrt{2}\tan^{-1}\sqrt{2}}{\ln 3} = 0.5907255.$$

For large u, we have

$$\ln\left(\frac{u+1}{u+\phi^2}\right) = \ln\left(1+\frac{1-\phi^2}{u+\phi^2}\right) \approx \frac{1-\phi^2}{u+\phi^2}.$$

Therefore,

$$y(\phi) \approx \frac{1-\phi^2}{u+\phi^2} - \frac{2\phi^2}{u+\phi^2} = \frac{1-3\phi^2}{u+\phi^2} = 0,$$

from which we get $\phi^* \approx 1/\sqrt{3} = 0.5773502$ and

$$\begin{aligned} h(\phi^*) &= \frac{\phi^*}{\ln(1+1/u)}\ln\left(\frac{u+1}{u+(\phi^*)^2}\right) \\ &\approx \frac{\phi^*}{1/u}\left(\frac{1-(\phi^*)^2}{u+(\phi^*)^2}\right) \\ &\quad (\text{since } \ln(1+1/u) \approx 1/u \text{ for large } u) \\ &= \frac{\phi^*(1-(\phi^*)^2)}{1+(\phi^*)^2/u} \\ &\approx \phi^*(1-(\phi^*)^2) \quad (\text{since } (\phi^*)^2/u \approx 0 \text{ for large } u) \\ &= \frac{2}{3\sqrt{3}} = 0.3849001, \end{aligned}$$

and

$$C = \frac{2(1-\sqrt{u}\tan^{-1}(1/\sqrt{u}))}{\ln(1+1/u)} \approx \frac{2}{3} = 0.6666667,$$

which are exactly the same as those of a uniform distribution. For instance, when $u = 10$, we have $\phi^* = 0.5712372$ and $h(\phi^*) = 0.3787868$ and $C = 0.6602845$.

Finally, we emphasize that what we have optimized is the average amount of energy consumed by all data transmissions and the network lifetime, not the energy consumption of an individual data transmission. Consider a sensor s_i in A_j with distance d from sink s_0, where $(j-1)r < d \le jr$. According to our previous discussion in Section 2, the minimum amount of energy consumed by a data transmission from s_i to s_0 is the smaller one of

$$a(j-1)\left(\left(\frac{d}{j-1}\right)^{\alpha} + c\right)$$

and

$$aj\left(\left(\frac{d}{j}\right)^{\alpha} + c\right),$$

where $j = \lceil d/r \rceil$. Therefore, the minimum amount of energy consumed by a data transmission from s_i to s_0 is no greater than

$$a\left\lceil\frac{d}{r}\right\rceil\left(\left(\frac{d}{\lceil d/r\rceil}\right)^{\alpha} + c\right) \le a\left\lceil\frac{d}{r}\right\rceil(r^{\alpha}+c),$$

where the right hand side of the above inequality is exactly the actual amount of energy consumed by a data transmission from s_i to s_0 and the equality holds only when s_i is on the boundary of A_j and A_{j+1}. Hence, energy consumptions for most sensors are not minimized.

9. Concluding remarks

While most existing sensor network lifetime maximization techniques focus on network design (i.e., various ways of sensor placement), assuming that all sensors are equipped with the same amount of initial energy, we take a different approach, i.e., incorporating optimal initial energy allocation into optimal network design. Since sensors closer to a sink deplete their energy more quickly, they need to be equipped with different energy levels. In our design, this is indeed the case, since $\beta_1 > \beta_2 > \cdots > \beta_m$, and $E_1 > E_2 > \cdots > E_m$. In fact, Figs. 7 and 13 demonstrate that the difference among the E_j's can be very significant.

Assuming that the available energy are allocated to the sensors such that the lifetime of a sensor network is maximized, we treat the lifetime of the sensor network as a function of the network design. For a circular area with multiple annuli, the lifetime of a sensor network is a function of the radii. When all the annuli have the same width, the lifetime of a sensor network is a function of the number of annuli. By representing the network lifetime as a function of the number m of annuli, we have shown that m has significant impact on the lifetime of WSNs. We have found the optimal number of annuli that maximizes the network lifetime for arbitrary sensor density functions.

The investigation in this paper assumes that all annuli have identical widths based on the observation that energy consumption of a data transmission is minimized when all hops have the same distance. The strength of the approach is that a closed-form expression of the optimal number of annuli can be found analytically. However, the weakness of the approach is that it is not clear whether the method yields an optimal network design. It is worth further investigation whether identical annulus widths give the maximum network lifetime. It is possible that an optimal network design has different widths of annuli. The results obtained along this direction should be compared with the results in this paper.

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