Discrete Mathematics 333 (2014) 39–55

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Ordered multiplicity lists for eigenvalues of symmetric matrices whose graph is a linear tree[☆]

Charles R. Johnson [a], Andrew A. Li [b,*], Andrew J. Walker [c]

[a] Department of Mathematics, College of William and Mary, Williamsburg, VA 23185, USA

[b] Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027, USA

[c] Department of Mathematics, Duke University, Durham, NC 27708, USA

ARTICLE INFO

Article history:
Received 6 August 2012
Received in revised form 28 April 2014
Accepted 29 April 2014
Available online 28 June 2014

Keywords:
Linear tree
Symmetric matrix
Eigenvalue
Multiplicity

ABSTRACT

We consider the class of trees for which all vertices of degree at least 3 lie on a single induced path of the tree. For such trees, a new superposition principle is proposed to generate all possible ordered multiplicity lists for the eigenvalues of symmetric (Hermitian) matrices whose graph is such a tree. It is shown that no multiplicity lists other than these can occur and that for two subclasses all such lists do occur. Important contrasts with trees outside the class are given, and it is shown that several prior conjectures about multiplicity lists, including the Degree Conjecture, follow from our superposition principle.

1. Introduction

Given an undirected graph on n vertices G, we say that an n-by-n real symmetric matrix $A = (a_{ij})$ has graph G if for all $i \neq j$, there is an edge between vertices i and j if and only if $a_{ij} \neq 0$. We are interested in the spectra of all n-by-n real symmetric matrices whose graph is a given G, and in particular, in the eigenvalue multiplicities that occur. When a real symmetric matrix A has been identified for some graph G, we will commonly refer to the eigenvalues of G or some subgraph of G, by which we mean the eigenvalues of A or of the principle submatrix of A whose rows and columns correspond to the vertices of the subgraph. For any real symmetric matrix, we refer to the *ordered multiplicities* as the list obtained by arranging the distinct eigenvalues in increasing order and listing their multiplicities. We may also arrange the multiplicities in non-increasing order, a list that we call the *unordered multiplicities*. The set of ordered (unordered) multiplicity lists of real symmetric matrices whose graph is G is denoted $\mathcal{L}_o(G)$ ($\mathcal{L}_u(G)$).

Our focus here will be upon trees, or connected graphs with $n - 1$ edges, and upon ordered multiplicity lists. (Since every complex Hermitian matrix whose graph is a tree is diagonally unitarily similar to a real symmetric matrix with the same graph, there is no loss of generality in considering only symmetric matrices in place of Hermitian matrices.) Multiplicity lists for certain classes of trees [8,9,13] and for trees on fewer than 12 vertices have been determined previously. Referring to any vertex of degree 3 or more as *high degree*, we consider a very rich class of trees, the linear trees (all high degree vertices lie on a path—see Definition 8), that includes some previously studied infinite classes of trees, as well as all but a few of the trees on fewer than 12 vertices. For linear trees, a combinatorial technique (involving a superposition principle) to generate all ordered multiplicity lists is proposed. The necessity of this proposal is proven in general and sufficiency is proven in

☆ This work was partially supported by NSF Grant DMS-0751964.

* Corresponding author.
E-mail addresses: crjohnso@math.wm.edu (C.R. Johnson), aali@mit.edu (A.A. Li), ajw32@duke.edu (A.J. Walker).

http://dx.doi.org/10.1016/j.disc.2014.04.030
0012-365X/

two major cases: (a) when there are just two high degree vertices and (b) when the linear tree is depth 1. The latter uses a new application of the implicit function theorem that leads to an independently interesting class of matrices as Jacobians. The superposition principle generalizes that of [9] used to treat double generalized stars. There seems to be no simpler way to describe all such multiplicity lists. Finally, the results given are used to verify special cases of some outstanding conjectures.

2. Background

For convenience, we use the standard submatrix notation. Given an index set $\alpha \subseteq \{1, \ldots, n\}$, we denote the principle submatrix of A lying in rows and columns $\{1, \ldots, n\} - \alpha$ (respectively, α) by $A(\alpha)$ (respectively, $A[\alpha]$). Additionally, $A(\{i\})$ is abbreviated by $A(i)$. If A is a matrix of graph G, we may use a subgraph of G to specify an index set. For example, $A[G]$ is simply the matrix A. For any real number λ, we use $m_A(\lambda)$ to denote the multiplicity of λ as an eigenvalue of the matrix A.

The classical Interlacing Theorem is very important to our discussion. We state it here and refer the reader to [4] for a more thorough description.

Theorem 1 (*Interlacing Theorem*)**.** *Let A be an n-by-n Hermitian matrix with (real) eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

and suppose $A(i)$ has eigenvalues

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$$

for some $i \in \{1, \ldots, n\}$. Then the eigenvalues satisfy the following inequalities:

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

A result following immediately from these inequalities is that, for any λ and $i = 1, \ldots, n$,

$$m_A(\lambda) - 1 \leq m_{A(i)}(\lambda) \leq m_A(\lambda) + 1.$$

In the case of trees, we have a very useful theorem coming from previous work in [14,15]. We state it without proof in the general form developed in [10].

Theorem 2. *Let A be a Hermitian matrix whose graph is a tree T, and suppose that there exists a vertex v of T and a real number λ such that λ is an eigenvalue of both A and $A(v)$. Then*

1. *there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$;*
2. *if $m_A(\lambda) \geq 2$, then v' may be chosen so that $\deg v' \geq 3$ and so that there are at least three components T_1, T_2, and T_3 of $T - v'$ such that $m_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$;*
3. *if $m_A(\lambda) = 1$, then v' may be chosen so that $\deg v' \geq 2$ and so that there are two components T_1 and T_2 of $T - v'$ such that $m_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

For any tree T, we define the *path cover number* of T to be the minimum number of induced paths of T that cover all vertices without intersecting. We also define the *diameter* to be the maximum length of an induced path of T, where the length of a path refers to the number of nodes in the path. Note that this definition of length differs from the standard definition which defines the length of a path as the number of edges (one less than the number of nodes), but throughout we will continue to use the modified definition of length to remain consistent with previous references (e.g. [7,9]).

To demonstrate the relationship between a graph's structure and its multiplicity lists, we present the following two theorems from [6,7], respectively:

Theorem 3. *Given a tree T, the maximum multiplicity for any single eigenvalue in the multiplicity lists for T is equal to the path cover number of T.*

Theorem 4. *Given a tree T, the minimum number of distinct eigenvalues among the real symmetric matrices with graph T is at least the maximum number of nodes in an induced path of T.*

Let $l = (l_1, \ldots, l_a)$ be a partition of some integer N, i.e., each l_i is a positive integer and $l_1 + \cdots + l_a = N$. Two concepts about partitions will be needed. First, we denote by $l^* = (l_1^*, \ldots, l_b^*)$ the *conjugate partition* of l, so l_j^* is the number of i's such that $l_i \geq j$. Note that l^* is a partition of N with $l_1^* \geq \cdots \geq l_j^*$. The second concept is *majorization*. Let $u = (u_1, \ldots, u_c)$, $u_1 \geq \cdots \geq u_c$, and $v = (v_1, \ldots, v_d)$, $v_1 \geq \cdots \geq v_d$, be ordered partitions of M and N, respectively. Suppose $u_1 + \cdots + u_s \leq v_1 + \cdots + v_s$ for all s, where u_s or v_s are interpreted as 0 when s is greater than c or d, respectively. Then we say that v *majorizes* u if $M = N$, denoting it as $u \preceq v$. If $M < N$, we create a partition u_e of N by appending $N - M$ 1's to u, and then we have $u_e \preceq v$.

We define a *generalized star* as a tree with at most one vertex of high degree. For each generalized star, we call the high degree vertex the *central vertex*, or let any vertex be the central vertex if there are none of high degree. The possible spectra of matrices whose graph is a generalized star were fully characterized in [9]. An additional result for generalized stars that we will use concerns the possible *upward* multiplicity lists. For a real symmetric matrix A with graph G, fix a vertex v. Then λ is an *upward eigenvalue* of A at v if $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, and the multiplicity of λ is called an *upward multiplicity*of A

at v. We denote a multiplicity q that is upward by $\hat{q}$. By a *list of upward multiplicities* for G at v, we mean an ordered list of multiplicities for G with upward designation for each eigenvalue whose multiplicity increases with the removal of v. The set of all (ordered) upward multiplicity lists of G at v is denoted $\hat{\mathcal{L}}_o(G)$, distinguished from the set of ordered multiplicity lists $\mathcal{L}_o(G)$ without upward designations. We do not include v in the notation here because it is usually clear which vertex is being considered. For example, when G is a generalized star, we will assume that the designated central vertex is the removed vertex for any upward eigenvalue or upward multiplicity. The following result from [9] characterizes the lists of upward multiplicities of generalized stars:

Theorem 5. *Let T be a generalized star on n vertices with central vertex v of degree k and arm lengths $l_1 \geq \cdots \geq l_k (\sum_{i=1}^{k} l_i = n-1)$. Let $\lambda_1 < \cdots < \lambda_r$ be any sequence of real numbers.*

Then there exists a real symmetric matrix A whose graph is T with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ and list of upward multiplicities $\hat{q} = (q_1, \ldots, q_r)$ if and only if $\hat{q}$ satisfies the following conditions:

1. $\sum_{i=1}^{r} q_i = n$;
2. *if q_i is an upward multiplicity in $\hat{q}$, then $1 < i < r$ and neither q_{i-1} nor q_{i+1} is an upward multiplicity in $\hat{q}$;*
3. $(q_{i_1}+1, \ldots, q_{i_h}+1)_e \preceq (l_1, \ldots, l_k)^*$, *where $q_{i_1} \geq \cdots \geq q_{i_h}$ are the upward multiplicities of $\hat{q}$.*

A more specific problem that arises in this context is to describe not just the possible multiplicity lists for a given graph, but which specific eigenvalues are realizable for each list, i.e., which numbers may be eigenvalues for a given multiplicity list. While there are many types of these inverse eigenvalue problems (see [3], e.g.), we will use the following definition:

Definition 6. Given a tree T on n vertices and real numbers $\lambda_1, \ldots, \lambda_n$, the *Inverse Eigenvalue Problem (IEP)* is to construct a real symmetric matrix whose graph is T with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Theorem 5 says that solving the IEP for generalized stars is equivalent to characterizing their ordered multiplicity lists, i.e., for any ordered multiplicity list of a generalized star, there is a real symmetric matrix with the given multiplicity list whose graph is the generalized star, subject to any choice of the distinct eigenvalues. For example, consider a path, which is a degenerate type of generalized star, so we can apply Theorem 5 to find its ordered multiplicity lists. Since there is no high degree vertex, we may choose any vertex to be the central vertex. For any choice of central vertex, there are at most two branches, so condition 3 guarantees that the only multiplicity list is the list of ones. In addition, Theorem 5 gives the classical result that for any n distinct real numbers, there is a real symmetric matrix whose eigenvalues are the given numbers and whose graph is a path on n vertices.

If T_1 and T_2 are generalized stars, then the graph consisting of T_1 and T_2 connected by an edge at their central vertices is called a *double generalized star*, which we denote by $D(T_1, T_2)$. The ordered multiplicity lists for double generalized stars were described in [9]. We refer to this theorem as the Original Superposition Principle.

Theorem 7 (*Original Superposition Principle*)**.** *Let $D(T_1, T_2)$ be a double generalized star, $\hat{b} = (b_1, \ldots, b_{s_1}) \in \hat{\mathcal{L}}_o(T_1)$, $\hat{c} = (c_1, \ldots, c_{s_2}) \in \hat{\mathcal{L}}_o(T_2)$. Construct any $b^+ = (b_1^+, \ldots, b_{s_1+t_1}^+)$, $c^+ = (c_1^+, \ldots, c_{s_2+t_2}^+)$ subject to the following conditions:*

0. $t_1, t_2 \in \mathbb{N}_0$ *and* $s_1 + t_1 = s_2 + t_2$
1. *b^+ (resp. c^+) is obtained from $\hat{b}$ (resp. $\hat{c}$) by inserting t_1 (resp. t_2) 0's;*
2. *b_i^+ and c_i^+, cannot both be 0;*
3. *if $b_i^+ > 0$ and $c_i^+ > 0$, then at least one of b_i^+ or c_i^+ must be an upward multiplicity of $\hat{b}$ or $\hat{c}$.*

Then we have $b^+ + c^+ \in \mathcal{L}_o(D(T_1, T_2))$. Moreover, $a \in \mathcal{L}_o(D(T_1, T_2))$ if and only if there are $\hat{b} \in \hat{\mathcal{L}}_o(T_1)$, $\hat{c} \in \hat{\mathcal{L}}_o(T_2)$ such that $a = b^+ + c^+$.

3. Linear trees

Each double generalized star has at most two high degree vertices, and these vertices are connected by an edge. A natural further generalization that we make is to allow for any number of high degree vertices and let them be connected by edges or paths of any length, as long as all of the high degree vertices lie on a single induced path of the tree.

Definition 8. A tree is called *k-linear* if the set of high degree vertices is a subset of k vertices that lie on a single induced path. Treating these k vertices as the central vertices of generalized stars, we may view any k-linear tree as a collection of k generalized stars, which we call the *components*, and the edges or paths that connect their central vertices. Let $L(T_1, s_1, T_2, s_2, \ldots, s_{k-1}, T_k)$ denote the k-linear tree consisting of the generalized stars $T_1, \ldots, T_k$, with T_i and T_{i+1} connected by a path of s_i vertices, not including the central vertices of T_i and T_{i+1}, $i = 1, \ldots, k-1$. Any tree that is k-*linear* for some k is called *linear*. A tree that is not linear is called *nonlinear*.

For example, a tree is 1-linear if and only if it is a generalized star. It is important to note that we do not require a k-linear tree to have exactly k high degree vertices, but only that there are k vertices lying on a single induced path, and among them is contained all of the high degree vertices. This means the components T_i are allowed to be single vertices. Also, an s_i may be 0.

Note also that while the classification of a tree as linear or nonlinear is unambiguous, a linear tree and its components might be classified in several ways. As a simple example, consider a path of four vertices. This can be called 4-linear, where

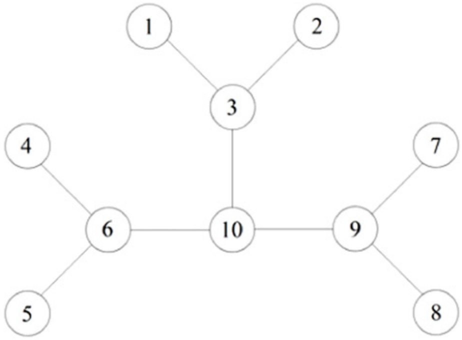

Fig. 1. A nonlinear tree on 10 vertices.

the components are each a single vertex and the connecting paths have length 0. It is also 2-linear, with a single vertex component and a star with a one vertex arm, and the components are connected by a path of length 1. There are even more classifications just for this simple example, and in general the choice of which classification to use can be made based on which is most convenient.

It should be emphasized that when classifying a linear tree, the components must be generalized stars, and they must be connected at their central vertices. The paths that connect these components are considered independent of the components and not as arms of the generalized stars.

When the number of high degree vertices is less than four, all of them must lie on a single induced path, so the set of linear trees contains all trees with fewer than four high degree vertices. However, nonlinear trees occur among trees with four or more high degree vertices. For example, Fig. 1 depicts a graph with four high degree vertices that is the smallest (fewest vertices) example of a nonlinear tree. This graph has many important characteristics and will be discussed further in Section 7.

For linear trees, we can further generalize the concept of superposition. To do so, it is helpful to consider the upward eigenvalues with upward multiplicity zero. These numbers appear as eigenvalues after removal of a vertex but are not eigenvalues of the original matrix. Now, fix a vertex v of a graph G. If we take a list of upward multiplicities for a real symmetric matrix A with graph G and augment it with upward zero multiplicities representing numbers that are eigenvalues of $A(v)$ but not A, then we have a *complete list of upward multiplicities* at v. Note that a complete list of upward multiplicities gives the ordered multiplicities of A and $A(v)$, so we will sometimes refer to the eigenvalues of A and $A(v)$ together.

We may now characterize the set of complete upward multiplicity lists of G, denoted $\hat{\mathcal{L}}_c(G)$, when G is a generalized star and v is its designated center. This slight extension of Theorem 5 will be most useful in defining our new (linear) superposition principle.

Theorem 9. *Let T be a generalized star on n vertices with central vertex v of degree k and arm lengths $l_1 \geq \cdots \geq l_k (\sum_{i=1}^{k} l_i = n-1)$. Let $\lambda_1 < \cdots < \lambda_r$ be any sequence of real numbers.*

Then there exists a real symmetric matrix A whose graph is T with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ of A and $A(v)$, and complete list of upward multiplicities $\hat{q} = (q_1, \ldots, q_r)$ if and only if $\hat{q}$ satisfies the following conditions:

1. $\sum_{i=1}^{r} q_i = n$;
2. $\sum_{j=1}^{\frac{r-1}{2}} (q_{2j} + 1) = n - 1$;
3. *q_i is an upward multiplicity in $\hat{q}$ if and only if i is even;*
4. $(q_{i_1} + 1, \ldots, q_{i_{\frac{r-1}{2}}} + 1) \preceq (l_1, \ldots, l_k)^*$, *where* $q_{i_1} \geq \cdots \geq q_{i_{\frac{r-1}{2}}}$ *are the upward multiplicities of $\hat{q}$.*

Proof. The proof closely follows the proof of Theorem 5 in [9]. In fact, the proof of the sufficiency of the stated conditions is similar to that for the proof of sufficiency for Theorem 5 in [9], so we omit it here. We now prove the necessity of the stated conditions. Conditions in (1) and (4) are also in Theorem 5, and their necessity is not affected by upward zeros. Condition (2) simply says that upon removal of the central vertex, the tree consisting of $n-1$ vertices will have $n-1$ eigenvalues. Condition (3) is an extension of Condition (2) in Theorem 5 with the observation from the interlacing theorem that between any two non-upward ones, there must be an upward multiplicity. □

The following proposes a characterization of the ordered multiplicity lists for linear trees using a generalization of the Original Superposition Principle. Of course, the Original Superposition Principle is a special case of this new superposition principle.

Definition 10. *(Linear Superposition Principle LSP)* Let $T_1, \ldots, T_k$ be generalized stars and $s_1, \ldots, s_{k-1}$ nonnegative integers. Given $\hat{b}_i$ a complete upward multiplicity list for T_i, $i = 1, \ldots, k$, and $\hat{c}_j$ a list of s_j non-upward ones, $j = 1, \ldots, k-1$, construct augmented lists b_i^+, $i = 1, \ldots, k$, and c_j^+, $j = 1, \ldots, k-1$, subject to the following conditions:

0. all b_i^+'s and c_j^+'s are the same length;
1. each b_i^+ and c_j^+ is obtained from its corresponding $\hat{b}_i$ and $\hat{c}_j$ by inserting non-upward 0's;
2. for each l, the lth element of the augmented lists, denoted $b_{i,l}^+$ and $c_{j,l}^+$, are not all non-upward zeros;
3. for each l, arranging the $b_{i,l}^+$'s and $c_{j,l}^+$'s in the order $b_{1,l}^+, c_{1,l}^+, b_{2,l}^+, c_{2,l}^+, \dots, b_{k-1,l}^+, c_{k-1,l}^+, b_{k,l}^+$, there is at least one upward multiplicity between any two non-upward ones.

Then $\sum_{i=1}^{k} b_i^+ + \sum_{j=1}^{k-1} c_j^+$, after removing zeros, is a multiplicity list for $L(T_1, s_1, T_2, s_2, \dots, s_{k-1}, T_k)$ generated by the *Linear Superposition Principle (LSP).*

For any linear tree, the LSP specifies conditions for multiplicity lists. We show that these conditions are necessary for an ordered multiplicity list of that tree (Theorem 14). Furthermore, we also show that the conditions are sufficient when $k = 2$ (Theorem 15) and for linear trees of depth 1 (Theorem 19).

4. Linear trees: necessity

In Sections 5 and 6, we show the sufficiency of the LSP conditions for two special classes of linear trees. This section will focus on the necessity of the LSP conditions. We will sometimes refer to a connected subgraph as *maximal*, by which we mean that no other vertex may be added to the subgraph while maintaining connectivity. The following facts about the relative position of eigenvalues will be useful.

Lemma 11. *Let T be a tree, and A a real symmetric matrix whose graph is T.*

1. *If λ_1 and λ_2, $\lambda_1 < \lambda_2$, are upward eigenvalues at some vertex v, then there is an eigenvalue λ of A such that $\lambda_1 < \lambda < \lambda_2$.*
2. *If λ is an upward eigenvalue at some vertex v, then it is not the smallest or largest eigenvalue of A.*

Proof. Both statements were shown in [11], except for the case of upward zero multiplicities. The proof of each statement follows identically when we allow for upward zeros.

1. If we remove v, the multiplicities of λ_1 and λ_2 increase, so by interlacing, there must be some eigenvalue between λ_1 and λ_2 whose multiplicity will decrease.
2. If we remove v, the multiplicity of λ must increase, so by interlacing, there must be eigenvalues greater than and less than λ whose multiplicity will decrease. □

Lemma 11 is used in proving the following two results which will be useful in proving the necessity of the LSP conditions.

Lemma 12. *Let $L = L(T_1, s_1, T_2, s_2, \dots, s_{k-1}, T_k)$ be a k-linear tree, and A a real symmetric matrix whose graph is L. If λ_1 and λ_2, $\lambda_1 < \lambda_2$, are upward eigenvalues for some T_i, there is some eigenvalue λ of L, $\lambda_1 < \lambda < \lambda_2$, that is not upward in T_i and is an eigenvalue of the maximal connected subtree containing T_i after all central vertices of the T_j's for which λ is upward are removed.*

Moreover, suppose we have an index set $\alpha \subseteq \{1, \dots, k\}$, and λ_{i1} and λ_{i2} are upward for T_i, $i \in \alpha$. If λ is the only eigenvalue guaranteed by the previous statement for each element of α, then for any two elements of α, there is an index j between them such that λ is upward for T_j.

Proof. We first suppose λ_1 and λ_2 are upward eigenvalues for some T_i. It will suffice to assume that there is no upward eigenvalue of T_i between λ_1 and λ_2. If the first statement is false, then only two situations may occur. First, there may be no eigenvalue of L between λ_1 and λ_2, but every upward eigenvalue of T_i is an upward eigenvalue of L, so this is not possible by Lemma 11(1). Second, there may be eigenvalues of L between λ_1 and λ_2, but each such eigenvalue is not an eigenvalue of the maximal connected subtree containing T_i after the central vertices of the T_j's for which the eigenvalue is upward are removed, i.e., the eigenvalues of L between λ_1 and λ_2 are upward, and their multiplicity in L comes from being an eigenvalue of subtrees not containing T_i. This is also not possible because removal of the central vertex of T_i must result in an increase in multiplicity of λ_1 and λ_2. Because of interlacing, this would result in a decrease in multiplicity of one of the eigenvalues in between, but as we have just seen, removal of the central vertex of T_i does not affect these multiplicities. The first statement is thus correct.

Now consider the second statement. It will again suffice to assume that there is no upward eigenvalue of T_i between λ_{i1} and λ_{i2}, for each $i \in \alpha$. If the second statement is false, then λ is the only eigenvalue guaranteed by the previous statement for each $i \in \alpha$, and there are two elements $\alpha_1, \alpha_2 \in \alpha$ such that λ is not upward for any T_j with j strictly between α_1 and α_2. However, if we remove the central vertices of T_{α_1} and T_{α_2}, then the multiplicity of λ must decrease by 2 because of its status as the only eigenvalue guaranteed by the previous statement, but just as in the previous discussion, this decrease in multiplicity has to come from λ being an eigenvalue of multiplicity at least 2 for the maximal connected subtree containing T_{α_1} and T_{α_2} after removal of the central vertices of the T_j's for which λ is upward. It follows that λ must be upward for some T_i in this maximal connected subtree, but we had already removed all central vertices of the T_i's at which λ is upward, so the second statement is correct. □

Lemma 13. *Let $L = L(T_1, s_1, T_2, s_2, \ldots, s_{k-1}, T_k)$ be a k-linear tree, and A a real symmetric matrix whose graph is L. If λ_1 is the smallest upward eigenvalue of some T_i, there is some eigenvalue λ of L, $\lambda < \lambda_1$, that is not upward in T_i and is an eigenvalue of the maximal connected subtree containing T_i after all central vertices of the T_j's for which λ is upward are removed.*

Moreover, suppose we have an index set $\alpha \subseteq \{1, \ldots, k\}$, and λ_i is the smallest upward eigenvalue for T_i, $i \in \alpha$. If λ is the only eigenvalue guaranteed by the previous statement for each element of α, then for any two elements of α, there is an index j between them such that λ is upward for T_j.

The corresponding statements for largest eigenvalues also hold.

Proof. The proof here uses the same ideas as in the proof of Lemma 12.

Suppose λ_1 is the smallest upward eigenvalue of some T_i. If the first statement is false, then two situations may occur. First, there may be no eigenvalue of L less than λ_1, but this is ruled out by Lemma 11(2). Second, there may be eigenvalues of L less than λ_1, but each such eigenvalue is not an eigenvalue of the maximal connected subtree containing T_i after the central vertices of the T_j's for which the eigenvalue is upward are removed. If we remove the central vertex of T_i, the multiplicity of λ_1 will increase in L, so by interlacing, the multiplicity of some eigenvalue or L less than λ_1 must decrease. However, by assumption, the multiplicity of the eigenvalues less than λ_1 are not affected by removal of this vertex because they are upward eigenvalues whose multiplicity in L is attributed to being an eigenvalue of subtrees not containing T_i. Therefore, the first statement is correct.

Now suppose λ_i is the smallest upward eigenvalue for T_i, $i \in \alpha$. If the second statement is false, then λ is the only eigenvalue guaranteed by the previous statement for all $i \in \alpha$, and there are two elements $\alpha_1, \alpha_2 \in \alpha$ such that λ is not upward for any T_j with j strictly between α_1 and α_2. If we now remove the central vertices of T_{α_1} and T_{α_2}, then the multiplicity of λ must decrease by 2 because of its status as the only eigenvalue guaranteed by the previous statement. However, this means λ has multiplicity at least 2 in the maximal connected subtree containing T_{α_1} and T_{α_2} after removal of the central vertices of the T_j's for which λ is upward. Then λ must be upward for some T_i in this maximal connected subtree, which contradicts our previous removal of all central vertices of the T_i's at which λ is upward, so the second statement is correct. □

Theorem 14. *For any k-linear tree $L = L(T_1, s_1, T_2, s_2, \ldots, s_{k-1}, T_k)$, $\mathcal{L}_o(L)$ is contained among those multiplicity lists generated by the LSP for L.*

Proof. Begin with an arbitrary ordered multiplicity list $a = (a_1, a_2, \ldots, a_u)$ of L. We need to show that a can be constructed from complete upward multiplicity lists of the T_i's and lists of ones, following the LSP conditions.

Because a is an ordered multiplicity list of L, there must be a matrix A, whose distinct eigenvalues we denote as $\lambda_1 < \lambda_2 < \cdots < \lambda_u$, which has graph L and ordered multiplicity list a. Our method of proof will be to use properties of this particular matrix A to select the required upward multiplicity lists of the T_i's and combine them with lists of ones according to the LSP conditions to construct a.

The LSP takes into account upward zero multiplicities, so we begin by augmenting the list of eigenvalues $\lambda_1, \ldots, \lambda_u$ with any numbers that have upward zero multiplicity in any of the T_i's, sorting and relabeling the list as $\lambda_1 < \lambda_2 < \cdots < \lambda_v$. The list a is also augmented with zeros where new eigenvalues were added, so that a_i is still the multiplicity of λ_i.

It is convenient to view the LSP construction in table form, so let us now construct a table as below, whose rows will correspond to $b_1^+, c_1^+, b_2^+, c_2^+, \ldots, b_{k-1}^+, c_{k-1}^+, b_k^+$, in that order, and whose columns will correspond to the distinct, ordered eigenvalues. Our goal is for column i to sum to a_i and each LSP condition to be met.

	λ_1	λ_2	$\cdots$	$\cdots$	λ_v
b_1^+					
c_1^+					
b_2^+					
c_2^+					
$\vdots$					
c_{k-1}^+					
b_k^+					
sum	a_1	a_2	$\cdots$	$\cdots$	a_v

In order to complete the proof, we must fill in this table while satisfying the following conditions:

1. (a) the row corresponding to b_i^+, ignoring non-upward 0's, is a complete upward multiplicity list for T_i;
   (b) the row corresponding to c_i^+ contains s_i non-upward 1's and $v - s_i$ non-upward 0's;
2. no column is made up entirely of non-upward 0's;
3. the column corresponding to λ_i sums to a_i;
4. in each column, any two non-upward ones are separated by at least one upward multiplicity.

The conditions above are labeled to emphasize how completing the table while satisfying the conditions is sufficient to complete the proof. To see this, note that conditions 1–3 above correspond exactly to the LSP conditions 1–3 in Definition 10. In addition, LSP condition 0 is automatically satisfied since each row of the table is the same length, and condition 4 above ensures that the multiplicity list a is indeed constructed.

We start filling in the table by completing the upward multiplicities in the b_i^+ rows: for each T_i and its corresponding row b_i^+, we complete the entries for all eigenvalues with upward multiplicity in T_i, filling these entries with the multiplicities marked as upward. To complete these rows while satisfying condition 1a, we need to place non-upward 1's alternating with the upward entries, and fill the remaining entries with non-upward 0's. It is not obvious that this is possible (upward entries may be directly adjacent or at the ends, thus preventing the insertion of non-upward 1's in between), but the first statements of Lemmas 12 and 13 below guarantee that it can be done.

In fact, Lemmas 12 and 13 provide a method of placing the non-upward 1's in the b_i^+ rows that ensures the successful completion of the rest of the construction: in each row b_i^+, the non-upward 1's should be entered into the entries for eigenvalues of L with the property of also being an eigenvalue of the maximal connected subtree containing T_i after the removal of all central vertices of the T_j's for which the eigenvalue is upward. It follows from the second statements of Lemmas 12 and 13 that doing this prevents condition 4 from being violated.

At this point, the b_i^+ rows are complete and satisfy condition 1a, without violating condition 4. Note that each column j currently sums to at most a_j because Theorem 2 would otherwise be violated, so condition 3 is also not violated. We will complete the table by filling in the columns with non-upward 1's and non-upward 0's in entries corresponding to the c_i's. We proceed by completing the columns that currently have an upward multiplicity in some entry. For each such column j and its corresponding eigenvalue λ_j, we remove the T_i's for which λ_j is upward and observe which of the remaining maximal connected subgraphs has λ_j as an eigenvalue. It follows from repeated use of Theorem 2 that the number of these maximal connected subgraphs having λ_j as an eigenvalue is equal to the number of non-upward 1's in the column, so we can complete these columns while satisfying conditions 3 and 4.

What remains is to finish the columns that have no upward multiplicities. These columns must sum to 1 and therefore consist of a single non-upward 1 and a number of non-upward 0's. The only concern is that we must simultaneously satisfy condition 1b, and it may be that in completing the previous columns, we placed more than s_i non-upward 1's in the row for c_i^+. However, note that every entry made so far has been the result of an eigenvalue existing for some subgraph, and because each graph has as many eigenvalues as vertices, no row c_i^+ too many non-upward 1's. We can therefore finish the columns and satisfy conditions 1b and 3. The final condition is condition 2, and this is satisfied by condition 3 and the fact that any column summing to zero must contain an upward 0 multiplicity because of how we began the table (i.e. every column corresponds to an eigenvalue of A or a number with upward zero multiplicity in any of the T_i's). □

5. Linear trees: sufficiency for $k = 2$

We now prove sufficiency of the LSP for generating all multiplicity lists when $k = 2$. When $k = 2$, the LSP has a simpler form without upward zero multiplicities that we will use here. Note the similarity of this form to the Original Superposition Principle.

Theorem 15. *Let $L = L(T_1, s, T_2)$ be a 2-linear tree. Given $\hat{b} = (b_1, \ldots, b_{s_1}) \in \hat{\mathcal{L}}_o(T_1)$, $\hat{c} = (c_1, \ldots, c_{s_2}) \in \hat{\mathcal{L}}_o(T_2)$, and $\hat{d} = (1, \ldots, 1)$ (list of s 1's), construct any $b^+ = (b_1^+, \ldots, b_{s_1+t_1}^+)$, $c^+ = (c_1^+, \ldots, c_{s_2+t_2}^+)$, and $d^+ = (d_1^+, \ldots, d_{s+t_3}^+)$ subject to the following conditions:*

0. *$t_1, t_2, t_3 \in \mathbb{N}_0$ and $s_1 + t_1 = s_2 + t_2 = s + t_3$,*
1. *$b^+ (c^+, d^+)$ is obtained from $\hat{b}(\hat{c}, \hat{d})$ by inserting $t_1 (t_2, t_3)$ 0's;*
2. *b_i^+, c_i^+, and d_i^+ cannot all be 0;*
3. *at most one of b_i^+, c_i^+, and d_i^+ can be nonzero and not an upward multiplicity of $\hat{b}$, $\hat{c}$, or $\hat{d}$.*

Then we have $b^+ + c^+ + d^+ \in \mathcal{L}_o(L)$.

Proof. Our goal is to show that any superposition of upward multiplicity lists constructed in accordance with the constraints set in Theorem 15 is an ordered multiplicity list of some real symmetric matrix A with graph L. In order to show this, we will consider each possibility for the elements of a given column (b_i^+, c_i^+, d_i^+) and show that their sum $b_i^+ + c_i^+ + d_i^+ = a_i$ is realizable in $\mathcal{L}_o(L(T_1, s, T_2))$. This will show that any column sum constructed by Theorem 15 is part of an ordered multiplicity list of L. Then, we will argue that it is possible to add all columns simultaneously in order to achieve a set of column sums that are an ordered multiplicity list of L. There are five possible cases for the columns generated according to the theorem:

1. b_i^+ and c_i^+ are upward and $d_i^+ = 1$;
2. b_i^+ and c_i^+ are upward and $d_i^+ = 0$;
3. one of b_i^+ and c_i^+ is upward and the column has exactly one non-upward one;

4. one of b_i^+ and c_i^+ is upward and the rest of the column entries are 0;
5. one of b_i^+, c_i^+, and d_i^+ is a non-upward one and the rest are 0.

For the discussion of these cases, let v_1 (v_2) denote the central vertex of the generalized star T_1 (T_2). Select $s_1 + t_1$ distinct real numbers $\lambda_1 < \cdots < \lambda_{s_1+t_1}$. We are able to ensure that the eigenvalues for columns of cases (1)–(4) match their corresponding λ_i.

In case (1), let λ_i be an eigenvalue of the connecting path (with multiplicity 1) and of $b_i^+ + 1$ branches of T_1 and $c_i^+ + 1$ branches of T_2. Then by Theorem 2,

$$m_A(\lambda_i) = b_i^+ + m_{A(T_1)}(\lambda_i) = b_i^+ + c_i^+ + m_{A(T_1,T_2)}(\lambda_i) = b_i^+ + c_i^+ + 1 = a_i.$$

In case (2), let λ_i be an eigenvalue of $b_i^+ + 1$ branches of T_1 and $c_i^+ + 1$ branches of T_2. By Theorem 2,

$$m_A(\lambda_i) = b_i^+ + m_{A(T_1)}(\lambda_i) = b_i^+ + c_i^+ + m_{A(T_1,T_2)}(\lambda_i) = b_i^+ + c_i^+ = a_i.$$

In case (3), given that b_i^+ is upward, let λ_i be an eigenvalue of $L - T_1$ with multiplicity 1 and of $b_i^+ + 1$ branches of T_1. By Theorem 2,

$$m_A(\lambda_i) = b_i^+ + m_{A(T_1)}(\lambda_i) = b_i^+ + 1 = a_i.$$

When c_i^+ is upward, the equivalent procedure is followed with T_2.

In case (4), given that b_i^+ is upward, let λ_i be an eigenvalue of $b_i^+ + 1$ branches of T_1. By Theorem 2,

$$m_A(\lambda_i) = b_i^+ + m_{A(T_1)}(\lambda_i) = b_i^+ = a_i.$$

When c_i^+ is upward, the equivalent procedure is followed with T_2.

In case (5), no action is needed, as L will have some eigenvalue with multiplicity $1 = a_i$.

We have now shown that each column is individually realizable; however, it remains to be shown that all of the columns can be realized simultaneously. Our procedure makes eigenvalue assignments to individual branches, the connecting path, and the connecting path combined with T_1 or T_2. Since Theorem 5 says that solving the IEP is equivalent to characterizing the ordered multiplicity lists for generalized stars, and each of the subtrees we are making assignments to is a generalized star, the only potential problem with realizing all columns simultaneously is that we might overload a subtree, i.e., we might make more assignments to a subtree than the number of vertices in the subtree. However, this problem does not occur, and therefore $b^+ + c^+ + d^+$ is an ordered multiplicity list for L. In fact, the branches are assigned only when an eigenvalue is upward, and since these upward multiplicities come from actual upward multiplicity lists for T_1 and T_2, there is no overloading. The path receives assignments for each column of case (1), but there are at most s of these columns, so the path is not overloaded. Finally, the connecting path combined with T_1 or T_2 receives assignments from columns of case (3), but for each column of case (3), there must be a vertex unaccounted for, so overloading still does not occur. □

6. Linear trees: sufficiency for depth 1

We now consider linear trees of depth 1. A linear tree is depth 1 if all vertices lie on or are connected by an edge to some induced path containing all high degree vertices (trees of this class were studied in [2], where they were called *centipedes*). Note that this is equivalent to limiting the T_i's to stars, i.e., generalized stars with all arms of length one. We will use the implicit function theorem method described in [13] to prove the sufficiency of the LSP conditions for this class of linear trees. We give a brief description of the method here and refer the reader to [13] for a more thorough exposition.

Our goal is to construct a matrix with a given graph and eigenvalues. The implicit function theorem allows us to take an initial matrix, whose graph is a subgraph of our target graph and that satisfies a set of determinant conditions, and change some of the entries of the matrix, called *manual entries*, so that the new matrix has the target graph while maintaining the same determinant conditions. With certain choices of determinant conditions, we can guarantee that a matrix has the exact eigenvalues we would like. However, the cost of having more conditions is that more *implicit entries* must be chosen. These implicit entries cannot be off-diagonal zeros, which limits our choice of initial matrices and makes the Jacobian more complicated. The following lemma from [13] will be instrumental in the difficult task of showing the nonsingularity of the Jacobian:

Lemma 16. *Let T be a tree on n vertices and $F = (f_k)$, $f_k(A) = \det(A[S_k] - \lambda_k I)$, a vector of r determinant conditions, with $S_k \subseteq \{1, \ldots, n\}$ an index subset and λ_k a real number, $k = 1, \ldots, r$. Assume r implicit entries have been identified. Suppose that a real symmetric matrix $A^{(0)}$, whose graph is a subgraph of T, is the direct sum of irreducible matrices $A_1^{(0)}, \ldots, A_p^{(0)}$. Let $J(A^{(0)})$ be the Jacobian matrix of F with respect to the implicit entries evaluated at $A^{(0)}$, and suppose*

1. *every off-diagonal implicit entry in $A^{(0)}$ has a nonzero value;*

2. *for each* $k = 1, \ldots, r, f_k(A_l^{(0)}) = 0$ *for exactly one* $l \in \{1, \ldots, p\}$*;*
3. *for each* $l = 1, \ldots, p$*, the columns of* $J(A^{(0)})$ *associated with the implicit entries of* $A_l^{(0)}$ *are linearly independent.*

Then $J(A^{(0)})$ *is nonsingular.*

The previous lemma suggests that choosing an initial matrix that is highly reducible will make it easier to show that the Jacobian is nonsingular. For our proof for depth 1 linear trees, we will choose an initial matrix that is reducible to the star components and each single vertex of the connecting paths. The problem will still exist of showing the nonsingularity of the Jacobian in terms of each star component, but the next lemma is helpful for this.

Lemma 17. *Consider an n-by-n matrix* ($n > 2$) *of the following form:*

$$A = \begin{bmatrix} \prod_{i=1}^{n-1}(a_i - x_1) & \prod_{\substack{i=1\\ i\neq 1}}^{n-1}(a_i - x_1) & \cdots & \prod_{\substack{i=1\\ i\neq n-1}}^{n-1}(a_i - x_1) \\ \vdots & \vdots & & \vdots \\ \prod_{i=1}^{n-1}(a_i - x_n) & \prod_{\substack{i=1\\ i\neq 1}}^{n-1}(a_i - x_n) & \cdots & \prod_{\substack{i=1\\ i\neq n-1}}^{n-1}(a_i - x_n) \end{bmatrix}.$$

The determinant of this matrix is the following:

$$\det A = \left(\prod_{\substack{i,j=1\\ i<j}}^{n-1} (a_i - a_j) \right) \left(\prod_{\substack{i,j=1\\ i>j}}^{n} (x_i - x_j) \right).$$

In particular, A is nonsingular if and only if $x_1, \ldots, x_n$ *are distinct and* $a_1, \ldots, a_{n-1}$ *are distinct.*

Proof. We begin by subtracting column 2 from columns 3 to n, which does not change the determinant. The matrix now has the form

$$\begin{bmatrix} \prod_{i=1}^{n-1}(a_i - x_1) & \prod_{\substack{i=1\\ i\neq 1}}^{n-1}(a_i - x_1) & (a_1 - a_2)\prod_{\substack{i=1\\ i\neq 1,2}}^{n-1}(a_i - x_1) & \cdots & (a_1 - a_{n-1})\prod_{\substack{i=1\\ i\neq 1,n-1}}^{n-1}(a_i - x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ \prod_{i=1}^{n-1}(a_i - x_n) & \prod_{\substack{i=1\\ i\neq 1}}^{n-1}(a_i - x_n) & (a_1 - a_2)\prod_{\substack{i=1\\ i\neq 1,2}}^{n-1}(a_i - x_n) & \cdots & (a_1 - a_{n-1})\prod_{\substack{i=1\\ i\neq 1,n-1}}^{n-1}(a_i - x_n) \end{bmatrix}.$$

We repeat this process with each column after column 2. The resulting matrix, which still has the same determinant as A, is

$$\begin{bmatrix} \prod_{i=1}^{n-1}(a_i - x_1) & \prod_{i=2}^{n-1}(a_i - x_1) & c_2\prod_{i=3}^{n-1}(a_i - x_1) & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \prod_{i=1}^{n-1}(a_i - x_n) & \prod_{i=2}^{n-1}(a_i - x_n) & c_2\prod_{i=3}^{n-1}(a_i - x_n) & \cdots & c_{n-1} \end{bmatrix},$$

where $c_j = \prod_{i=1}^{j-1}(a_i - a_j), j = 2, \ldots, n-1$.

If $c_j = 0$ for any j, then $a_i = a_j$ for some $i < j$ and one of the columns is zero, so the matrix has zero determinant and the theorem is correct. Otherwise, we perform a final set of column manipulations. For convenience, we rewrite the matrix as

$$\begin{bmatrix} (-x_1)^{n-1} + f_1(x_1) & (-x_1)^{n-2} + f_2(x_1) & c_2((-x_1)^{n-3} + f_3(x_1)) & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ (-x_n)^{n-1} + f_1(x_n) & (-x_n)^{n-2} + f_2(x_n) & c_2((-x_n)^{n-3} + f_3(x_n)) & \cdots & c_{n-1} \end{bmatrix},$$

where f_i is a polynomial of degree at most $n-(i+1)$, $i=1,\ldots,n-1$. We may now remove the f_i's without changing the determinant: we start by using column n to remove the f_{n-1}'s in column $n-1$, and proceed similarly to the left, each time using the columns to the right. This operation leaves the determinant unchanged, and we are left with

$$\begin{bmatrix} (-x_1)^{n-1} & (-x_1)^{n-2} & c_2(-x_1)^{n-3} & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ (-x_n)^{n-1} & (-x_n)^{n-2} & c_2(-x_n)^{n-3} & \cdots & c_{n-1} \end{bmatrix}.$$

The determinant of this matrix is equal to the determinant of the Vandermonde matrix with nodes $-x_1,\ldots,-x_n$ multiplied by the c_j's, i.e.,

$$\begin{aligned} \det A &= \left(\prod_{j=2}^{n-1} c_j\right)\left(\prod_{\substack{i,j=1\\ i<j}}^{n} ((-x_i)-(-x_j))\right) \\ &= \left(\prod_{j=2}^{n-1}\prod_{i=1}^{j-1}(a_i-a_j)\right)\left(\prod_{\substack{i,j=1\\ i<j}}^{n}(x_j-x_i)\right) \\ &= \left(\prod_{\substack{i,j=1\\ i<j}}^{n-1}(a_i-a_j)\right)\left(\prod_{\substack{i,j=1\\ i>j}}^{n}(x_i-x_j)\right). \quad \square \end{aligned}$$

We can now show that the Jacobian at each star component is nonsingular.

Lemma 18. *Let T be a star on n vertices, and suppose we have a matrix-valued function $A(a_1,a_2,\ldots,a_n)$, defined on n real variables, whose range is the set of real symmetric matrices with graph T (or some subgraph of T), complete upward multiplicity list $\hat{b}=(b_1,\ldots,b_u)$, and distinct upward eigenvalues $\mu_2,\mu_4,\ldots,\mu_{u-1}$, and a matrix $A^{(0)}$ in the range of A with graph T and non-upward eigenvalues $\lambda_1,\lambda_3,\ldots,\lambda_u$. We may select $\frac{u+1}{2}$ variables of A so that the Jacobian of the function $F(A)=(\det(A-\lambda_1 I),\det(A-\lambda_3 I),\ldots,\det(A-\lambda_u I))$ is nonsingular at $A^{(0)}$.*

Proof. Without loss of generality, assume the function A takes the following form:

$$A=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & & & \\ \vdots & & D & \\ a_n & & & \end{bmatrix},$$

where $D=\operatorname{diag}(\mu_2,\ldots,\mu_2,\mu_4,\ldots,\mu_4,\ldots,\mu_{u-1},\ldots,\mu_{u-1})$, with each μ_j appearing $\hat{b}_j$ times.

Let $i_k=1+\sum_{j=1}^{k-1}\hat{b}_{2j}$, $k=1,\ldots,\frac{u+1}{2}$. Then we construct the Jacobian with respect to the variables $a_{i_1},\ldots,a_{i_{\frac{u+1}{2}}}$. Note that we have selected a_1 and exactly one a_i appearing in the rows and columns of each μ_j. Letting J denote the Jacobian, we have

$$J=\begin{bmatrix} \prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_1) & \frac{-2a_1}{\mu_2-\lambda_1}\prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_1) & \cdots & \frac{-2a_1}{\mu_{u-1}-\lambda_1}\prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_1) \\ \prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_3) & \frac{-2a_1}{\mu_2-\lambda_3}\prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_3) & \cdots & \frac{-2a_1}{\mu_{u-1}-\lambda_3}\prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_3) \\ \vdots & \vdots & & \vdots \\ \prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_u) & \frac{-2a_1}{\mu_2-\lambda_u}\prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_u) & \cdots & \frac{-2a_1}{\mu_{u-1}-\lambda_u}\prod_{j=1}^{\frac{u-1}{2}}\prod_{k=1}^{\hat{b}_{2j}}(\mu_{2j}-\lambda_u) \end{bmatrix}.$$

We would like to show that J is nonsingular at $A^{(0)}$. Because $A^{(0)}$ has graph T, each $a_i^{(0)}$ is nonzero, so we can simplify this matrix by multiplying columns by nonzero constants, which produces the following matrix:

$$J' = \begin{bmatrix} \prod_{j=1}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_1) & \prod_{\substack{j=1\\ j\neq 1}}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_1) & \cdots & \prod_{\substack{j=1\\ j\neq \frac{u-1}{2}}}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_1) \\ \prod_{j=1}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_3) & \prod_{\substack{j=1\\ j\neq 1}}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_3) & \cdots & \prod_{\substack{j=1\\ j\neq \frac{u-1}{2}}}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_3) \\ \vdots & \vdots & & \vdots \\ \prod_{j=1}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_u) & \prod_{\substack{j=1\\ j\neq 1}}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_u) & \cdots & \prod_{\substack{j=1\\ j\neq \frac{u-1}{2}}}^{\frac{u-1}{2}}(\mu_{2j}-\lambda_u) \end{bmatrix}.$$

Sincethe μ_j's and λ_i's are distinct, J' is nonsingular by Lemma 17, so J is also nonsingular at $A^{(0)}$. □

Theorem 19. *For any depth 1 k-linear tree $L = L(T_1, s_1, T_2, s_2, \ldots, s_{k-1}, T_k)$, $\mathcal{L}_o(L)$ contains the set of all multiplicity lists generated by the LSP for L.*

Proof. Assume $T_1, \ldots, T_k$ are stars. Our goal is to show that any combination of complete upward multiplicity lists and lists of ones following the stated conditions is an ordered multiplicity list for L.

Our initial matrix $A^{(0)}$ will be the direct sum of k matrices whose graphs are the T_i's and $\sum_{j=1}^{k-1} s_j$ 1-by-1 matrices that will correspond to the connecting vertices. Let u be the length of the augmented lists, and select any real numbers $\lambda_1 < \cdots < \lambda_u$, which will be the eigenvalues of our matrix. For each $i = 1, \ldots, k$, select a matrix with complete upward multiplicity list $\hat{b}_i$ and eigenvalues corresponding to the λ_l's based on how $\hat{b}_i$ is spaced in b_i^+; these matrices will be the components of $A^{(0)}$ corresponding to the T_i's. Similarly, for each $j = 1, \ldots, k-1$, the s_j 1-by-1 matrices corresponding to the connecting path will be the λ_l's such that the lth element of c_j^+ is one.

For each $l = 1, \ldots, u$, we have a set of determinant conditions to ensure the correct multiplicity in the final matrix. For each b_i^+ in which λ_l is upward, we place λ_l on $b_i^+ + 1$ of the branches of T_i, and for each non-upward one, we have a determinant condition on the subtree containing that vertex after the upward vertices are removed. We need as many implicit entries as we have determinant conditions, i.e., the number of non-upward ones. The 1-by-1 matrix components will all be implicit entries, and the implicit entries in the T_i components will be chosen as in Lemma 18. Note that the following properties now hold:

1. all off-diagonal implicit entries are nonzero;
2. each determinant condition is satisfied by exactly one component of $A^{(0)}$;
3. for each component of $A^{(0)}$, the columns of the Jacobian at $A^{(0)}$ associated with its implicit entries are linearly independent.

By Lemma 16, the Jacobian at $A^{(0)}$ is nonsingular. The implicit function theorem now shows that there is some real symmetric matrix with graph L that satisfies all of the determinant and branch conditions, so this guaranteed matrix has the given ordered multiplicity list and our chosen eigenvalues. □

For depth 1 linear trees, we have actually solved the IEP while proving sufficiency because the implicit function theorem method allowed us to choose any real numbers to be the eigenvalues.

Corollary 20. *Characterizing the ordered multiplicity lists for depth 1 linear trees is equivalent to the IEP for such trees.*

It is important to note that the implicit function theorem framework could be used to complete the proof of sufficiency when generalized stars are allowed. This extension would require proving the more general version of Lemma 18 for generalized stars where the Jacobian has a more complicated form.

7. Applications

We present some examples demonstrating explicitly how to apply the LSP. We then apply the LSP to proving some previously conjectured statements about multiplicity lists for (general) trees in the case of linear trees and conclude with a discussion of nonlinear trees.

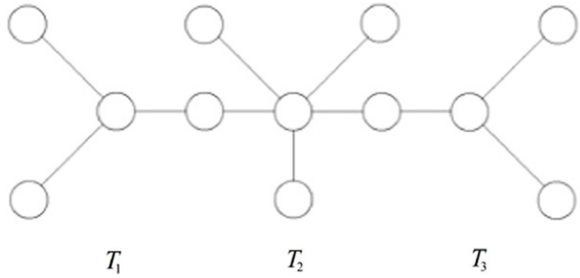

Fig. 2. A 3-linear tree on 12 vertices.

7.1. Examples

Example 21. Consider the following depth 1 3-linear tree (see Fig. 2) and the complete upward multiplicity lists of each component:

$$\hat{\mathcal{L}}_c(T_1) = \{(1, \hat{1}, 1), (1, \hat{0}, 1, \hat{0}, 1)\}$$
$$\hat{\mathcal{L}}_c(T_2) = \{(1, \hat{2}, 1), (1, \hat{1}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{1}, 1), (1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1)\}$$
$$\hat{\mathcal{L}}_c(T_3) = \{(1, \hat{1}, 1), (1, \hat{0}, 1, \hat{0}, 1)\}.$$

It is important to note that the vertices connecting T_1 to T_2 and T_2 to T_3 can be viewed as generalized stars, and therefore the entire tree treated as a 5-linear tree. However, it is more convenient to treat them as connecting paths and the whole tree as 3-linear. While the LSP produces all of the ordered multiplicity lists, we will for the sake of space give only the complete set of unordered lists for this linear tree. In listing the unordered multiplicity lists, we will use an abbreviated notation that omits ones. Since the sum of the multiplicities in a list must equal the total number of vertices in the tree, this shorter notation is unambiguous. For example, for this tree, 52 refers to the unordered list (5, 2, 1, 1, 1, 1, 1). Note that the list of all ones is therefore omitted entirely.

$$\mathcal{L}_u(L(T_1, s_1, T_2, s_2, T_3)) = \{6, 52, 5, 43, 422, 42, 4, 332, 33, 322, 32, 3, 22222, 2222, 222, 22, 2\}.$$

To demonstrate actual constructions of ordered multiplicity lists using the LSP, we provide examples of possible constructions for some of the more interesting cases. In each of the tables below, there are five rows and their superposition. Rows 1, 3, and 5 are augmented complete upward multiplicity lists of T_1, T_2, and T_3, respectively, and rows 2 and 4 are augmented lists of a single one, representing the one vertex paths connecting T_1 to T_2 and T_2 to T_3. We will use this construction notation for all of the remaining examples and applications.

1	0	0	$\hat{1}$	0	0	1
0	0	0	1	0	0	0
0	1	0	$\hat{2}$	0	1	0
0	0	0	1	0	0	0
0	0	1	$\hat{1}$	1	0	0
1	1	1	6	1	1	1

0	1	0	$\hat{1}$	0	1	0
0	0	0	1	0	0	0
0	0	1	$\hat{1}$	1	$\hat{0}$	1
0	0	0	1	0	0	0
1	0	0	$\hat{1}$	0	1	0
1	1	1	5	1	2	1

0	1	0	$\hat{1}$	0	1	0
0	0	0	1	0	0	0
0	0	1	$\hat{0}$	1	$\hat{1}$	1
0	0	0	1	0	0	0
1	0	0	$\hat{1}$	0	1	0
1	1	1	4	1	3	1

0	1	0	$\hat{1}$	0	1	0
0	0	0	1	0	0	0
1	$\hat{0}$	1	$\hat{0}$	1	$\hat{0}$	1
0	0	0	1	0	0	0
0	1	0	$\hat{1}$	0	1	0
1	2	1	4	1	2	1

0	0	1	$\hat{1}$	0	0	1
0	0	0	0	1	0	0
0	1	$\hat{0}$	1	$\hat{1}$	1	0
0	0	1	0	0	0	0
1	0	0	$\hat{1}$	1	0	0
1	1	2	3	3	1	1

0	1	0	$\hat{1}$	0	1	0	0
0	0	0	1	0	0	0	0
0	0	1	$\hat{0}$	1	$\hat{1}$	1	0
0	0	0	1	0	0	0	0
1	0	0	0	0	$\hat{1}$	0	1
1	1	1	3	1	3	1	1

It is possible for different upward multiplicity lists of a given generalized star in a linear tree to contribute to the same multiplicity list. For example, using the list $(1, \hat{2}, 1)$ from $\hat{\mathcal{L}}_c(T_2)$ instead of $(1, \hat{0}, 1, \hat{1}, 1)$ can still lead to the ordered list, (1, 1, 1, 3, 1, 3, 1, 1) as shown below.

0	1	0	$\hat{1}$	0	1	0	0
0	0	1	0	0	0	0	0
0	0	0	1	0	$\hat{2}$	1	0
0	0	0	0	1	0	0	0
1	0	0	$\hat{1}$	0	0	0	1
1	1	1	3	1	3	1	1

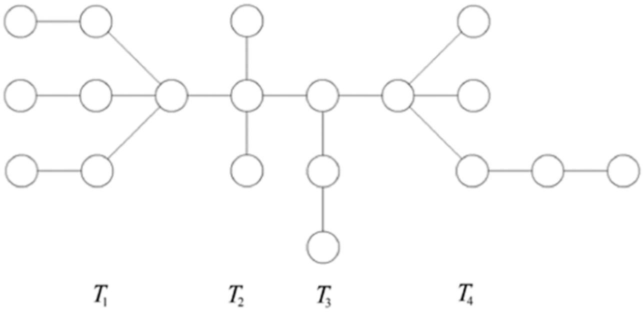

Fig. 3. A 4-linear tree on 19 vertices.

Example 22. Here we present a more complicated example. Consider the 4-linear tree below and the complete upward multiplicity lists of its 4 components (see Fig. 3):

$$\begin{aligned}
\hat{\mathcal{L}}_c(T_1) = \{&(1, \hat{2}, 1, \hat{2}, 1), (1, \hat{2}, 1, \hat{1}, 1, \hat{0}, 1), (1, \hat{2}, 1, \hat{0}, 1, \hat{1}, 1), (1, \hat{1}, 1, \hat{2}, 1, \hat{0}, 1), (1, \hat{1}, 1, \hat{0}, 1, \hat{2}, 1)\\
&(1, \hat{0}, 1, \hat{2}, 1, \hat{1}, 1), (1, \hat{0}, 1, \hat{1}, 1, \hat{2}, 1), (1, \hat{2}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{2}, 1, \hat{0}, 1, \hat{0}, 1),\\
&(1, \hat{0}, 1, \hat{0}, 1, \hat{2}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{2}, 1), (1, \hat{1}, 1, \hat{1}, 1, \hat{1}, 1), (1, \hat{1}, 1, \hat{0}, 1, \hat{0}, 1, \hat{1}, 1),\\
&(1, \hat{1}, 1, \hat{0}, 1, \hat{1}, 1, \hat{0}, 1), (1, \hat{1}, 1, \hat{1}, 1, \hat{0}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{1}, 1, \hat{1}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{1}, 1, \hat{0}, 1, \hat{1}, 1),\\
&(1, \hat{0}, 1, \hat{0}, 1, \hat{1}, 1, \hat{1}, 1), (1, \hat{1}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{1}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1),\\
&(1, \hat{0}, 1, \hat{0}, 1, \hat{1}, 1, \hat{0}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{1}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{1}, 1),\\
&(1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1)\}\\
\hat{\mathcal{L}}_c(T_2) = \{&(1, \hat{1}, 1), (1, \hat{0}, 1, \hat{0}, 1)\}\\
\hat{\mathcal{L}}_c(T_3) = \{&(1, \hat{0}, 1, \hat{0}, 1)\}\\
\hat{\mathcal{L}}_c(T_4) = \{&(1, \hat{2}, 1, \hat{0}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{2}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{0}, 1, \hat{2}, 1), (1, \hat{1}, 1, \hat{1}, 1, \hat{0}, 1), (1, \hat{1}, 1, \hat{0}, 1, \hat{1}, 1),\\
&(1, \hat{0}, 1, \hat{1}, 1, \hat{1}, 1), (1, \hat{1}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{1}, 1, \hat{0}, 1, \hat{0}, 1), (1, \hat{0}, 1, \hat{0}, 1, \hat{1}, 1, \hat{0}, 1),\\
&(1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{1}, 1), (1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1, \hat{0}, 1)\}.
\end{aligned}$$

There is a subtle point about generalized stars with no high degree vertex that is highlighted by T_2 and T_3. While these two components are both paths of length 3, we see that their complete upward multiplicity lists are different. This is because they are connected to the linear tree at different vertices, so they have different central vertices.

We complete this example by showing how the LSP works with the structural properties of trees discussed in Theorems 3 and 4. Recall that the path cover number of a tree is equal to the maximum multiplicity over all multiplicity lists for the tree (Theorem 3), and the diameter is a lower bound on the minimum number of distinct eigenvalues (Theorem 4). The path cover number of this linear tree is 6, and the following construction shows one way to achieve this:

0	0	1	$\hat{2}$	1	$\hat{2}$	0	1	0	0
0	1	0	$\hat{1}$	0	1	0	0	0	0
0	0	0	1	0	$\hat{0}$	1	$\hat{0}$	1	0
1	0	0	$\hat{2}$	0	1	$\hat{0}$	1	$\hat{0}$	1
1	1	1	6	1	4	1	2	1	1

The diameter of this linear tree is 9, so all multiplicity lists must have at least 9 elements. The following multiplicity list has 9 elements, including only two 1's:

0	1	$\hat{2}$	1	0	$\hat{2}$	1	0	0
0	0	1	$\hat{1}$	1	0	0	0	0
0	0	0	1	$\hat{0}$	1	$\hat{0}$	1	0
1	$\hat{1}$	0	0	1	$\hat{0}$	1	$\hat{1}$	1
1	2	3	3	2	3	2	2	1

Of course, the sufficiency of the LSP conditions has not been proven for this tree.

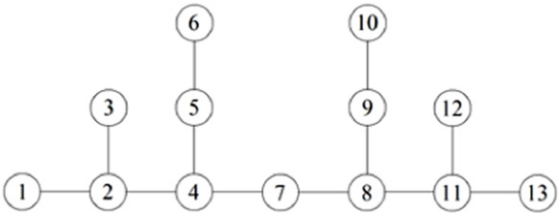

Fig. 4. A linear tree discussed in [5].

Example 23. Our final example is the 13 vertex linear tree discussed in [5] and reproduced below with the same vertex numbering (see Fig. 4).

Informally, the assignment process for a given multiplicity list and given tree involves identifying subtrees that will have certain eigenvalues so that the whole tree will have the given multiplicity list (see [5] for a more formal discussion of assignments). A valid assignment is a necessary condition for a multiplicity list to exist, but the linear tree above was used in [5] to demonstrate that a valid assignment is not a sufficient condition. In fact, for the unordered multiplicity list (3, 3, 3, 1, 1, 1, 1), if we denote the three eigenvalues of multiplicity 3 as α, β, and γ, then the only valid assignment places these eigenvalues in the subtrees containing the following sets of vertices:

$$\alpha : \{1\}, \{3\}, \{4, 5, 6, 7\}, \{9, 10\}, \{11, 12, 13\}$$
$$\beta : \{1, 2, 3\}, \{5, 6\}, \{7, 8, 9, 10\}, \{12\}, \{13\}$$
$$\gamma : \{1, 2, 3\}, \{5, 6\}, \{7\}, \{9, 10\}, \{11, 12, 13\}.$$

Unfortunately, this assignment does not lead to a multiplicity list. The subtree containing the vertices $\{1, 2, 3\}$ has α, β, and γ as eigenvalues, and because α is an upward eigenvalue, α is not the smallest or largest eigenvalue. Similarly, the subtree containing the vertices $\{11, 12, 13\}$ has α, β, and γ as eigenvalues, but since β is an upward eigenvalue, β is not the smallest or largest eigenvalue. Since these conditions cannot happen simultaneously, and this was the only valid assignment for the multiplicity list (3, 3, 3, 1, 1, 1, 1), we conclude that (3, 3, 3, 1, 1, 1, 1) is not a multiplicity list.

The LSP is similar to the assignment process, as can be seen from the proof of Theorem 15, so we present this example to illustrate how the LSP resolves these situations. Similar to the procedure used in proof of Theorem 14, we will work backwards from the given assignment and try to create the invalid multiplicity list by superposition. We will treat the tree as a 4-linear tree, where vertices 2, 4, 8, and 11 are the central vertices of the components T_1, T_2, T_3, and T_4 respectively, and vertex 7 is a path connecting T_2 to T_3. Based on the assignment, the columns for α, β, and γ in the superposition would be the following:

α	β	γ
$\hat{1}$	1	1
1	$\hat{0}$	$\hat{0}$
0	0	1
$\hat{0}$	1	$\hat{0}$
1	$\hat{1}$	1
3	3	3

However, the problem of ordering that kept the assignment from producing a valid multiplicity list appears identically in this superposition. Rows 1 and 5 must come from complete upward multiplicity lists of T_1 and T_4, but since each upward multiplicity must be between two non-upward multiplicities in any complete upward multiplicity list of a generalized star, there is no arrangement of the three columns that gives valid complete upward multiplicity lists for T_1 and T_4 simultaneously.

7.2. Previous conjectures

We now consider a set of prior conjectures about multiplicity lists for trees in the special case of linear trees. We begin with the Degree Conjecture, which claims that each tree with k high degree vertices of degrees $d_1, \ldots, d_k$ has a multiplicity list whose only entries greater than 1 are $d_1 - 1, \ldots, d_k - 1$. For further discussion on the Degree Conjecture, see [12] where it is proved for diametric trees. We extend this now to any linear trees satisfying the sufficiency of the LSP conditions.

Corollary 24. *The Degree Conjecture holds for linear trees satisfying the sufficiency of the LSP conditions.*

Proof. We will treat T as a k-linear tree $L(T_1, s_1, T_2, s_2, \ldots, s_{k-1}, T_k)$, where the center vertex of T_i has degree d_i when viewed as a vertex of T (as opposed to just T_i). For $i = 2, \ldots, k-1$, T_i has $d_i - 2$ branches, so $(1, \widehat{d_i - 3}, 1, \hat{0}, 1, \hat{0}, 1, \ldots)$ is a complete upward multiplicity list for T_i by Theorem 9. Similarly, for $i = 1, k$, T_i has $d_i - 1$ branches, so $T_i$has $(1, \widehat{d_i - 2}, 1, \hat{0}, 1, \hat{0}, 1, \ldots)$

Fig. 5. A linear complete binary tree.

as a complete upward multiplicity list. We combine the first three elements of each of these lists as follows:

T_1	1	$\widehat{d_1-2}$	1	0	0	0	0	0	0
T_2	0	1	$\widehat{d_2-3}$	1	0	0	0	0	0
T_3	0	0	1	$\widehat{d_3-3}$	1	0	0	0	0
$\vdots$	0	0	0	$\ddots$	$\ddots$	$\ddots$	0	0	0
T_{k-2}	0	0	0	0	1	$\widehat{d_{k-2}-3}$	1	0	0
T_{k-1}	0	0	0	0	0	1	$\widehat{d_{k-1}-3}$	1	0
T_k	0	0	0	0	0	0	1	$\widehat{d_k-2}$	1
	1	d_1-1	d_2-1	$\cdots$	$\cdots$	$\cdots$	$d_{k-1}-1$	d_k-1	1

The remaining ones in the multiplicity lists of the T_i's along with the lists of ones corresponding to the connecting paths may be included with no overlap, leaving us with a multiplicity list for T containing $d_1-1, \dots, d_k-1$ and ones. □

Two results follow immediately from the Degree Conjecture. The first is regarding a multiplicity list for complete binary trees. Note that our definition of complete binary tree may differ from other sources. We use the following definition.

Definition 25. A tree is called *complete binary* if each vertex has degree one or three.

An example of a complete binary tree is the graph shown in Fig. 1. For this class of trees, there is a conjecture proposed by Johnson, Leal-Duarte, and Saiago stating that any complete binary tree with k vertices of degree three has $(1, 2, 2, \dots, 2, 2, 1)$, i.e. the list consisting of k two's and a one at each end, as an ordered multiplicity list. Using the LSP, it is possible to prove this conjecture for complete binary trees that are also linear.

Corollary 26. *Suppose T is a complete binary tree with k vertices of degree three, and all degree three vertices lie on a single induced path. Then $(1, 2, 2, \dots, 2, 2, 1)$, i.e. the list consisting of k two's and a one at each end, is an ordered multiplicity list for T.*

Proof. While this follows directly from the Degree Conjecture, we offer an alternative proof using the LSP directly. Within the class of complete binary trees with k degree three vertices, there is exactly one tree that is also k-linear. We may represent this tree as in Fig. 5:

We can interpret this tree as a k-linear tree with k components on two vertices, 2 single vertex components, and connecting paths of length zero. In this case, $(1, \hat{0}, 1)$ is the only complete upward multiplicity list for the two vertex components, and (1) is the only list for the single vertices. We augment each list to length $k+2$ and combine them as follows:

0	1	0	0	0	0	0
1	$\hat{0}$	1	0	0	0	0
0	1	$\hat{0}$	1	0	0	0
0	0	$\ddots$	$\ddots$	$\ddots$	0	0
0	0	0	1	$\hat{0}$	1	0
0	0	0	0	1	$\hat{0}$	1
0	0	0	0	0	1	0
1	2	2	$\cdots$	2	2	1

□

For a tree T, let $U(T)$ denote the minimum number of ones among the multiplicity lists of T. From [10], we know that $U(T) \geq 2$. The second result to follow from the Degree Conjecture is an upper bound for $U(T)$. For any tree T, let $D_2(T)$ be the number of degree 2 vertices of T. Then we have

$$U(T) \leq 2 + D_2(T).$$

It was shown in [12] that this follows from the Degree Conjecture.

Corollary 27. *For any linear tree T satisfying the sufficiency of the LSP conditions,*

$$U(T) \leq 2 + D_2(T).$$

7.3. Nonlinear trees

We conclude with a discussion of some of the differences in determining multiplicity lists of nonlinear trees compared to linear trees. A fundamental difference occurs between linear and nonlinear trees that makes superposition difficult to use for characterizing the multiplicity lists of nonlinear trees. For any high degree vertex of a linear tree, all other high degree vertices are contained in at most two of its branches. This gives linear trees two important properties that superposition makes use of: (a) any linear tree can be viewed as a collection of generalized stars connected at their central vertices and (b) there is a natural ordering of these generalized star components. For nonlinear trees, there is no way to satisfy both properties. For example, consider Fig. 1, previously mentioned to be the smallest example of a nonlinear tree. This tree has four high degree vertices, and at one of the vertices, there is a high degree vertex in three branches. If we would like to interpret the tree as four generalized stars connected by edges, there is no obvious ordering. On the other hand, if we divide it into components with a natural ordering, we no longer have just generalized star components connected at their central vertices.

Fig. 1 achieves the ordered multiplicity list (1, 2, 4, 2, 1) by assigning each of the three branches of the central vertex with the same three eigenvalues, one of which having an upward multiplicity. Thus, by removing the central vertices of the three branches we must see some eigenvalue a total of six times, and thus if we also have this eigenvalue on the central vertex of the nonlinear tree, then its multiplicity must be $7-3=4$. The two other eigenvalues from the induced 3-paths will be seen three times upon removing the central vertex of the tree, insuring that we have two eigenvalues of multiplicity two. Let us attempt to produce this list while trying to follow the LSP. We might end up with the following as the non-upward eigenvalues from the 3-path must be the same.

$$\begin{array}{ccc} 1 & \hat{1} & 1 \\ 1 & \hat{1} & 1 \\ 1 & \hat{1} & 1 \\ 0 & 1 & 0 \\ \hline 3 & 4 & 3 \end{array}$$

However, this is not a possible multiplicity list for any tree as interlacing requires that the first and last eigenvalue be distinct. Now let us attempt to force the list (1, 2, 4, 2, 1) to occur.

$$\begin{array}{ccccc} 1 & 0 & \hat{1} & 0 & 1 \\ 0 & 1 & \hat{1} & 1 & 0 \\ 0 & 1 & \hat{1} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 2 & 4 & 2 & 1 \end{array}$$

While this does seem to achieve the list (1, 2, 4, 2, 1) there are two problems with the above superposition. First, the non-upward eigenvalues of the 3-paths are the same in all three cases, yet the summation shows a difference for one of the eigenvalues. Second, the sum adds two non-upward ones without an upward vertex between them. This addition without upward multiplicity is not allowed in the LSP, and thus this example raises the question if it is necessary in the nonlinear case. These issues lead to the necessity to devise some new manner to understand multiplicity lists for nonlinear trees.

To generalize the superposition concept to nonlinear trees, there seems to be two possibilities. First, interpret any nonlinear tree as a collection of generalized stars connected at their central vertices, but create a more elaborate structure for combining multiplicity lists. Second, allow for components other than generalized stars.

Another difficulty with nonlinear trees concerns the inverse eigenvalue problem. We showed in Section 6 that for any ordered multiplicity list of a depth 1 linear tree, the numerical values of the eigenvalues are arbitrary, subject to order. Unfortunately, the property does not generally hold for nonlinear trees, as shown in [1]. For example, for Fig. 1 (a similar argument was first presented in [1]), the ordered multiplicity list (1, 2, 4, 2, 1) does not allow all choices of eigenvalues. In fact, denoting the eigenvalues of a real symmetric matrix A whose graph is that of Fig. 1 with ordered multiplicity list (1, 2, 4, 2, 1) by $\lambda_1, \ldots, \lambda_5$, we must have $A[i] = \lambda_3$ for $i = 1, 2, 4, 5, 7, 8, 10$, and the submatrices $A[1, 2, 3]$, $A[4, 5, 6]$, and $A[7, 8, 9]$ each have eigenvalues $\lambda_2, \lambda_3, \lambda_4$. The trace of each of these submatrices is therefore $\lambda_2 + \lambda_3 + \lambda_4$, and the trace of A is $3(\lambda_2+\lambda_3+\lambda_4)+\lambda_3$. The trace of A can also be expressed as the sum of its eigenvalues: $\lambda_1+2\lambda_2+4\lambda_3+2\lambda_4+\lambda_5$. Setting equal the two expressions for the trace of A gives the following numerical restriction on the eigenvalues:

$$\lambda_2 + \lambda_4 = \lambda_1 + \lambda_5.$$

Because of this difficulty with even the smallest nonlinear tree, it may be that the likely relationship between the inverse eigenvalue problem and the ordered multiplicity lists is unique to the linear trees.

References

[1] F. Barioli, S.M. Fallat, On two conjectures regarding an inverse eigenvalue problem for acyclic symmetric matrices, Electron. J. Linear Algebra 11 (2004) 41–50.
[2] F. Barioli, S.M. Fallat, R.L. Smith, On acyclic and unicyclic graphs whose minimum rank equals the diameter, Linear Algebra Appl. 429 (2008) 1568–1578.

[3] M.T. Chu, G.H. Golub, Structured inverse eigenvalue problems, Acta Numer. 11 (2002) 1–71.
[4] R. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[5] C.R. Johnson, C. Jordan-Squire, D.A. Sher, Eigenvalue assignments and the two largest multiplicities in a Hermitian matrix whose graph is a tree, Discrete Appl. Math. 158 (2010) 681–691.
[6] C.R. Johnson, A. Leal-Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, Linear Multilinear Algebra 46 (1999) 139–144.
[7] C.R. Johnson, A. Leal-Duarte, On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree, Math. Inequal. Appl. 5 (2) (2002) 175–180.
[8] C.R. Johnson, A. Leal-Duarte, On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree, Linear Algebra Appl. 348 (2002) 7–21.
[9] C.R. Johnson, A. Leal-Duarte, C.M. Saiago, Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices whose graph is a tree: the case of generalized stars and double generalized stars, Linear Algebra Appl. 373 (2003) 311–330.
[10] C.R. Johnson, A. Leal-Duarte, C.M. Saiago, The Parter–Weiner theorem: refinement and generalization, SIAM J. Matrix Anal. Appl. 25 (2) (2003) 352–361.
[11] C.R. Johnson, A. Leal-Duarte, C.M. Saiago, B.D. Sutton, A.J. Witt, On the relative position of multiple eigenvalues in the spectrum of an Hermitian matrix with a given graph, Linear Algebra Appl. 363 (2003) 147–159.
[12] C.R. Johnson, J. Nuckols, C. Spicer, The implicit construction of multiplicity lists for classes of trees and verification of some conjectures, Linear Algebra Appl. 438 (2013) 1990–2003.
[13] C.R. Johnson, B.D. Sutton, A.J. Witt, Implicit construction of multiple eigenvalues for trees, Linear Multilinear Algebra 57 (4) (2009) 409–420.
[14] S. Parter, On the eigenvalues and eigenvectors of a class of matrices, J. Soci. Ind. Appl. Math. 8 (1960) 376–388.
[15] G. Wiener, Spectral multiplicity and splitting results for a class of qualitative matrices, Linear Algebra Appl. 61 (1984) 15–29.