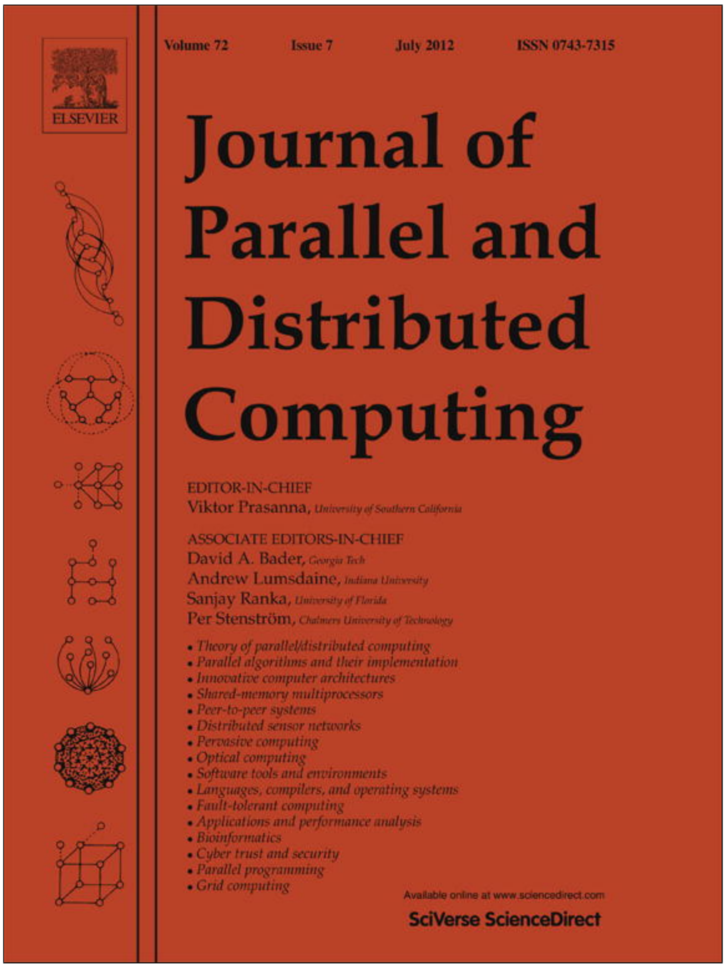
ELSEVIER
Volume 72
Issue 7
July 2012
ISSN 0743-7315
Journal of Parallel and Distributed Computing
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J. Parallel Distrib. Comput. 72 (2012) 902–916

Contents lists available at SciVerse ScienceDirect

J. Parallel Distrib. Comput.

journal homepage: www.elsevier.com/locate/jpdc

Optimal energy allocation in heterogeneous wireless sensor networks for lifetime maximization

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ARTICLE INFO

Article history:
Received 2 August 2011
Received in revised form
17 February 2012
Accepted 7 March 2012
Available online 23 March 2012

Keywords:
Data sensing
Data transmission
Energy consumption
Heterogeneous wireless sensor network
Lifetime maximization
Optimal energy allocation

ABSTRACT

We consider the problem of optimal energy allocation and lifetime maximization in heterogeneous wireless sensor networks. We construct a probabilistic model for heterogeneous wireless sensor networks where sensors can have different sensing range, different transmission range, different energy consumption for data sensing, and different energy consumption for data transmission, and the stream of data sensed and transmitted from a sensor and the stream of data relayed by a sensor to a base station are all treated as Poisson streams. We derive the probability distribution and the expectation of the number of data transmissions during the lifetime of each sensor and the probability distribution and the expectation of the lifetime of each sensor. In all these analysis, energy consumption of data sensing and data transmission and data relay are all taken into consideration. We develop an algorithm to find an optimal initial energy allocation to the sensors such that the network lifetime in the sense of the identical expected sensor lifetime is maximized. We show how to deal with a large amount of energy budget that may cause excessive computational time by developing accurate closed form approximate expressions of sensor lifetime and network lifetime and optimal initial energy allocation. We derive the expected number of working sensors at any time. Based on such results, we can find the latest time such that the expected number of sensors that are still functioning up to that time is above certain threshold.

1. Introduction

A wireless sensor network consists of dozens to thousands of low cost, low power, and energy constrained sensors. These sensors are inexpensive and highly integrated electronic devices capable of sensing, processing, receiving, and transmitting data. Wireless sensor networks have been widely used in numerous applications. These applications include monitoring climate (temperature and humidity) and environment (habitat, moving objects, water pollution), detecting physical phenomenon (acoustic or seismic waves), sensing and diagnosing faults (factory production, industrial supply lines, power grids), measuring data (human and vehicle traffic intensity), monitoring security (human and vehicle intrusion), military surveillance, and battlefield information collection [8,9].

The lifetime of a wireless sensor network has been an important research and application issue due to limited battery power in sensors and infeasibility of replacing and recharging sensor batteries. Energy consumption in these networks contains two components, namely, data sensing and data transmission. The first component includes the energy required for a sensor to be in an active sensing mode and the energy used in computing and processing sensed data and images. The second component includes energy consumed to transmit sensed data to a base station (also called sink, collector, access point, gateway) and to relay data transmission from other sensors to the base station.

There are three different perspectives in studying the lifetime maximization problem of wireless sensor networks. The first perspective is to consider only the first component of energy consumption. In [4,5,10,22], improving and maximizing network lifetime becomes a disjoint set covers problem. The second perspective is to consider only the second component of energy consumption. In [6,11,16,17,19], the problem of network lifetime maximization is a data transmission scheduling and energy consumption minimization problem. The third perspective is to consider both components of energy consumption. In [1,2,26], both data sensing and transmission costs are taken into consideration in assigning different roles to the sensors for sensing, relaying, and aggregating data. In all the above studies, the lifetime maximization problem of wireless sensor networks is formulated as an integer programming or linear programming problem that can be solved by using various heuristics. We argue that the lifetime maximization problem of wireless sensor networks should be studied

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doi:10.1016/j.jpdc.2012.03.001

from the third perspective. On the one hand, transmitting a datum costs much more energy than sensing a datum. On the other hand, a sensor consumes energy even though it does not transmit any datum.

A number of strategies have been proposed in the literature to increase the lifetime of a sensor network. The first method is to use redundant sensors [25]. At any moment, only a subset of sensors (called a sub-network [1]) are active and other sensors are in the sleeping mode to save energy consumption, as long as the active sensors can provide necessary connectivity among them and appropriate coverage of an area of interest. The second method is to deploy sensors according to a nonuniform distribution, such that sensors are distributed more densely in the region closer to a base station [23]. Such a nonuniform distribution reduces the burden of data transmission on the sensors closer to the base station and increases the lifetime of these sensors and also the lifetime of the entire sensor network. The third method is to use aggregation and forwarding nodes instead of or in addition to sensing nodes for data transmission to a base station [13,24]. These relay nodes not only mitigate geometric deficiency of a network, but also provide extra energy for data fusion and transmission.

While all these approaches have their merit, each has its disadvantages. The sensor redundancy method is costly, since multiples of necessary number of sensors are employed. Furthermore, sensors in sub-networks activated after staying for a long time of sleeping mode in severe weather conditions may not function as properly as newly deployed sensors and are more likely to be faulty sensors. A nonuniform distribution of sensors causes overly coverage of the region closer to a base station, while the region far away from the base station is not well covered. Using relay nodes simply reduces the lifetime maximization problem of the original network to the lifetime maximization problem of the sub-network of relay nodes.

A different approach can be taken to maximize the lifetime of a sensor network. It is clear that once sensors are deployed and activated, the lifetime of the sensors as well as the entire sensor network are determined by the initial battery energy capacities. It is also clear that sensors closer to a base station should be given more energy than sensors further away from the base station; otherwise, when sensors near the base station have exhausted their energy, a network becomes nonfunctional even if other sensors have significant remaining energy [15]. It has been found that if sensors are equipped with the same initial energy, after the lifetime of a network is over, up to 90% of the initial energy remains unused [18]. Therefore, the question to be answered is, "Given a total energy budget, how to allocate the energy to the sensors such that the lifetime of a sensor network is maximized?" This problem has been addressed by several researchers. In [1], the problem of optimal initial energy allocation to sensors with known sensing and transmission rates are defined and solved as a mixed integer linear programming problem. In [20], a wireless sensor network is divided into coronas and optimal initial energy allocation to sensors in different coronas are obtained. In [21], optimal initial energy allocations to sensors in linear and planar sensor networks are derived.

We notice that to properly solve the problem of optimal initial energy allocation and network lifetime maximization, we need to have a precise definition of network lifetime and an analytical expression of network lifetime. Unfortunately, there has been no clear and unified definition of the lifetime of a sensor network. One definition is that the lifetime of a sensor network is the time when the first sensor runs out of battery power, or, all sensors that have direct connection with a base station run out of battery power, since these sensors play critical roles and tend to die sooner than other sensors. Therefore, the lifetime of a sensor network is optimized when all sensors in the network die at the same time. Our approach in this paper is to allocate an initial energy budget to sensors in a sensor network such that all sensors have the same expected lifetime, and the lifetime of the entire sensor network is defined as the identical expected sensor lifetime.

Another definition of the lifetime of a sensor network is the latest time such that the number of sensors that are still functioning up to that time is above certain threshold. A low number of working sensors implies reduced coverage of an area of interest, reduced connectivity among the sensors and a base station, increased data transmission failures, and nonfunctionality of a sensor network. We find that the probability distribution function of the random number of working sensors at any time is extremely difficult to characterize analytically. The reason is that the lifetime of the sensors are correlated. Consider the path from a sensor s to a base station. When a sensed datum is transmitted from s to the base station, the datum is transmitted by all the sensors on the path. Therefore, more data sensed by s not only consume battery power of s and reduce the lifetime of s, but also consume battery powers of all the sensors on the path and reduce the lifetime of all these sensors.

To solve the problem of optimal initial energy allocation and network lifetime maximization, we need to know clearly and exactly the relationship between the lifetime of a sensor and its initial energy and the relationship between the network lifetime and total energy budget. Unfortunately, there has been little such analytical results on the lifetime of sensors and sensor networks. In [7], a fairly general expression of expected sensor network lifetime is given. However, the expression does not include individual sensors and cannot be used for optimal energy allocation. In [21], expressions of expected sensor network lifetime are derived for linear and planar sensor networks. However, it is not clear how the expected lifetime of an arbitrary sensor network can be obtained.

In this paper, we consider the problem of optimal energy allocation and lifetime maximization in heterogeneous wireless sensor networks. Our main contributions are summarized as follows.

- Heterogeneous Sensor Network Modeling—We construct a heterogeneous wireless sensor network model (Sections 2 and 3). Virtually all existing research use a homogeneous sensor network model (with few exceptions [12]), i.e., all sensors have the same sensing range, the same transmission range, the same energy consumption for data sensing, and the same energy consumption for data transmission. In our model, sensors can have different sensing range, different transmission range, different energy consumption for data sensing, and different energy consumption for data transmission. Furthermore, the locations of sensors can have arbitrary distributions and a sensor network can have an arbitrary topology. Sensors can have different initial battery energy capacities.
- Probabilistic Data Sensing and Transmission Modeling—Virtually all existing research use a deterministic data sensing and transmission model, i.e., the number of data sensed and transmitted from a sensor and the number of data relayed by a sensor to a base station in one unit of time are all constants. In our model, the stream of data sensed and transmitted from a sensor and the stream of data relayed by a sensor to a base station are all treated as Poisson streams defined in Section 2.
- Sensor and Network Lifetime Analysis—We are able to derive the probability distribution and the expectation of the number of data transmissions during the lifetime of each sensor (Theorem 1 in Section 4). Based on these results, we are able to derive the probability distribution and the expectation of the lifetime of each sensor (Theorem 2 in Section 4). These results form the basis of our optimal energy allocation, sensor network lifetime maximization, and analysis of the expected number of

working sensors. In all these analysis, energy consumption of data sensing and data transmission and data relay are all taken into consideration.

- Optimal Energy Allocation and Network Lifetime Maximization—We develop an algorithm to find an optimal initial energy allocation to the sensors such that the network lifetime in the sense of the identical expected sensor lifetime is maximized (Section 5). We also show how to deal with a large amount of energy budget that may cause excessive computational time by developing accurate closed form approximate expressions of sensor lifetime and network lifetime and optimal initial energy allocation (Theorems 3–6 in Section 6 and Theorem 8 in Appendix).
- Working Sensors—We are able to derive the expected number of working sensors at any time (Theorem 7 in Section 7). Based on such results, we can find the latest time such that the expected number of sensors that are still functioning up to that time is above certain threshold.

To the best of our knowledge, there has been no probabilistic modeling of heterogeneous wireless sensor networks, no analytical expression of sensor and network lifetime of an arbitrary heterogeneous wireless sensor network, and no optimal energy allocation method which maximizes the lifetime of a heterogeneous wireless sensor network. In this sense, this paper makes significant progress in and contribution to the study of wireless sensor network lifetime.

2. The network model

We first describe our model of heterogeneous wireless sensor networks and introduce the notations used in this paper.

Throughout the paper, we use $\mathbb{P}[e]$ to denote the probability of an event e. For a random variable X, we use $f_X(x)$ to represent the probability density function (pdf) of X, $F_X(x)$ to represent the cumulative distribution function (cdf) of X, and $\overline{X}$ to represent the expectation of X.

A heterogeneous wireless sensor network contains a set S of N heterogeneous sensors $S = \{s_1, s_2, \ldots, s_N\}$, which cover an area of interest A in a two dimensional Euclidean space. We are not interested in the size and shape of A and assume that the sensors are distributed in A such that the sensors are connected and A is well covered by S.

An area of interest A is characterized by a parameter ρ, which is the rate of data generation per unit of area in A, assuming that A is homogeneous. The times T between successive data generated per unit of area are independent and identically distributed (i.i.d.) random variables with a common exponential pdf,

$$f_T(t) = \rho e^{-\rho t},$$

and the expectation of T is $\overline{T} = 1/\rho$, where ρ is the expected number of data generated within one unit of area during one unit of time. The sequence of data generated from each unit of area form a Poisson stream of data with arrival rate ρ. Such a data generation model is applicable to sensing areas where events to be detected happen randomly with exponentially distributed inter-event times, such as motion detection, traffic intensity measuring, and intrusion monitoring. For applications where data are reported periodically, the lifetime of a sensor is predictable and the wireless sensor network lifetime maximization problem becomes trivial. All sensed data have the same size (i.e., the number of bits or bytes).

A sensor s_i, where $1 \le i \le N$, is specified as a septuple $s_i = (x_i, y_i, B_i, R_i, p_i, q_i, E_i)$, whose components are described as follows. The pair (x_i, y_i) is the location of s_i in a two dimensional Euclidean space, i.e., the x- and y-coordinates. The communication range of s_i (i.e., the set of locations that can be reached by the communication signals of s_i) is B_i, which can be of any size and shape, regular or irregular. A typical case is a circle with radius c_i. The sensing range of s_i is R_i, which can be of any size and shape, regular or irregular. A typical case is a circle with radius r_i. The amount of energy consumed by s_i during sensing and receiving data per unit of time is p_i. The amount of energy consumed by s_i to transmit one sensed datum is q_i. The initial energy allocated to s_i is E_i.

A base station, denoted by s_0, has location (x_0, y_0) and transmission range B_0, which is large enough such that s_i is in B_0 as long as s_0 is in B_i. The (x_i, y_i, B_i)'s, where $0 \le i \le N$, define an undirected graph G, which has $N+1$ nodes, namely, the N sensors $s_1, s_2, \ldots, s_N$ and a base station s_0. There is an edge between s_i and s_j if and only if s_i and s_j are within the communication range of each other. For circular communication ranges, this means that

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \le \min(c_i, c_j),$$

where $0 \le i, j \le N$.

It is clear that the sequence of data sensed by a sensor s_i form a Poisson stream of data with arrival rate $\mu_i = R_i \rho$, which is actually a combination of R_i Poisson streams with arrival rate ρ, where for convenience, we have used R_i to denote the size of the sensing range of s_i. For circular sensing ranges, we have $R_i = \pi r_i^2$. Hence, the times T_i between successive data sensed by s_i are i.i.d. random variables with a common exponential pdf,

$$f_{T_i}(t) = \mu_i e^{-\mu_i t}.$$

The expectation of T_i is $\overline{T}_i = 1/\mu_i$, where μ_i is the expected number of data sensed by s_i per unit of time to be sent to a base station.

Two randomized routing methods can be used in a heterogeneous wireless sensor network. The first method is based on a randomized breadth-first search (RBFS) tree of G with root s_0 established for data transmission from the sensors $s_1, s_2, \ldots, s_N$ to the base station s_0 (see Section 3). Each s_j has a unique path from s_j to s_0, namely, $(s_j, s_{i_1}, s_{i_2}, \ldots, s_{i_k}, s_0)$, where $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ are $k \ge 0$ intermediate nodes. Each sensed datum collected by s_j is sent to s_0 along this path. This implies that sensor s_j contributes a Poisson stream of data with arrival rate μ_j to each sensor in $\{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}$. Let D_i denote the set of sensors s_j such that s_i is on the path from s_j to s_0, i.e., the set of descendants of s_i in the RBFS tree. Then,

$$\beta_i = \mu_i + \sum_{s_j \in D_i} \mu_j$$

is the rate of the actual Poisson stream (a combination of $|D_i| + 1$ Poisson streams) of data transmitted and relayed by s_i. Equivalently, β_i can be represented as

$$\beta_i = \mu_i + \sum_{s_j \in C_i} \beta_j,$$

where C_i is the set of children of s_i in an RBFS tree. The second randomized routing method is described in Section 3.

Notice that the above model can be easily extended to wireless links that are unreliable and unstable. Assume that data losses on a link occur independently. Furthermore, if a data loss occurs, there is no retransmission. To consider possible data loss, we use ξ_{ij} to denote the reliability of the link between s_i and s_j. Since the intensity of a Poisson stream can be scaled by a factor between 0 and 1 (notice that ξ_{ij} is such a value, which reduces the actual rate of a Poisson stream), we have

$$\beta_i = \mu_i + \sum_{s_j \in C_i} \xi_{ij} \beta_j,$$

where ξ_{ij} is the probability that a datum from s_j is relayed by s_i, i.e., the chance that the datum is successfully transmitted on the link between s_i and s_j.

For the purpose of lifetime analysis, the specification of a sensor s_i can be simplified as a quartet $s_i = (\beta_i, p_i, q_i, E_i)$, where β_i is determined by x_i, y_i, B_i, and R_i.

Algorithm: Calculating Data Transmission Rates

Input: An undirected graph $G = (S \cup \{s_0\}, E)$, where $S = \{s_1, s_2, ..., s_N\}$, and sensor s_i has data sensing rate μ_i, for all $1 \le i \le N$.
Output: Data transmission rate β_i of sensor s_i, for all $1 \le i \le N$.

```
(1)  for (i ← 0; i ≤ N; i++) do
(2)      visited[s_i] ← false;
(3)      potential_parents[s_i] ← ∅;
(4)  end for;
(5)  Initialize a queue Q to be empty;
(6)  visited[s_0] ← true;
(7)  level[s_0] ← 0;
(8)  enqueue(s_0, Q);
(9)  repeat
(10)     s ← dequeue(Q);
(11)     for (all s's neighbor s') do
(12)         if (visited[s'])
(13)             if (level[s'] > level[s])
(14)                 potential_parents[s'] ← potential_parent[s'] ∪ {s};
(15)         else
(16)             visited[s'] ← true;
(17)             potential_parents[s'] ← {s};
(18)             level[s'] ← level[s] + 1;
(19)             enqueue(s', Q);
(20)         end if;
(21)     end for;
(22) until (Q is empty);
(23) for (i ← 1; i ≤ N; i++) do
(24)     parent[s_i] ← a member chosen randomly and uniformly from potential_parents[s_i];
(25) end for;
(26) for (i ← 1; i ≤ N; i++) do
(27)     β_i ← μ_i;
(28) end for;
(29) for (j ← 1; j ≤ N; j++) do
(30)     s ← parent[s_j];
(31)     while (s ≠ s_0) do
(32)         β_i ← β_i + μ_j; //assume that s = s_i
(33)         s ← parent[s];
(34)     end do;
(35) end for.
```

Fig. 1. An algorithm to calculate data transmission rates.

3. Data transmission rate

As mentioned earlier, the N sensors in a heterogeneous wireless sensor network plus a base station define an undirected graph $G = (S \cup \{s_0\}, E)$, where $S = \{s_1, s_2, \ldots, s_N\}$ contains N heterogeneous sensors; s_0 is a base station; and there is an edge $\{s_i, s_j\} \in E$ if and only if s_i and s_j are within the communication range of each other, for all $0 \le i, j \le N$. The $N + 1$ nodes in G can be divided into levels. A node s_i is in level l if its shortest distance to s_0 is l, that is, the minimum number of links on a path from s_i to s_0 is l. Hence, level 0 only contains s_0. Level 1 contains neighbors of s_0. In general, for all $l \ge 1$, level $l + 1$ contains the neighbors of nodes in level l that are not in levels $l - 1$ and l. For each node s_i in level $l \ge 1$, a neighbor of s_i in level $l - 1$ is called a *potential parent* of s_i. One of the potential parents of s_i, which is chosen randomly from the set of all potential parents of s_i with a uniform distribution, is called the parent of s_i and is represented by $parent[s_i]$. Such a set of parent–child relationship define a *randomized breadth-first search* (RBFS) tree.

When a datum is transmitted from s_j to s_0, it is sent to the parent of s_j, and then from the parent of s_j to s_0. Hence, the path from a sensor s_j to s_0 is

$$(s_j, parent[s_j], parent[parent[s_j]], \ldots, s_0),$$

which is a shortest path from s_j to s_0. This implies that s_j contributes a Poisson stream of data transmission with arrival rate μ_j to each sensor s_i on the path from s_j to s_0. The random selection of the parent of a node among its potential parents ensures that data transmissions are evenly distributed among the sensors, especially those close to a base station.

Our optimal energy allocation algorithm needs the data transmission rate β_i of each sensor s_i, where $1 \le i \le N$. In Fig. 1, we describe our algorithm to calculate the data transmission rates $\beta_1, \beta_2, \ldots, \beta_N$, when given an undirected graph $G = (S \cup \{s_0\}, E)$, where $S = \{s_1, s_2, \ldots, s_N\}$, and sensor s_i has data sensing rate μ_i, for all $1 \le i \le N$.

An RBFS tree can be constructed by using the standard *breadth-first search* (BFS) algorithm (lines (1)–(25)). An array *visited*[] and a queue Q are used as in the standard BFS algorithm (lines (1)–(25)). The root of the tree is s_0 (lines (6)–(8)). An array *potential_parents*[] is used to save the set of potential parents of each node. Every time a new potential parent s of a node s' is detected, s is added to *potential_parents*[s'] (lines (14) and (17)). When a node s detects a neighbor s' which has been visited (line (12)) and has a higher level number (line (13)), multiple potential parents exist. In this case, s is added to *potential_parents*[s'] (line (14)). When a node s detects an unvisited neighbor s' (line (15)), s becomes a potential parent of s' and is added to *potential_parents*[s'] (line (17)). A *level*[] array is used to record the level number of each node in the RBFS tree, where the level of a node is its distance to the root s_0 (lines (7) and (18)).

A *parent*[] array is used to record the RBFS tree. Every node s_i remembers its parent in the RBFS tree (line (24)), where *parent*[s_i] is randomly and uniformly chosen from *potential_parents*[s_i]. The *parent*[] array is used to recover the path from each node to the root (lines (30)–(34)). The data transmission rate β_i of sensor s_i includes its data sensing rate μ_i (line (27)) and the sensing rate μ_j (line (32)) if s_i is on the path from s_j to s_0 (lines (30) and (33)), for all $1 \le i \le N$ (line (26)) and for all $1 \le j \le N$ (line (29)).

The second randomized routing method in a heterogeneous wireless sensor network is based on the potential parents of a node. Each time when a sensed datum collected by s_j is sent to s_0, the datum is sent to one of the potential parents s_i of s_j, where s_i is chosen from *potential_parents*[s_j] randomly and uniformly. For different datum, different s_i may be selected. A sensor s_j is called a *potential child* of s_i if s_i is a potential parent of s_j. Let C_i be the set of potential children of s_i and $P_j = |potential\_parents[s_j]|$ be the number of potential parents of s_j in an RBFS tree. Then, we have

$$\beta_i = \mu_i + \sum_{s_j \in C_i} \frac{\beta_j}{P_j},$$

for all s_i in level l, where $1 \le l \le h - 1$, and h is the height of an RBFS tree. The above equation means that sensor s_j contributes a Poisson stream of data with arrival rate β_j/P_j to each sensor in *potential_parents*[s_j]. To calculate data transmission rates for the second routing method, lines (29)–(35) in Fig. 1 are modified as follows (assuming that $h = \max(level[s_1], level[s_2], \ldots, level[s_N])$):

for $(l \leftarrow h; l \ge 2; l{+}{+})$ **do** (29)
  for (all s_j's in level l) **do** (30)
    for (all $s_i \in potential\_parents[s_j]$) **do** (31)
      $\beta_i \leftarrow \beta_i + \beta_j / |potential\_parents[s_j]|$; (32)
    end for; (33)
  end for; (34)
end for. (35)

Again, the last equation for β_i can be extended to incorporate link reliability, i.e.,

$$\beta_i = \mu_i + \sum_{s_j \in C_i} \xi_{ij} \left(\frac{\beta_j}{P_j} \right),$$

as we did in the last section.

An example sensor network is shown in Fig. 2, which has $N = 8$ sensors. An RBFS tree of the network is given in Fig. 3, where we have *potential_parents*[s_3] = *potential_parents*[s_4] = $\{s_1, s_2\}$. Assume that s_3 chooses *parent*[s_3] = s_1, and s_4 chooses *parent*[s_4] = s_2. Therefore, in the first randomized routing method, we have $C_1 = \{s_3, s_5, s_6\}$, $C_2 = \{s_4, s_7, s_8\}$, $\beta_1 = \mu_1 + \beta_3 + \beta_5 + \beta_6$, and $\beta_2 = \mu_2 + \beta_4 + \beta_7 + \beta_8$. On the other hand, in the second randomized routing method, we have $C_1 = \{s_3, s_4, s_5, s_6\}$, $C_2 = \{s_3, s_4, s_7, s_8\}$, $\beta_1 = \mu_1 + 0.5\beta_3 + 0.5\beta_4 + \beta_5 + \beta_6$, and $\beta_2 = \mu_2 + 0.5\beta_3 + 0.5\beta_4 + \beta_7 + \beta_8$.

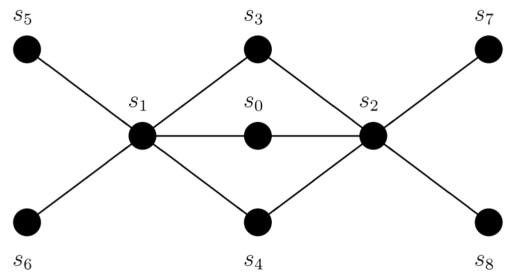

Fig. 2. A sensor network.

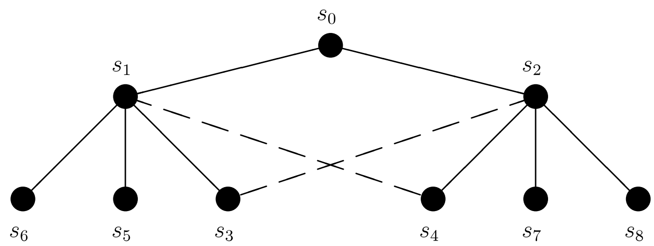

Fig. 3. An RBFS tree.

4. Sensor lifetime

Assume that a sensor network starts operation at time zero.

Let M_i denote the random number of data transmissions performed by s_i during its lifetime. Let $T_{i,1}, T_{i,2}, T_{i,3}, \ldots$ be a sequence of inter-transmission times of s_i, and

$$S_j = T_{i,1} + T_{i,2} + \cdots + T_{i,j},$$

where $j \ge 1$. Clearly, S_j is the time when s_i performs the j's data transmission. Notice that the maximum number of data transmissions that can be performed by s_i is

$$m_i = \left\lfloor \frac{E_i}{q_i} \right\rfloor,$$

assuming that all these data are available at time zero. Therefore, M_i is a random variable in the range $0 \le M_i \le m_i$.

The following theorem gives the probability distribution and the expectation of M_i.

Theorem 1. *The probability distribution of M_i is given by*

$$\mathbb{P}[M_i = j] = \begin{cases} 1 - F_{S_1}\left(\dfrac{E_i - q_i}{p_i}\right), & j = 0; \\ F_{S_j}\left(\dfrac{E_i - jq_i}{p_i}\right) - F_{S_{j+1}}\left(\dfrac{E_i - (j+1)q_i}{p_i}\right), & 1 \le j < m_i; \\ F_{S_{m_i}}\left(\dfrac{E_i - m_i q_i}{p_i}\right), & j = m_i; \end{cases}$$

and the expectation of M_i is

$$\overline{M}_i = \sum_{j=1}^{m_i} \left(1 - \exp\left(-\beta_i \left(\frac{E_i - jq_i}{p_i} \right) \right) \times \sum_{k=0}^{j-1} \frac{1}{k!} \left(\beta_i \left(\frac{E_i - jq_i}{p_i} \right) \right)^k \right),$$

where $m_i = \lfloor E_i/q_i \rfloor$.

Proof. It is easy to see that the reason that $M_i = 0$ is that it takes too long to wait for the first sensed datum, or, by the time S_1 the first sensed datum is generated, the battery of s_i does not have enough power to transmit the datum, since S_1p_i amount of energy has been consumed. This means that $E_i - S_1p_i < q_i$, that is,

$$\mathbb{P}[M_i = 0] = \mathbb{P}\left[S_1 > \frac{E_i - q_i}{p_i}\right] = 1 - F_{S_1}\left(\frac{E_i - q_i}{p_i}\right).$$

For s_i to transmit at least j sensed data, where $1 \le j \le m_i$, we need the condition

$$S_jp_i + jq_i \le E_i,$$

where S_jp_i is the amount of energy required to wait for j sensed data and jq_i is the amount of energy to transmit the j sensed data. Consequently, for all $1 \le j \le m_i$, we have

$$\mathbb{P}[M_i \ge j] = \mathbb{P}\left[S_j \le \frac{E_i - jq_i}{p_i}\right] = F_{S_j}\left(\frac{E_i - jq_i}{p_i}\right).$$

The last equation implies that

$$\begin{aligned}\mathbb{P}[M_i = j] &= \mathbb{P}[M_i \ge j] - \mathbb{P}[M_i \ge j+1] \\ &= F_{S_j}\left(\frac{E_i - jq_i}{p_i}\right) - F_{S_{j+1}}\left(\frac{E_i - (j+1)q_i}{p_i}\right),\end{aligned}$$

where $1 \le j < m_i$.

Based on $\mathbb{P}[M_i = j]$, we get

$$\overline{M}_i = \sum_{j=1}^{m_i} j\mathbb{P}[M_i = j] = \sum_{j=1}^{m_i} F_{S_j}\left(\frac{E_i - jq_i}{p_i}\right).$$

To obtain the cdf of S_j, we notice that the inter-transmission times of sensor s_i, that is, $T_{i,1}, T_{i,2}, T_{i,3}, \ldots$, are i.i.d. random variables with a common exponential pdf

$$f_{T_i}(t) = \beta_i e^{-\beta_i t}.$$

This implies that S_j has an Erlang distribution with pdf

$$f_{S_j}(t) = \frac{\beta_i e^{-\beta_i t}(\beta_i t)^{j-1}}{(j-1)!},$$

and cdf [14]

$$F_{S_j}(t) = 1 - e^{-\beta_i t}\left(1 + \frac{\beta_i t}{1!} + \frac{(\beta_i t)^2}{2!} + \cdots + \frac{(\beta_i t)^{j-1}}{(j-1)!}\right).$$

We can also represent $F_{S_j}(t)$ as

$$F_{S_j}(t) = \frac{\gamma(j, \beta_i t)}{(j-1)!},$$

where $\gamma(j, t)$ is the lower incomplete gamma function defined as [14]

$$\gamma(j, t) = \int_0^t x^{j-1}e^{-x}dx.$$

Therefore, the expectation of M_i is

$$\begin{aligned}\overline{M}_i = \sum_{j=1}^{m_i}\Bigg(1 - \exp\left(-\beta_i\left(\frac{E_i - jq_i}{p_i}\right)\right) \\ \times \sum_{k=0}^{j-1}\frac{1}{k!}\left(\beta_i\left(\frac{E_i - jq_i}{p_i}\right)\right)^k\Bigg),\end{aligned}$$

where $m_i = \lfloor E_i/q_i \rfloor$. □

Let L_i denote the lifetime of s_i. We have

$$L_i \le \frac{E_i}{p_i},$$

where the equality is achieved when s_i does not transmit any datum. Hence, L_i is a random variable in the range $(0, E_i/p_i]$. The following theorem gives the probability distribution of L_i and the expected lifetime $\overline{L}_i$ of sensor s_i.

Theorem 2. *The probability distribution of L_i is given by*

$$\mathbb{P}\left[L_i = \frac{E_i - jq_i}{p_i}\right] = \begin{cases} 1 - F_{S_1}\left(\dfrac{E_i - q_i}{p_i}\right), & j = 0; \\ F_{S_j}\left(\dfrac{E_i - jq_i}{p_i}\right) - F_{S_{j+1}}\left(\dfrac{E_i - (j+1)q_i}{p_i}\right), & 1 \le j < m_i; \\ F_{S_{m_i}}\left(\dfrac{E_i - m_iq_i}{p_i}\right), & j = m_i; \end{cases}$$

where $m_i = \lfloor E_i/q_i \rfloor$. The expected lifetime of s_i is

$$\overline{L}_i = \frac{E_i - \overline{M}_iq_i}{p_i},$$

where $\overline{M}_i$ is given by Theorem 1.

Proof. Notice that L_i and M_i have the following relationship:

$$L_i = \frac{E_i - M_iq_i}{p_i}.$$

It is clear that if s_i performs M_i data transmissions during its lifetime, we have

$$L_i = S_{M_i} + \frac{E_i - S_{M_i}p_i - M_iq_i}{p_i} = \frac{E_i - M_iq_i}{p_i},$$

where S_{M_i} is the time to wait for M_i sensed data, $S_{M_i}p_i$ is the amount of energy required to wait for M_i sensed data, M_iq_i is the amount of energy to transmit the M_i sensed data, and $(E_i - S_{M_i}p_i - M_iq_i)/p_i$ is the remaining survival time. In other words, except the amount of energy M_iq_i used to transmit the M_i sensed data, the remaining energy $E_i - M_iq_i$ is used to sense and receive data, which gives the sensor s_i lifetime $(E_i - M_iq_i)/p_i$.

Since M_i is a discrete random variable having values in the set $\{0, 1, 2, \ldots, m_i\}$, the lifetime L_i of sensor s_i is also a discrete random variable which can only have $m_i + 1$ possible values, i.e.,

$$L_i = \frac{E_i - jq_i}{p_i},$$

for all $0 \le j \le m_i$. It is clear that

$$\mathbb{P}\left[L_i = \frac{E_i - jq_i}{p_i}\right] = \mathbb{P}[M_i = j],$$

for all $0 \le j \le m_i$, where $\mathbb{P}[M_i = j]$ is given by Theorem 1. The equation for $\overline{L}_i$ is a direct consequence of the above discussion. □

5. Optimal energy allocation

The lifetime of a sensor network is optimized when all the sensors have the same expected lifetime. This can be achieved by an initial energy allocation $(E_1, E_2, \ldots, E_N)$ such that $\overline{L}_1 = \overline{L}_2 = \cdots = \overline{L}_N = L$ and $E_1 + E_2 + \cdots + E_N = E$, where $\overline{L}_i$ is given by Theorem 2 and E is the total energy budget. The lifetime of a sensor network is defined as the identical expected sensor lifetime L.

In Fig. 4, we describe our algorithm for finding an optimal initial energy allocation $(E_1, E_2, \ldots, E_N)$ as well as the lifetime L of a sensor network for a given set of N heterogeneous sensors $s_1, s_2, \ldots, s_N$ with β_i, p_i, q_i, where $1 \le i \le N$, and a total energy budget E.

It is clear that $\overline{L}_i$ given by Theorem 2 is an increasing function of E_i, since more energy results in longer lifetime. Hence, given L, there is unique E_i such that the expected lifetime of s_i is L when given E_i. Our strategy to find the network lifetime L is to search L using the standard bisection method [3, p. 22] in a well

Algorithm: Optimal Energy Allocation

Input: N heterogeneous sensors $s_1, s_2, ..., s_N$ with β_i, p_i, q_i, where $1 \le i \le N$, and a total energy budget E for a sensor network.
Output: An initial energy allocation $(E_1, E_2, ..., E_N)$ and the network lifetime L, such that $\overline{L}_1 = \overline{L}_2 = \cdots = \overline{L}_N = L$ and $E_1 + E_2 + \cdots + E_N = E$.

for $(i \leftarrow 1; i \le N; i{+}{+})$ **do** (1)
  $E_i \leftarrow E/N$; (2)
  Calculate $\overline{L}_i$ based on E_i by using Theorem 2; (3)
end for; (4)
$L_{lb} \leftarrow \min(\overline{L}_1, \overline{L}_2, ..., \overline{L}_N)$; (5)
$L_{ub} \leftarrow \max(\overline{L}_1, \overline{L}_2, ..., \overline{L}_N)$; (6)
$L \leftarrow \overline{L}_i$ for any i; (7)
while $(L_{ub} - L_{lb} > \epsilon)$ **do** (8)
  $L \leftarrow (L_{lb} + L_{ub})/2$; (9)
  for $(i \leftarrow 1; i \le N; i{+}{+})$ **do** (10)
    $E_{lb} \leftarrow (L < \overline{L}_i)\ ?\ 0 : E_i$; (11)
    $E_{ub} \leftarrow (L < \overline{L}_i)\ ?\ E_i : E$; (12)
    while $(E_{ub} - E_{lb} > \epsilon)$ **do** (13)
      $E_i \leftarrow (E_{lb} + E_{ub})/2$; (14)
      Calculate $\overline{L}_i$ based on E_i by using Theorem 2; (15)
      if $(\overline{L}_i < L)$ (16)
        $E_{lb} \leftarrow E_i$; (17)
      else (18)
        $E_{ub} \leftarrow E_i$; (19)
      end if; (20)
    end do; (21)
  end for; (22)
  $E' \leftarrow E_1 + E_2 + \cdots + E_N$; (23)
  if $(E' < E)$ (24)
    $L_{lb} \leftarrow L$; (25)
  else (26)
    $L_{ub} \leftarrow L$; (27)
  end if; (28)
end do; (29)
return $(E_1, E_2, ..., E_N)$ and L. (30)

Fig. 4. An algorithm for optimal energy allocation.

chosen interval $[L_{lb}, L_{ub}]$, such that $L \in [L_{lb}, L_{ub}]$ (lines (8)–(29)). To properly set L_{lb} and L_{ub}, we first assign equal amount of energy to each sensor, i.e., let $E_i = E/N$, for all $1 \le i \le N$, and calculate $\overline{L}_i$ based on E_i by using Theorem 2 (lines (1)–(4)). Due to heterogeneity of the sensors, the $\overline{L}_i$'s computed in line (3) are different. To obtain identical sensor lifetime, we must move energy from the sensor with the largest $\overline{L}_i$ to the sensor with the smallest $\overline{L}_i$. Hence, we know that $L \in [L_{lb}, L_{ub}]$, where $L_{lb} = \min(\overline{L}_1, \overline{L}_2, \ldots, \overline{L}_N)$ (line (5)) and $L_{ub} = \max(\overline{L}_1, \overline{L}_2, \ldots, \overline{L}_N)$ (line (6)). Before we go to search for L, L is given a value in line (7) in case sensors are not too heterogeneous and the rest of the algorithm is not executed.

The interval $[L_{lb}, L_{ub}]$ is divided into two halves (line (9)) in each repetition of the while-loop of lines (8)–(29), and the loop is repeated until the length of the interval is sufficiently small and the required numerical accuracy is achieved (see line (8), where the value of ϵ is used to control numerical precision). For each L given in line (9), we find the unique E_i such that the expected lifetime of s_i is L when given E_i, for all $1 \le i \le N$ (lines (10)–(22)). Based on the E_i's, we check $E' = E_1 + E_2 + \cdots + E_N$ (line (23)) and adjust the search interval $[L_{lb}, L_{ub}]$ based on the relationship between E' and E (lines (24)–(28)).

Our strategy to find E_i for a given L is also to search E_i using the standard bisection method in a well chosen interval $[E_{lb}, E_{ub}]$, such that $E_i \in [E_{lb}, E_{ub}]$ (lines (11)–(21)). It is noticed that each time when the algorithm reaches lines (11) and (12), there has been a matching pair of E_i and $\overline{L}_i$ in the sense that the expected lifetime of s_i is $\overline{L}_i$ when given E_i, for all $1 \le i \le N$. Such a pair of E_i and $\overline{L}_i$ is initialized in line (3) and later updated in line (15). If $L < \overline{L}_i$, the new E_i that matches with L must be in the interval $[0, E_i]$, and we should set $E_{lb} = 0$ (line (11)) and $E_{ub} = E_i$ (line (12)). If $L \ge \overline{L}_i$, the new E_i that matches with L must be in the interval $[E_i, E]$, and we should set $E_{lb} = E_i$ (line (11)) and $E_{ub} = E$ (line (12)). A conditional expression $(c)\ ?\ u : v$ means that if a Boolean condition c is true, the value of the expression is u; otherwise, the value of the expression is v. The while-loop of lines (13)–(21) is repeated until the length of the interval $[E_{lb}, E_{ub}]$ is sufficiently small and the required numerical accuracy is achieved (see line (13), where the value of ϵ is used to control numerical precision).

6. Large energy budget

The main problem of the above algorithm in Fig. 4 is that the running time is too long when E_i is large. Furthermore, there is also a problem that a numerical value may exceed the data range represented by a computer even when E_i is not large and when m_i is, say, 30, due to such calculations as power and factorial in Theorem 1. Fortunately, we find that both $\overline{M}_i$ and $\overline{L}_i$ are approximately linear functions of E_i when E_i is not too small. Let $\overline{L}_i(E_i)$ be a function of E_i. Then, we have

$$\overline{L}_i(E_i) = \overline{L}_i(E_i^{(1)}) + \frac{\overline{L}_i(E_i^{(2)}) - \overline{L}_i(E_i^{(1)})}{E_i^{(2)} - E_i^{(1)}} \cdot (E_i - E_i^{(1)}),$$

where $E_i^{(1)}$ and $E_i^{(2)}$ are small values which are able to provide accurate approximation of $\overline{L}_i(E_i)$ for large E_i.

Let us consider an ideal case when the inter-transmission time is fixed at $T_i = 1/\beta_i$, that is, a sensor s_i transmits a datum every T_i units of time. Also, assume that the lifetime L_i can be evenly divided by T_i, that is, s_i transmits $M_i = L_i/T_i = L_i\beta_i$ data during its lifetime. It is clear that

$$E_i = L_i p_i + M_i q_i = L_i p_i + L_i\beta_i q_i = L_i(p_i + \beta_i q_i),$$

which implies that

$$L_i = \frac{E_i}{p_i + \beta_i q_i}.$$

To have an identical lifetime L for all the sensors, i.e.,

$$\frac{E_i}{p_i + \beta_i q_i} = L,$$

we need

$$E_i = L(p_i + \beta_i q_i),$$

for all $1 \le i \le N$. Since $E_1 + E_2 + \cdots + E_N = E$, i.e.,

$$\sum_{i=1}^{N} L(p_i + \beta_i q_i) = E,$$

we get

$$L = \frac{E}{\sum_{i=1}^{N}(p_i + \beta_i q_i)},$$

and

$$E_i = \frac{(p_i + \beta_i q_i)E}{\sum_{i=1}^{N}(p_i + \beta_i q_i)}.$$

The above discussion shows that the sensor lifetime L_i is a linear function of E_i and the network lifetime L is a linear function of E.

The above analysis reveals that in an ideal case, the lifetime of a sensor network is the initial total energy divided by the total amount of energy consumed by all sensors in one unit of time. We expect such a relation roughly holds for any sensor network. In the following, we show that in a heterogeneous wireless sensor network, if E_i is not too small, the expected number $\overline{M}_i$ of data transmissions performed by s_i as well as the expected lifetime $\overline{L}_i$ of sensor s_i are linear functions of E_i. Furthermore, for sufficiently large E, the network lifetime L as well as the optimal energy allocation E_i are linear functions of E.

Let us define

$$t_j = \frac{E_i - jq_i}{p_i},$$

where $1 \le j \le m_i$. Then, by Theorem 1, we have

$$\overline{M}_i = F_{S_1}(t_1) + F_{S_2}(t_2) + F_{S_3}(t_3) + \cdots + F_{S_{m_i}}(t_{m_i}),$$

where

$$F_{S_1}(t_1) = 1 - e^{-\beta_i t_1},$$

$$F_{S_2}(t_2) = 1 - e^{-\beta_i t_2}\left(1 + \frac{\beta_i t_2}{1!}\right),$$

$$F_{S_3}(t_3) = 1 - e^{-\beta_i t_3}\left(1 + \frac{\beta_i t_3}{1!} + \frac{(\beta_i t_3)^2}{2!}\right),$$

$$\vdots$$

$$F_{S_{m_i}}(t_{m_i}) = 1 - e^{-\beta_i t_{m_i}}\left(1 + \frac{\beta_i t_{m_i}}{1!} + \frac{(\beta_i t_{m_i})^2}{2!} + \cdots + \frac{(\beta_i t_{m_i})^{m_i-1}}{(m_i-1)!}\right).$$

It is clear that $\overline{M}_i$ can be viewed as a function $\overline{M}_i(E_i)$ of E_i. The following theorem states that $\overline{M}_i(E_i)$ is almost a linear function of E_i for sufficiently large E_i.

Theorem 3. *For any sensor s_i, where $1 \le i \le N$, we have*

$$\frac{d\overline{M}_i}{dE_i} = \frac{\beta_i}{p_i}\left(e^{-\beta_i t_1} + e^{-\beta_i t_2}\frac{\beta_i t_2}{1!} + e^{-\beta_i t_3}\frac{(\beta_i t_3)^2}{2!} + \cdots + e^{-\beta_i t_{m_i}}\frac{(\beta_i t_{m_i})^{m_i-1}}{(m_i-1)!}\right),$$

and

$$\lim_{E_i\to\infty}\frac{d\overline{M}_i}{dE_i} = \frac{\beta_i}{\beta_i q_i + p_i}.$$

Consequently, for sufficiently large E_i, we have

$$\overline{M}_i = a_i E_i + b_i$$

for some constants a_i and b_i.

Proof. It is clear that

$$\frac{d\overline{M}_i}{dE_i} = \sum_{j=1}^{m_i}\frac{dF_{S_j}(t_j)}{dE_i}.$$

It is easy to verify that

$$\frac{dF_{S_j}(t_j)}{dE_i} = \left(\frac{\beta_i}{p_i}\right)e^{-\beta_i t_j}\frac{(\beta_i t_j)^{j-1}}{(j-1)!},$$

for all $1 \le j \le m_i$. Therefore, we have

$$\frac{d\overline{M}_i}{dE_i} = \frac{\beta_i}{p_i}\sum_{j=1}^{m_i} e^{-\beta_i t_j}\frac{(\beta_i t_j)^{j-1}}{(j-1)!}.$$

Without loss of generality, we only consider those E_i with $E_i = m_i q_i$, where m_i is an integer, and let $m_i \to \infty$. If $\lim_{m_i\to\infty} d\overline{M}_i/dE_i$ is some constant, then $\lim_{E_i\to\infty} d\overline{M}_i/dE_i$ is also the same constant, since $d\overline{M}_i/dE_i$ is a continuous function of E_i. Notice that

$$\beta_i t_j = \beta_i\left(\frac{E_i - jq_i}{p_i}\right) = \beta_i\left(\frac{m_i q_i - jq_i}{p_i}\right) = \frac{\beta_i q_i}{p_i}(m_i - j) = w_i(m_i - j),$$

where

$$w_i = \frac{\beta_i q_i}{p_i},$$

for all $1 \le j \le m_i$. (Since $\beta_i q_i$ is the expected amount of energy consumed by s_i for data transmission in one unit of time, and p_i is the amount of energy consumed by s_i for data sensing in one unit of time, w_i is the *data transmission to data sensing ratio* of sensor s_i.) Hence, we get

$$\frac{d\overline{M}_i}{dE_i} = \frac{\beta_i}{p_i} \sum_{j=1}^{m_i-1} e^{-w_i(m_i-j)} \cdot \frac{(w_i(m_i-j))^{j-1}}{(j-1)!}.$$

We can show that for any $w_i > 0$, we have

$$\lim_{m_i\to\infty} \sum_{j=1}^{m_i-1} e^{-w_i(m_i-j)} \cdot \frac{(w_i(m_i-j))^{j-1}}{(j-1)!} = \frac{1}{w_i+1}.$$

The proof of the above identity is given in Appendix. Consequently,

$$\lim_{m_i\to\infty} \frac{d\overline{M}_i}{dE_i} = \frac{\beta_i}{p_i} \cdot \frac{1}{w_i+1} = \frac{\beta_i}{\beta_i q_i + p_i}.$$

Based on the fact that

$$\frac{d\overline{M}_i}{dE_i} = \frac{\beta_i}{\beta_i q_i + p_i},$$

for large E_i, we get

$$\begin{aligned}\overline{M}_i(E_i) &= \overline{M}_i(E_i^*) + \left(\frac{\beta_i}{\beta_i q_i + p_i}\right)(E_i - E_i^*) \\ &= \left(\frac{\beta_i}{\beta_i q_i + p_i}\right) E_i + \overline{M}_i(E_i^*) - \left(\frac{\beta_i}{\beta_i q_i + p_i}\right) E_i^* \\ &= a_i E_i + b_i,\end{aligned}$$

for all $E_i \ge E_i^*$, where E_i^* is sufficiently large, and

$$a_i = \frac{\beta_i}{\beta_i q_i + p_i}$$

and

$$b_i = \overline{M}_i(E_i^*) - \left(\frac{\beta_i}{\beta_i q_i + p_i}\right) E_i^*$$

are two constants. □

The expected sensor lifetime $\overline{L}_i$ can be viewed as a function of E_i. The following theorem states that $\overline{L}_i(E_i)$ is almost a linear function of E_i for sufficiently large E_i.

Theorem 4. *For sufficiently large E_i, we have $\overline{L}_i = u_i E_i + v_i$, where u_i and v_i are some constants, for all $1 \le i \le N$.*

Proof. Since $\overline{M}_i = a_i E_i + b_i$ for sufficiently large E_i, we have

$$\begin{aligned}\overline{L}_i &= \frac{E_i - \overline{M}_i q_i}{p_i} \\ &= \frac{E_i - (a_i E_i + b_i) q_i}{p_i} \\ &= \left(\frac{1 - a_i q_i}{p_i}\right) E_i - \frac{b_i q_i}{p_i} \\ &= \frac{E_i}{\beta_i q_i + p_i} + \frac{q_i}{p_i}\left(\left(\frac{\beta_i}{\beta_i q_i + p_i}\right) E_i^* - \overline{M}_i(E_i^*)\right) \quad \text{(by Theorem 3)} \\ &= u_i E_i + v_i,\end{aligned}$$

where

$$u_i = \frac{1}{\beta_i q_i + p_i}$$

and

$$v_i = \frac{q_i}{p_i}\left(\left(\frac{\beta_i}{\beta_i q_i + p_i}\right) E_i^* - \overline{M}_i(E_i^*)\right)$$

are two constants. □

The network lifetime L and the optimal energy allocation E_i can be viewed as a function of E. The following theorem shows that both L and E_i are linear functions of E.

Theorem 5. *For sufficiently large energy budget E, we have*

$$L = \frac{E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{\sum_{i=1}^{N} \frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)},$$

and

$$E_i = \frac{(\beta_i q_i + p_i) E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{(\beta_i q_i + p_i)\sum_{i=1}^{N} \frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} - \frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right),$$

for all $1 \le i \le N$.

Proof. Since $\overline{L}_i = u_i E_i + v_i = L$, we have

$$E_i = \frac{L - v_i}{u_i} = \frac{L}{u_i} - \frac{v_i}{u_i},$$

for all $1 \le i \le N$. Since $E_1 + E_2 + \cdots + E_N = E$, i.e.,

$$\left(\sum_{i=1}^{N} \frac{1}{u_i}\right) L - \sum_{i=1}^{N} \frac{v_i}{u_i} = E,$$

we get

$$L = \frac{E + \sum_{i=1}^{N} \frac{v_i}{u_i}}{\sum_{i=1}^{N} \frac{1}{u_i}}.$$

By using u_i and v_i from the proof of Theorem 4, we obtain

$$L = \frac{E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{\sum_{i=1}^{N} \frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)}.$$

Furthermore, since $E_i = L/u_i - v_i/u_i$, we get

$$E_i = \frac{(\beta_i q_i + p_i) E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{(\beta_i q_i + p_i)\sum_{i=1}^{N} \frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} - \frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right),$$

for all $1 \le i \le N$. □

The following theorem gives our effective way to calculate network lifetime and an optimal energy allocation for large energy budget.

Theorem 6. *If $E^* = E_1^* + E_2^* + \cdots + E_N^*$ is certain (sufficiently large) amount of energy budget allocated to the N sensors such that s_i gets E_i^* and all the sensors have the same expected lifetime*

$$L^* = \frac{E_i^* - \overline{M}_i(E_i^*)q_i}{p_i},$$

where L^* is also the network life and the E_i^*'s are obtained by the optimal energy allocation algorithm, then for large energy budget E, we have

$$L = \frac{E - E^*}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + L^*,$$

and

$$E_i = \frac{(\beta_i q_i + p_i)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)}(E - E^*) + E_i^*,$$

for all $1 \le i \le N$.

Proof. Notice that

$$\frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right) = q_i(\beta_i L^* - \overline{M}_i(E_i^*)).$$

For very large energy budget E, by Theorem 5, the network lifetime is

$$\begin{aligned}
L &= \frac{E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{\sum_{i=1}^{N} q_i(\beta_i L^* - \overline{M}_i(E_i^*))}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} \\
&= \frac{E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{\sum_{i=1}^{N} \beta_i q_i L^* - \sum_{i=1}^{N} \overline{M}_i(E_i^*) q_i}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} \\
&= \frac{E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{\sum_{i=1}^{N} \beta_i q_i L^* - \sum_{i=1}^{N}(E_i^* - p_i L^*)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} \\
&= \frac{E}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + \frac{\sum_{i=1}^{N}(\beta_i q_i + p_i) L^* - \sum_{i=1}^{N} E_i^*}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} \\
&= \frac{E - E^*}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + L^*.
\end{aligned}$$

Similarly, by the fact that $E_i = L/u_i - v_i/u_i$, the optimal energy allocation is

$$\begin{aligned}
E_i &= \frac{(\beta_i q_i + p_i)(E - E^*)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} \\
&\quad + (\beta_i q_i + p_i)L^* - \frac{q_i}{p_i}\left(\beta_i E_i^* - \overline{M}_i(E_i^*)(\beta_i q_i + p_i)\right) \\
&= \frac{(\beta_i q_i + p_i)(E - E^*)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + (\beta_i q_i + p_i)L^* - q_i(\beta_i L^* - \overline{M}_i(E_i^*)) \\
&= \frac{(\beta_i q_i + p_i)(E - E^*)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)} + p_i L^* + \overline{M}_i(E_i^*) q_i \\
&= \frac{(\beta_i q_i + p_i)}{\sum_{i=1}^{N}(\beta_i q_i + p_i)}(E - E^*) + E_i^*,
\end{aligned}$$

for all $1 \le i \le N$. □

7. Working sensors

Let $W(t)$ denote the number of sensors which are still working at time t. The lifetime of two sensors s_i and s_j are correlated if s_i is on the path from s_j to s_0. Consequently, $W(t)$ is an extremely complicated random variable whose pdf and cdf are unknown, due to the extensive correlation among sensor lifetime. Fortunately, we can find the expectation of $W(t)$ without knowing its pdf.

The following theorem gives the expectation of $W(t)$.

Theorem 7. *The expected number of sensors which are still working at time $t \ge 0$ is*

$$\overline{W(t)} = N - \sum_{i=1}^{N} F_{L_i}(t),$$

where

$$F_{L_i}(t) = \begin{cases} F_{S_{n_i}}\left(\dfrac{E_i - n_i q_i}{p_i}\right), & 0 \le t < \dfrac{E_i}{p_i}; \\ 1, & t \ge \dfrac{E_i}{p_i}; \end{cases}$$

with

$$n_i = \left\lceil \frac{E_i - p_i t}{q_i} \right\rceil.$$

Proof. It is clear that $\overline{W(t)}$ is the summation of the probability that s_i is still functioning at time t, for all $1 \le i \le N$, that is,

$$\overline{W(t)} = \sum_{i=1}^{N} \mathbb{P}[s_i \text{ is still functioning at time } t].$$

Notice that the probability that s_i is still functioning at time t is equivalent to the probability that the lifetime of s_i is at least t, i.e., $L_i \ge t$. Thus, we get

$$\begin{aligned}
\overline{W(t)} &= \sum_{i=1}^{N} \mathbb{P}[L_i \ge t] \\
&= \sum_{i=1}^{N}(1 - \mathbb{P}[L_i \le t]) \\
&= \sum_{i=1}^{N}(1 - F_{L_i}(t)) \\
&= N - \sum_{i=1}^{N} F_{L_i}(t).
\end{aligned}$$

When $0 \le t < E_i/p_i$, we have

$$\begin{aligned}F_{L_i}(t) &= \mathbb{P}\left[L_i \le t\right]\\ &= \mathbb{P}\left[\frac{E_i - M_i q_i}{p_i} \le t\right]\\ &= \mathbb{P}\left[M_i \ge \frac{E_i - p_i t}{q_i}\right]\\ &= \mathbb{P}\left[M_i \ge n_i\right]\\ &= \sum_{j=n_i}^{m_i} \mathbb{P}\left[M_i = j\right],\end{aligned}$$

where

$$n_i = \left\lceil \frac{E_i - p_i t}{q_i} \right\rceil \ge 1.$$

By Theorem 1, the last summation is

$$F_{L_i}(t) = F_{S_{n_i}}\left(\frac{E_i - n_i q_i}{p_i}\right).$$

Finally, we notice that $F_{L_i}(t) = 1$ when $t \ge E_i/p_i$. □

Notice that $\overline{W(t)}$ is a nonincreasing function of t. Therefore, for any threshold N^*, we can find the largest t^* such that $\overline{W(t)} \ge N^*$ for all $t \le t^*$, and t^* can be considered as the lifetime of a sensor network with respect to N^*.

8. Numerical examples

Let us consider an area of interest A which is a square of size $d \times d$ m^2 with $d = 50$ m. The area A is divided into $d^2 = 2500$ identical square cells of size 1 m^2. A base station is in the center of A.

There are $N = 100$ heterogeneous sensors randomly distributed in A. We consider two probability distributions of the sensors. The first distribution is a uniform distribution, i.e., all the x_i's and the y_i's are uniformly and independently distributed in the range $[0, d]$. The second distribution is a truncated normal distribution, i.e., all the x_i's and the y_i's are independently distributed in the range $[0, d]$ according to a normal distribution with mean $\mu = d/2$ and variance $\sigma^2 = d^2/16$. In our numerical examples, a normal random number is discarded if the number is not in the range $[0, d]$.

The sensor communication radii, i.e., the c_i's, are uniformly and independently distributed in the range $[c', c''] = [10, 15]$ with mean $\bar{c} = (c' + c'')/2 = 12.5$ m. The sensing range of s_i is set as $r_i = c_i/2$, for all $1 \le i \le N$, namely, r_i is in the range $[r', r''] = [5.0, 7.5]$ with mean $\bar{r} = (r' + r'')/2 = 6.25$ m and linearly proportional to the communication radius.

The parameter p_i is in the range [0.00050, 0.00075] (in Watts) with $p_i = 0.0001 r_i$, that is, p_i is in the range 0.50–0.75 mW with mean $\bar{p} = 0.0001\bar{r} = 0.625$ mW and linearly proportional to the sensing range. The parameter q_i is in the range [0.025, 0.050] (in Joules) with $q_i = 0.005 + 0.0002 c_i^2$, that is, q_i is in the range 25–50 mJ with mean

$$\bar{q} = \frac{1}{c'' - c'}\int_{c'}^{c''} (0.005 + 0.0002c^2)dc = 36.67 \text{ mJ},$$

and proportional to the square of the communication radius c_i.

The (x_i, y_i, c_i)'s determine the topology G of a heterogeneous wireless sensor network. Two random graphs G_1 and G_2 are produced for the uniform and the normal distributions respectively. An RBFS tree T_i is constructed for G_i by using the algorithm in Fig. 1, where $i = 1, 2$. In addition to the root, T_1 has four levels and the number of nodes on levels 1–4 are (19, 48, 28, 5), and T_2 has three levels and the number of nodes on levels 1–3 are (44, 44, 12). It is observed that due to the nature of a normal distribution, sensors in G_2 are deployed more densely in areas closer to a base station than in areas further away from the base station.

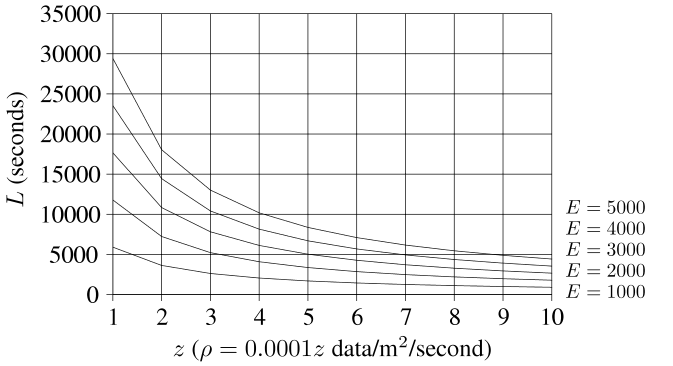

Fig. 5. Network lifetime L vs. data generation rate ρ (uniform distribution).

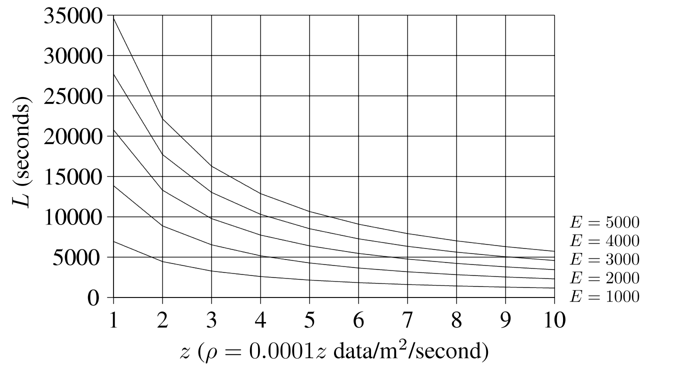

Fig. 6. Network lifetime L vs. data generation rate ρ (normal distribution).

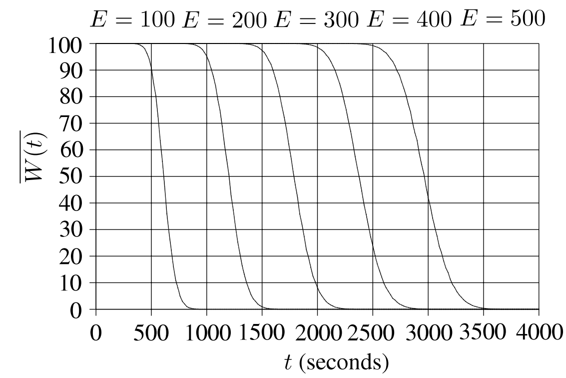

Fig. 7. Expected number of working sensors at time t (uniform distribution).

The (x_i, y_i, r_i)'s determine the coverage of A. A normal distribution leads to less coverage of areas far away from the base station. It is observed that 99.16% cells are covered by at least one sensor in G_1, while only 94.36% cells are covered by at least one sensor in G_2.

We would like to mention that the difference of the numerical data in Figs. 5–8 between the two randomized routing methods is too small to be noticed. The two routing methods yield about the same network lifetime and about the same expected number of working sensors.

In Figs. 5 and 6, we show network lifetime L (in seconds) as a function of ρ for the two probability distributions, where $\rho = 0.0001, 0.0002, 0.0003, \ldots, 0.0010$ data/m^2/s, and $E = 1000, 2000, \ldots, 5000$ J. The number of sensed data received by the base station per second is $d^2\rho$, which is in the range [0.25,2.50]. For a typical sensor with $r_i = \bar{r} = 6.25$ m, $p_i = \bar{p} = 0.000625$ W, and $q_i = \bar{q} = 0.03667$ J, in a typical sensing environment

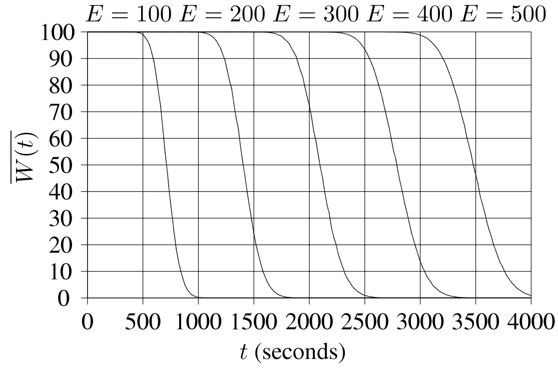

Fig. 8. Expected number of working sensors at time t (normal distribution).

with $\rho = 0.0005$ data/m^2/s, and $\beta_i = \mu_i = \pi \overline{r}^2 \rho = 0.06135923$ data/s (by ignoring the data relayed), we find that we can calculate $\overline{M}_i$ for $E_i = 1$ J with $m_i = 27$. Beyond that, we will encounter numerical inaccuracy. Hence, the data in the figures are calculated by using Theorem 6 with $E^* = 100$ J, such that the average amount of energy allocated to each sensor is approximately $E^*/N = 1$ J, which is enough for $m = \lfloor (E^*/N)/\overline{q} \rfloor = 27$ data transmissions. Our numerical computations reveal that $d\overline{M}_i/dE_i$ is almost a constant when $E_i \geq 20q_i$, which is roughly 0.7334 J, i.e., $E^* = 100$ is sufficiently large to apply Theorem 6. The parameter ϵ in Fig. 4 is set as $\epsilon = 10^{-7}$.

It is observed that the normal distribution of sensors leads to less traffic burden on sensors closer to the base station and results in longer network lifetime. It is also observed that as ρ increases, the percentage of energy devoted to data sensing, i.e.,

$$L(p_1 + p_2 + \cdots + p_N)/E \approx LN\overline{p}/E,$$

decreases (equivalently, the percentage of energy devoted to data transmission increases) noticeably. For instance, for the uniform distribution, when $\rho = 0.0001$, the percentage of energy for data sensing is at least 36%; however, when $\rho = 0.001$, the percentage of energy for data sensing is no more than 6%.

In Figs. 7 and 8, we show the expected number of working sensors $\overline{W(t)}$ for the two probability distributions, where $\rho = 0.0001$ and $E = 100, 200, \ldots, 500$ J. For the first routing method and the five values of E, the network lifetime L is 607, 1189, 1771, 2353, and 2935, respectively, for the uniform distribution, and 717, 1411, 2105, 2798, and 3492, respectively, for the normal distribution. For the second routing method and the five values of E, the network lifetime L is 613, 1201, 1789, 2377, and 2965, respectively, for the uniform distribution, and 715, 1407, 2098, 2790, and 3481, respectively, for the normal distribution. It is observed that as t increases, $\overline{W(t)}$ remains at N for most of the time and drops to zero in a very short period of time. This means that all sensors die at about the same time. It is also observed that $\overline{W(t)}$ is about $N/2$ at time $t = L$. The significance of these data is that we can easily determine the network lifetime with respect to a threshold N^*. For instance, for the uniform distribution, when $E = 500$ and $N^* = 80$, the network lifetime is $t^* = 2770$ for the first routing method and $t^* = 2800$ for the second routing method, since we have $\overline{W(t)} \geq N^*$ for all $t \leq t^*$.

Additional data are demonstrated in Tables 1–3 for a typical sensor with $\beta_i = 0.06135923$ data/s, $p_i = 0.000625$ W, $q_i = 0.03667$ J, $E_i = 1$ J, and $m_i = 27$. In Table 1, we show the probability distribution of M_i calculated by using Theorem 1, where a zero probability means that the probability is too small to be shown within the given range of accuracy. The above probability distribution yields $\overline{M}_i = 20.865650614068464$ (by Theorem 1) and $\overline{L}_i = 375.770547171375028$ (by Theorem 2). In Table 2, we show the derivative $d\overline{M}_i/dE_i$ calculated by using Theorem 3.

Table 1
The probability distribution of M_i.

j	$\mathbb{P}[M_i = j]$
0	0.000000000000000
1	0.000000000000000
2	0.000000000000000
3	0.000000000000000
4	0.000000000000000
5	0.000000000000000
6	0.000000000000000
7	0.000000000000000
8	0.000000000000000
9	0.000000000000000
10	0.000000000000006
11	0.000000000000570
12	0.000000000044048
13	0.000000002591118
14	0.000000114027680
15	0.000003666395794
16	0.000083495370354
17	0.001292809458138
18	0.012895057526243
19	0.077092922488284
20	0.250456358120700
21	0.385764127695034
22	0.231385848163540
23	0.039874819472959
24	0.001149079567642
25	0.000001699073387
26	0.000000000004503
27	0.000000000000000

Table 2
The derivative $d\overline{M}_i/dE_i$.

E_i	$d\overline{M}_i/dE_i$
0.1	18.9528223466
0.2	18.7645478475
0.3	21.7335949666
0.4	21.4597651289
0.5	21.3103069680
0.6	21.3376806818
0.7	21.3441760714
0.8	21.3421062901
0.9	21.3418918811
1.0	21.3420271870
1.1	21.3420291302
1.2	21.3420212103
1.3	21.3420217057
1.4	21.3420221226
1.5	21.3420220622
1.6	21.3420220429
1.7	21.3420220478
1.8	21.3420220486
1.9	21.3420220482
2.0	21.3420220482

It is observed that $d\overline{M}_i/dE_i$ quickly approaches a constant as E_i increases. In Table 3, we compare the exact value of the expected sensor lifetime $\overline{L}_i$ obtained by using Theorem 2 and the approximate sensor lifetime $\overline{L}_i$ obtained by using Theorem 4. In Theorem 4, we set $E_i^* = 1$, which gives $u_i = 347.8208823862$, and $v_i = 27.9496647852$. It is observed that our approximate values are extremely close to the exact values with negligible differences. These data validate the legitimacy of our approach developed in Section 6.

9. Concluding remarks

We have addressed the problem of optimal energy allocation and lifetime maximization in heterogeneous wireless sensor networks. We construct a probabilistic model for heterogeneous wireless sensor networks such that the lifetime of sensors as well as an entire network can be studied analytically. We are able to find

Table 3
The expected sensor lifetime.

E_i	$\bar{L}_i$ (exact)	$\bar{L}_i$ (approximate)
1.0	375.7705471714	375.7705471714
1.1	410.5526220144	410.5526354100
1.2	445.3347102611	445.3347236486
1.3	480.1167992685	480.1168118872
1.4	514.8988874477	514.8989001258
1.5	549.6809756467	549.6809883645
1.6	584.4630638918	584.4630766031
1.7	619.2451521322	619.2451648417
1.8	654.0272403703	654.0272530803
1.9	688.8093286088	688.8093413189
2.0	723.5914168475	723.5914295575
2.1	758.3735050861	758.3735177961
2.2	793.1555933247	793.1556060348
2.3	827.9376815633	827.9376942734
2.4	862.7197698019	862.7197825120
2.5	897.5018580406	897.5018707506
2.6	932.2839462792	932.2839589892
2.7	967.0660345178	967.0660472278
2.8	1001.8481227564	1001.8481354665
2.9	1036.6302109950	1036.6302237051
3.0	1071.4122992336	1071.4123119437

optimal initial energy allocation such that all sensors exhaust their battery power at about the same time and the lifetime of a sensor network is significantly prolonged. We also know the expected number of working sensors at any time, so that the lifetime of a wireless sensor network can be predicted.

We would like to mention that our optimal energy allocation algorithm developed in this paper is a centralized and offline algorithm. While it is applicable to many static and stable sensor networks which are optimally designed and carefully deployed, there are many other sensor networks which are randomly deployed and even dynamically changing. Therefore, extension and modification of our method into a distributed, online, and adaptive algorithm becomes an interesting and important topic for further investigation.

Acknowledgments

The valuable comments and suggestions from four anonymous referees substantially improved the paper.

Appendix

In the proof of Theorem 3, we need the following result.

Theorem 8. *For any $w_i > 0$, we have*

$$\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1} e^{-w_i(m_i-j)}\cdot\frac{(w_i(m_i-j))^{j-1}}{(j-1)!}=\frac{1}{w_i+1}. \qquad (1)$$

Proof. We are going to show the following,

$$\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{(m_i-j)^{j-1}}{(j-1)!}\cdot w_i^{j-1}e^{-w_i(m_i-j)} = 1-w_i+w_i^2-w_i^3+w_i^4-\cdots.$$

Notice that

$$\sum_{j=1}^{m_i-1}\frac{(m_i-j)^{j-1}}{(j-1)!}\cdot w_i^{j-1}e^{-w_i(m_i-j)} = \sum_{j=1}^{m_i-1}\frac{(m_i-j)^{j-1}}{(j-1)!}\cdot w_i^{j-1}\sum_{k=0}^{\infty}(-1)^k\frac{(m_i-j)^k}{k!}\cdot w_i^k = \sum_{j=1}^{m_i-1}\sum_{k=0}^{\infty}(-1)^k\frac{(m_i-j)^{j-1+k}}{(j-1)!k!}\cdot w_i^{j-1+k}.$$

It is easy to see that for all $n\ge 0$, if $m_i > n+1$ (which is really the case since $m_i\to\infty$), the coefficient of w_i^n is

$$(-1)^n\left(\frac{(m_i-1)^n}{0!n!}-\frac{(m_i-2)^n}{1!(n-1)!}+\frac{(m_i-3)^n}{2!(n-2)!}-\cdots+(-1)^n\frac{(m_i-n-1)^n}{n!0!}\right),$$

by setting $j=1,2,3,\ldots,n+1$ and $k=n,n-1,n-2,\ldots,0$.

Now we need to show that

$$\frac{(m_i-1)^n}{0!n!}-\frac{(m_i-2)^n}{1!(n-1)!}+\frac{(m_i-3)^n}{2!(n-2)!}-\cdots+(-1)^n\frac{(m_i-n-1)^n}{n!0!}=1,$$

for all integers $n\ge 0$ and $m_i > n+1$. By replacing m_i-1 by m, the above equation becomes

$$\frac{m^n}{0!n!}-\frac{(m-1)^n}{1!(n-1)!}+\frac{(m-2)^n}{2!(n-2)!}-\cdots+(-1)^n\frac{(m-n)^n}{n!0!}=1, \qquad (2)$$

for all integers $n\ge 0$ and $m>n$. Eq. (2) is equivalent to

$$\binom{n}{0}m^n-\binom{n}{1}(m-1)^n+\binom{n}{2}(m-2)^n-\cdots+(-1)^n\binom{n}{n}(m-n)^n=n!. \qquad (3)$$

To prove Eq. (3), let us assume that we have n candies to be given to n boys and $(m-n)$ girls. We are going to give the candies to the kids such that each boy gets exactly one candy. The number W of ways to give the candies is obviously $n!$. We can figure out W by following another reasoning. Let W_i denote the number of ways to give the candies to the m kids such that a kid may receive more than one candy but i particularly chosen boys do not get any candy. Clearly, we have

$$W_i=\binom{n}{i}(m-i)^n,$$

where $\binom{n}{i}$ is the number of ways to choose the i boys who do not get any candy and $(m-i)^n$ is the number of ways to give the n candies to the remaining $(m-i)$ kids such that a kid may receive more than one candy. It is clear by the Principle of Inclusion and Exclusion,

$$W=W_0-W_1+W_2-\cdots+(-1)^nW_n,$$

where the right hand side guarantees that each boy receives one candy. The last equation is exactly Eq. (3) we need to prove. □

Remark 1. We would like to mention that although Eqs. (2) and (3) are proven for integers $m>n$, they hold for all reals m, even negative m. In fact, Eq. (3) is a special case of the following general result. Let

$$S(n,m,k)=\binom{n}{0}m^k-\binom{n}{1}(m-1)^k+\binom{n}{2}(m-2)^k-\cdots+(-1)^n\binom{n}{n}(m-n)^k,$$

where $n\ge 0$ is an integer, m is any real number, and $0\le k\le n$ is an integer. Then, we have

$$S(n,m,k)=\begin{cases}0, & 0\le k\le n-1;\\ n!, & k=n.\end{cases}$$

The above result can be proven by induction on both $n \geq 0$ and $0 \leq k \leq n$. The base cases of $n \leq 1$ and $0 \leq k \leq n$, and $n \geq 0$ and $k = 0$ can be verified easily. For the general case of $n > 1$ and $1 \leq k \leq n$, we notice that for all $1 \leq j \leq n$,

$$\begin{aligned}\binom{n}{j}(m-j)^k &= \binom{n}{j}(m-j)(m-j)^{k-1}\\ &= m\binom{n}{j}(m-j)^{k-1} - j\binom{n}{j}(m-j)^{k-1}\\ &= m\binom{n}{j}(m-j)^{k-1} - n\binom{n-1}{j-1}\\ &\quad \times ((m-1)-(j-1))^{k-1}.\end{aligned}$$

The above equation implies that

$$\begin{aligned}&\binom{n}{0}m^k - \binom{n}{1}(m-1)^k + \binom{n}{2}(m-2)^k\\ &\quad - \cdots + (-1)^n\binom{n}{n}(m-n)^k\\ &= m\left[\binom{n}{0}m^{k-1} - \binom{n}{1}(m-1)^{k-1}\right.\\ &\quad \left. + \binom{n}{2}(m-2)^{k-1} - \cdots + (-1)^n\binom{n}{n}(m-n)^{k-1}\right]\\ &\quad + n\left[\binom{n-1}{0}(m-1)^{k-1} - \binom{n-1}{1}((m-1)-1)^{k-1}\right.\\ &\quad \left. + \cdots + (-1)^{n-1}\binom{n-1}{n-1}((m-1)-(n-1))^{k-1}\right],\end{aligned}$$

namely,

$$S(n, m, k) = mS(n, m, k-1) + nS(n-1, m-1, k-1).$$

By the induction hypothesis, we get

$$S(n, m, k) = \begin{cases} m \times 0 + n \times 0 = 0, & 0 \leq k \leq n-1;\\ m \times 0 + n(n-1)! = n!, & k = n.\end{cases}$$

Remark 2. We replace w_i by x and take integration on both sides of Eq. (1) in $[0, w_i]$. Then, by the Fundamental Theorem of Calculus, Eq. (1) is equivalent to

$$\begin{aligned}&\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\int_0^{w_i}\left(x^{j-1}e^{(j-m_i)x}\cdot\frac{(m_i-j)^{j-1}}{(j-1)!}\right)dx\\ &= \int_0^{w_i}\left(\frac{1}{x+1}\right)dx. \qquad (4)\end{aligned}$$

We notice that

$$\begin{aligned}&\int_0^{w_i}\left(x^{j-1}e^{(j-m_i)x}\cdot\frac{(m_i-j)^{j-1}}{(j-1)!}\right)dx\\ &= \frac{(m_i-j)^{j-1}}{(j-1)!}\int_0^{w_i}x^{j-1}e^{(j-m_i)x}dx.\end{aligned}$$

Since

$$\begin{aligned}&\int x^{j-1}e^{(j-m_i)x}dx\\ &= e^{(j-m_i)x}\sum_{k=0}^{j-1}(-1)^k\frac{(j-1)!x^{j-1-k}}{(j-1-k)!(j-m_i)^{k+1}}\\ &= -(j-1)!e^{(j-m_i)x}\sum_{k=0}^{j-1}\frac{x^{j-1-k}}{(j-1-k)!(m_i-j)^{k+1}},\end{aligned}$$

we have

$$\begin{aligned}&\int_0^{w_i}x^{j-1}e^{(j-m_i)x}dx\\ &= (j-1)!\left(\frac{1}{(m_i-j)^j} - e^{(j-m_i)w_i}\right.\\ &\quad \left.\times\sum_{k=0}^{j-1}\frac{w_i^{j-1-k}}{(j-1-k)!(m_i-j)^{k+1}}\right)\\ &= \frac{(j-1)!}{(m_i-j)^j}\left(1 - \frac{1}{e^{(m_i-j)w_i}}\sum_{k=0}^{j-1}\frac{((m_i-j)w_i)^{j-1-k}}{(j-1-k)!}\right),\end{aligned}$$

and

$$\begin{aligned}&\int_0^{w_i}\left(x^{j-1}e^{(j-m_i)x}\cdot\frac{(m_i-j)^{j-1}}{(j-1)!}\right)dx\\ &= \frac{1}{m_i-j}\left(1 - \frac{1}{e^{(m_i-j)w_i}}\sum_{k=0}^{j-1}\frac{((m_i-j)w_i)^{j-1-k}}{(j-1-k)!}\right).\end{aligned}$$

The last equation implies that Eq. (4) is identical to

$$\begin{aligned}&\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{1}{m_i-j}\left(1 - \frac{1}{e^{(m_i-j)w_i}}\sum_{k=0}^{j-1}\frac{((m_i-j)w_i)^k}{k!}\right)\\ &= \ln(w_i+1), \qquad (5)\end{aligned}$$

or, equivalently, replacing j by $m_i - j$,

$$\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{1}{j}\left(1 - \frac{1}{e^{jw_i}}\sum_{k=0}^{m_i-j-1}\frac{(jw_i)^k}{k!}\right) = \ln(w_i+1). \qquad (6)$$

Eq. (5) is identical to

$$\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{\gamma(j, (m_i-j)w_i)}{(m_i-j)(j-1)!} = \ln(w_i+1), \qquad (7)$$

or, equivalently, replacing j by $m_i - j$,

$$\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{\gamma(m_i-j, jw_i)}{j(m_i-j-1)!} = \ln(w_i+1). \qquad (8)$$

Eq. (7) is identical to

$$\begin{aligned}&\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{1}{(m_i-j)(j-1)!}\int_0^{(m_i-j)w_i}x^{j-1}e^{-x}dx\\ &= \ln(w_i+1), \qquad (9)\end{aligned}$$

or, equivalently, replacing j by $m_i - j$,

$$\begin{aligned}&\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{1}{j(m_i-j-1)!}\int_0^{jw_i}x^{m_i-j-1}e^{-x}dx\\ &= \ln(w_i+1). \qquad (10)\end{aligned}$$

Let $T_1, T_2, T_3, \ldots$ be i.i.d. random variables with a common exponential pdf

$$f_T(t) = w_ie^{-w_it}.$$

Then, $S_j = T_1 + T_2 + \cdots + T_j$ has an Erlang distribution with pdf

$$f_{S_j}(t) = \frac{w_ie^{-w_it}(w_it)^{j-1}}{(j-1)!}.$$

Eq. (5) is also identical to

$$\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{F_{S_j}(m_i-j)}{(m_i-j)}=\ln(w_i+1), \tag{11}$$

or, equivalently, replacing j by m_i-j,

$$\lim_{m_i\to\infty}\sum_{j=1}^{m_i-1}\frac{F_{S_{m_i-j}}(j)}{j}=\ln(w_i+1). \tag{12}$$

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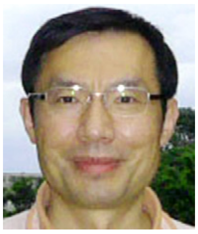

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