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Integral Equations and Operator Theory

Truncated complex moment problems with a $\bar{z}z$ relation

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Dedicated to the memory of my parents

Abstract. We solve the truncated complex moment problem for measures supported on the variety $\mathcal{K} \equiv \{z \in \mathbf{C} : z\bar{z} = A + Bz + C\bar{z} + Dz^2,\ D \neq 0\}$. Given a doubly indexed finite sequence of complex numbers $\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \gamma_{1,2n-1}, \ldots, \gamma_{2n-1,1}, \gamma_{2n,0}$, there exists a positive Borel measure μ supported in $\mathcal{K}$ such that $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu$ $(0 \leq i + j \leq 2n)$ if and only if the moment matrix $M(n)(\gamma)$ is positive, recursively generated, with a column dependence relation $Z\bar{Z} = A1 + BZ + C\bar{Z} + DZ^2$, and $card\ \mathcal{V}(\gamma) \geq rank\ M(n)$, where $\mathcal{V}(\gamma)$ is the variety associated to γ. The last condition may be replaced by the condition that there exists a complex number $\gamma_{n,n+1}$ satisfying $\gamma_{n+1,n} \equiv \bar{\gamma}_{n,n+1} = A\gamma_{n,n-1} + B\gamma_{n,n} + C\gamma_{n+1,n-1} + D\gamma_{n,n+1}$. We combine these results with a recent theorem of J. Stochel to solve the full complex moment problem for $\mathcal{K}$, and we illustrate the connection between the truncated and full moment problems for other varieties as well, including the variety $z^k = p(z, \bar{z})$, $deg\ p < k$.

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1. Introduction

Given a doubly indexed finite sequence of complex numbers
$\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \gamma_{1,2n-1}, \ldots, \gamma_{2n-1,1}, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the truncated complex moment problem entails finding a positive Borel

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measure μ supported in the complex plane $\mathbf{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \ (0 \leq i + j \leq 2n);$$

γ is called a *truncated moment sequence* (of order $2n$) and μ is a *representing measure* for γ. In [C][CF1]-[CF7] [F1]-[F3], R.E. Curto and the author studied conditions for the existence of representing measures expressed in terms of positivity and extension properties of the *moment matrix* $M(n) \equiv M(n)(\gamma)$ associated to γ. As we discuss in the sequel, if γ has a representing measure, then $M(n)$ is positive semidefinite and *recursively generated* (see below for terminology and notation). For truncated moment problems in one real variable, where the moment matrix for a real sequence $\beta^{(2n)}$ is the Hankel matrix $H(n) = (\beta_{i+j})_{0 \leq i,j \leq n}$, a representing measure for $\beta^{(2n)}$ exists if and only if $H(n)$ is positive and recursively generated [CF1]. By contrast, positivity and recursiveness are *not* sufficent in multivariable problems: in [CF3] we exhibited a positive, invertible $M(3)$ having no representing measure, and $M(3)$ is (vacuously) recursively generated.

There is a close connection between the existence of representing measures supported in a prescribed algebraic variety and the presence of corresponding dependence relations in the columns of $M(n)$. For $n \geq 2$, we denote the successive columns of $M(n)$ by $1, Z, \bar{Z}, Z^2, Z\bar{Z}, \bar{Z}^2, \ldots, Z^n, Z^{n-1}\bar{Z}, \ldots, Z\bar{Z}^{n-1}, \bar{Z}^n$. For a complex polynomial $p \in \mathcal{P}_n$, $p(z, \bar{z}) = \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$, a representing measure μ is supported in $\mathcal{Z}(p) \equiv \{z \in \mathbf{C} : p(z, \bar{z}) = 0\}$ if and only if there is a dependence relation $p(Z, \bar{Z}) \equiv \sum a_{ij} \bar{Z}^i Z^j = 0$ in $C_{M(n)}$ (the column space of $M(n)$) [CF2, Proposition 3.1]; it follows that if μ is a representing measure, then *card supp* $\mu \geq$ *rank* $M(n)$ [CF2, Corollary 3.7]. Moreover, if $M(n)$ is positive and recursively generated, then any of the following types of dependence relations in $C_{M(n)}$ implies the existence of a *rank* $M(n)$-atomic representing measure: $Z = A1$ (for measures concentrated at a point); $\bar{Z} = A1 + BZ$ (measures supported on a line) [CF3,Theorem 2.1]; $Z^2 = A1 + BZ + C\bar{Z}$ (measures on the intersections of two hyperbolas) [CF3, Theorem 3.1]; $\bar{Z}Z = A1 + BZ + C\bar{Z}$ (measures on a circle) [CF7, Theorem 1.1]; or $Z^k = p(Z, \bar{Z})$ with *deg* $p < k \leq [n/2] + 1$ [CF3,Theorem 3.1]. In view of these results, in studying conditions for the existence of representing measures, we may assume without loss of generality that $M(n)$ is positive and recursively generated, and that $\{1, Z, \bar{Z}, Z^2\}$ is independent.

The precise relationship between the column structure of $M(n)$ and the existence of representing measures for $\gamma^{(2n)}$ is not well understood. The basic existence theorem of [CF2] shows that a *rank* $M(n)$-atomic representing measure exists in the case of *flat* data, when $M(n)$ is positive and *rank* $M(n) =$ *rank* $M(n-1)$. For the case when there is an *analytic* column relation, of the form $Z^k = p(Z, \bar{Z})$ with $p \in \mathcal{P}_{k-1}$ for some $k \leq n$, an algorithm of [F2] determines whether or not a finitely atomic representing measure exists. In the present note we solve the truncated moment problem for a moment matrix $M(n)$ with a column relation

$$\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2, \ D \neq 0. \tag{1.1}$$

Clearly, this relation falls beyond the scope of [CF2] [CF3] [CF7] [F2]. Our main result, which follows, gives a concrete necessary and sufficient condition for the existence of a finitely atomic representing measure (necessarily supported in the variety $\mathcal{K} \equiv \{z \in \mathbf{C} : z\bar{z} = A + Bz + C\bar{z} + Dz^2,\ D \neq 0\}$).

Theorem 1.1. *Let $n \geq 2$ and assume that $M(n)(\gamma)$ is positive and recursively generated. Suppose that $\{1, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(n)}$ and that there is a dependence relation of the form $\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2$, $D \neq 0$. The following are equivalent for $\gamma \equiv \gamma^{(2n)}$:*
i) γ admits a finitely atomic representing measure;
ii) γ has a representing measure with finite moments up to at least order $2n+2$;
iii) $M(n)$ admits a positive moment matrix extension $M(n+1)$;
iv) $M(n)$ admits an extension $M(n+1)$ satisfying rank $M(n+1) =$ rank $M(n)$;
v) γ has a rank $M(n)$-atomic representing measure;
vi) There exists $\gamma_{n,n+1} \in \mathbf{C}$ such that

$$\gamma_{n+1,n} \equiv \bar{\gamma}_{n,n+1} = A\gamma_{n,n-1} + B\gamma_{n,n} + C\gamma_{n+1,n-1} + D\gamma_{n,n+1}.$$

For $n = 2$, Theorem 1.1 was proved in [CF7, Theorem 1.3] as part of a solution to the quartic complex moment problem. The main implication, vi) $\Rightarrow$ v), provides a simple numerical test for the existence of a minimal finitely atomic representing measure (which may then be explicitly computed using the Flat Extension Theorem (Theorem 2.1 below)). This type of numerical test is new in truncated moment problems and indicates the kind of auxiliary condition (beyond positivity and recursiveness) that may be required to solve moment problems that are neither flat nor analytic (problems in which there is a column dependence relation with more than one term of highest degree).

Let $\mathcal{V}(\gamma) \equiv \bigcap_{\substack{p \in \mathcal{P}_n \\ p(Z,\bar{Z})=0}} \mathcal{Z}(p)$, the *variety* associated to $\gamma^{(2n)}$; [CF4, (1.7)] implies that if μ is a representing measure, then *card* $\mathcal{V}(\gamma) \geq$ *card supp* $\mu \geq$ *rank* $M(n)$. As we noted above, there exists a positive invertible $M(3)$ with no representing measure, and in this example $\mathcal{V}(\gamma)(= \mathbf{C})$ is infinite. By contrast, in all of the examples of [CF4] [CF7] in which a *singular*, positive, recursively generated moment matrix $M(n)$ fails to have a representing measure, it transpires that *card* $V(\gamma) <$ *rank* $M(n)$. These results suggest the following solution to the singular truncated complex moment problem.

Conjecture 1.2. Suppose $M(n)$ is singular. $\gamma^{(2n)}$ admits a representing measure if and only if $M(n)$ is positive, recursively generated, and *card* $\mathcal{V}(\gamma) \geq$ *rank* $M(n)$.

An affirmation of Conjecture 1.2 would also solve the singular *full* complex moment problem for $\gamma^{(\infty)}$. Indeed, as we discuss in Section 4, if Conjecture 1.2 is true, then for $M(\infty)$ singular, it would follow that $\gamma^{(\infty)}$ has a representing measure if and only if $M(\infty) \geq 0$ and *card* $\mathcal{V}(\gamma^{(\infty)}) \geq$ *rank* $M(\infty)$. Theorem 1.1 yields the following result in support of Conjecture 1.2.

Theorem 1.3. *Under the hypotheses of Theorem 1.1, the following are equivalent for $\gamma \equiv \gamma^{(2n)}$:*

i) γ admits a rank $M(n)$-atomic representing measure;
ii) γ admits a representing measure;
iii) card $\mathcal{V}(\gamma) \geq$ rank $M(n)$.

Examples 1.7 and 1.9 (below) illustrate Theorem 1.3 in negative and, respectively, positive cases. Theorem 1.3 shows that the conditions of Theorem 1.1 for a finitely atomic representing measure are actually equivalent to the existence of an arbitrary representing measure. Whether the existence of a representing measure implies the existence of a finitely atomic representing measure in a general truncated moment problem is an open question that we address in [CF8] (cf. [P3] and Question 2.4 below). By combining Theorem 1.3 with the above-mentioned results of [CF3] [CF7], we have the following partial affirmation of Conjecture 1.2.

Corollary 1.4. *Suppose $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is dependent in $\mathcal{C}_{M(n)}$. Then $\gamma^{(2n)}$ admits a representing measure if and only if $M(n)$ is positive, recursively generated, and card $\mathcal{V}(\gamma) \geq$ rank $M(n)$.*

Our earlier remarks imply that the variety condition is superfluous if $\{1, Z, \bar{Z}, Z^2\}$ is dependent; with respect to the lexicographic ordering of the columns, the $\bar{Z}Z$ relation is the first case in which the variety condition is indispensible.

It is instructive to compare [CF7, Theorem 1.3] and Theorem 1.1 to results of J. Stochel [St1] concerning the *full moment problem*, in which moments of *all* orders are prescribed (cf. [AK] [Akh] [PV2] [ST] [SS1] [SS2]). Stochel's results for the 2-dimensional real full moment problem are stated in terms of positivity properties of the Riesz functional, but we may paraphrase them in the language of real moment matrices $M_{\mathbf{R}}(\infty)$, with columns $1,\ X,\ Y,\ X^2,\ XY,\ Y^2, \ldots$. Paraphrasing [St1], we say that a real polynomial $p(x,y)$ is of *type A* if it satisfies the following property: a full real moment sequence $\beta^{(\infty)} \equiv \{\beta_{ij}\}_{i,j\geq 0}$ has a representing measure supported in $\mathcal{V}(p) \equiv \{(x,y) \in \mathbf{R}^2 : p(x,y) = 0\}$ if and only if $M_{\mathbf{R}}(\infty)(\beta) \geq 0$ and $p(X,Y) = 0$ in $\mathcal{C}_{M_{\mathbf{R}}(\infty)(\beta)}$. In particular, Stochel proved that if *deg* $p \leq 2$, then p is of type A [St1, Theorem 5.4] . Using the equivalence between the 2-dimensional real moment problem and the complex moment problem (cf. [CF7, Section 1] [SS2, Appendix]), one may conclude that if $p \in \mathbf{C}[z,\bar{z}]$ and *deg* $p(z,\bar{z}) \leq 2$, then $\gamma^{(\infty)}$ has a representing measure supported in $\mathcal{Z}(p) \equiv \{z \in \mathbf{C} : p(z,\bar{z}) = 0\} \iff M(\infty)(\gamma) \geq 0$ and $p(Z,\bar{Z}) = 0$ in $C_{M(\infty)(\gamma)}$. By contrast, in the truncated moment problem, an example of [CF7] (Example 1.7 below) illustrates that for certain column relations of degree 2, such as (1.1), additional conditions beyond positivity and recursiveness may be required to insure the existence of a representing measure. As we show in Section 4, the explanation of this phenomenon is that in the full moment problem subordinate to (1.1), positivity of $M(\infty)(\gamma)$ actually subsumes, for each n, the recursiveness of $M(n)(\gamma)$ and the auxiliary condition of Theorem 1.1-vi).

In Section 4 we describe a technique for solving a full multidimensional moment problem which entails combining the solution of the corresponding truncated moment problem with a recent convergence theorem of Stochel [St2] (Theorem 4.1 below). For moment problems subordinate to $\mathcal{K} \equiv \{z \in \mathbf{C} : z\bar{z} =$

$A + Bz + C\bar{z} + Dz^2, D \neq 0\}$), this technique shows that if $M(\infty)$ has a column relation of the form (1.1), then $\gamma^{(\infty)}$ has a representing measure (necessarily supported in $\mathcal{K}$) if and only if $M(\infty) \geq 0$ (Proposition 4.4). Of course, this result also follows from [St1, Theorem 5.4] via the equivalence between the $\mathbf{R}^2$ and $\mathbf{C}$ moment problems. In Section 4 we also illustrate the convergence technique for other types of moment problems, including those subordinate to an analytic relation; as we discuss in Section 4, the following result can also be derived from [SS1].

Proposition 1.5. *If $M(\infty)$ has a column relation of the form $Z^k = p(Z, \bar{Z})$, deg $p < k$, then $\gamma^{(\infty)}$ has a representing measure if and only if $M(\infty) \geq 0$.*

In [St1], Stochel showed that not every real polynomial of degree 3 is of type A, and recently Powers and Scheiderer [PS] gave a criterion for solving the full moment problem on (non-compact) semialgebraic subsets of the plane (cf. [KM] [PV1] [PV2] [Sch] [SS1] [SS2]). In view of Theorem 1.1 and [CF3] [CF7], to complete the theory of the truncated complex moment problem for degree 2 curves, it suffices to consider the case when $M(n)$ is positive, recursively generated, $\{1,\ Z,\ \bar{Z},\ Z^2,\ Z\bar{Z}\}$ is independent, and there is a column relation $\bar{Z}^2 = A1 + BZ + C\bar{Z} + DZ^2 + EZ\bar{Z}$, $E \neq 0$. This problem is solved in [CF7] for $n = 2$ and we are currently studying the general case.

Theorem 1.1-vi) provides a concrete test for the existence of a finitely atomic representing measure supported in the variety $\mathcal{K}$ corresponding to (1.1). If $|D| \neq 1$, it is elementary that the equation of Theorem 1.1-vi) always admits a unique solution. Alternately, in Section 2 we prove the following result independently of Theorem 1.1, using instead only the $n = 2$ case [CF7] and moment matrix extension results of [CF2] [CF4].

Proposition 1.6. *If $\gamma^{(2n)}$ satisfies the hypothesis of Theorem 1.1 and $|D| \neq 1$, then there exists a unique finitely atomic representing measure μ, and card supp $\mu = 4$.*

For $|D| = 1$, $\mathcal{K}$ admits diverse possibilities, and may be finite or infinite; further, in this case there need not exist any representing measure, as the following example of [CF7] shows.

Example 1.7. *([CF7, Example 3.8]) For $f > 1$, let*

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & f & f-1 & f-1 \\ 1 & 0 & 0 & f-1 & f & f-1 \\ 0 & 1 & 0 & f-1 & f-1 & f \end{pmatrix}.$$

$M(2)$ is positive, recursively generated (vacuously), $\{1, Z, \bar{Z}, Z^2\}$ is a basis for $C_{M(n)}$, and $\bar{Z}Z = 1 - \bar{Z} + Z^2$. Theorem 1.1-vi) requires γ_{23} such that $\bar{\gamma}_{23} = \gamma_{21} - \gamma_{31} + \gamma_{23}$, or $iIm\gamma_{23} = (f-1)/2$ (> 0). Thus $\gamma^{(4)}$ admits no finitely atomic representing measure; moreover, since card $\{z : \bar{z}z = 1 - \bar{z} + z^2\} = 3 < 4 =$

rank $M(2)$, it follows from [CF4, (1.7)] that there is no representing measure whatsoever. Note that this example also illustrates Theorem 1.3 in a case in which Theorem 1.3-iii) fails. $\square$

One element in the proof of Theorem 1.1 is the following description of a basis for the column space of a recursively generated moment matrix $M(n)$ satisfying (1.1).

Proposition 1.8. *Suppose $n \geq 2$, $M(n)$ is positive and recursively generated, $\{1, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(n)}$, and there is a dependence relation of the form $\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2$, $D \neq 0$. Then there is a unique integer q, $2 \leq q \leq n$, such that a basis for $\mathcal{C}_{M(n)}$ consists of $\{1, Z, \bar{Z}, Z^2, \ldots, Z^i, \ldots, Z^q\}$.*

Note that if $q < n$, then *rank* $M(n) =$ *rank* $M(n-1)$, so the existence of a unique finitely atomic representing measure (which is *rank* $M(n)$-atomic) follows immediately from [CF2, Corollary 5.14] (cf. Theorem 1.13 below). Example 1.7 (above) illustrates a case with $q = n$ $(= 2)$ in which there is no representing measure. We next illustrate the existence of a representing measure in a case in which $q = n = 3$ (and $|D| = 1$).

Example 1.9. *For $r > 1$, let $M(3) =$*

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1-r & r & 1-r & r-1 \\
0 & 0 & 1 & 1 & 0 & 0 & r-1 & 1-r & r & 1-r \\
0 & 0 & 1 & r & 1-r & r-1 & \frac{1-r}{2} & \frac{1-r}{2} & \frac{3r-1}{2} & \frac{5(1-r)}{2} \\
1 & 0 & 0 & 1-r & r & 1-r & \frac{3r-1}{2} & \frac{1-r}{2} & \frac{1-r}{2} & \frac{3r-1}{2} \\
0 & 1 & 0 & r-1 & 1-r & r & \frac{5(1-r)}{2} & \frac{3r-1}{2} & \frac{1-r}{2} & \frac{1-r}{2} \\
1 & 1-r & r-1 & \frac{1-r}{2} & \frac{3r-1}{2} & \frac{5(1-r)}{2} & x & y & z & w \\
0 & r & 1-r & \frac{1-r}{2} & \frac{1-r}{2} & \frac{3r-1}{2} & y & x & y & z \\
0 & 1-r & r & \frac{3r-1}{2} & \frac{1-r}{2} & \frac{1-r}{2} & z & y & x & y \\
1 & r-1 & 1-r & \frac{5(1-r)}{2} & \frac{3r-1}{2} & \frac{1-r}{2} & w & z & y & x
\end{pmatrix}.$$

Here, we choose $x > (3 - 7r + 8r^2)/4$ to insure that $M(3)$ is positive and that $\{1, Z, \bar{Z}, Z^2, Z^3\}$ is independent. Now $\bar{Z}Z = 1 + \bar{Z} - Z^2$, so we have $A = 1$, $B = 0$, $C = 1$, $D = -1$. Recursiveness requires $\bar{Z}Z^2 = Z + Z\bar{Z} - Z^3$, which in turn successively determines y, z, and w as $y = (r+1)/2 - x$, $z = 3(1-r)/2 - y = 1 - 2r + x$, $w = (5r-3)/2 - z = (9r - 2x - 5)/2$. With these definitions, $M(3)$ satisfies the hypotheses of Theorem 1.1 and Proposition 1.8, with $q = n = 3$. The condition of Theorem 1.1-vi), based on the recursive relation $\bar{Z}Z^3 = Z^2 + Z^2\bar{Z} - Z^4$, is that there exists γ_{34} satisfying $\bar{\gamma}_{34} = \gamma_{32} + \gamma_{42} - \gamma_{34}$, or Re $\gamma_{34} = (1-x)/2$. Thus, corresponding to each choice of γ_{34} with Re $\gamma_{34} = (1-x)/2$, there is a distinct 5-atomic representing measure for $\gamma^{(6)}$. We compute one such measure explicitly in Example 2.2 (below). Alternately, the existence of a 5-atomic representing measure follows from Theorem 1.3, since the variety associated to $M(3)$ is the vertical line $x = -1/2$, and is thus infinite. $\square$

We devote the remainder of this section to some background results concerning moment matrices and representing measures. In Section 2 we prove most of the implications of Theorem 1.1, as well as Theorem 1.3 and Proposition 1.6. In Section 3 we prove Proposition 1.8 and we complete the proof of Theorem 1.1 by proving vi) $\Rightarrow$ iv). In Section 4 we discuss a recent theorem of J. Stochel which connects the full and truncated moment problems; we show how to re-derive the solution of certain one- and two-dimensional full moment problems from our previous results on the truncated moment problem in [CF1] [CF4] [CF7] [F2], or from Theorem 1.1.

Let $\mathcal{P}_n$ denote the complex polynomials $q(z,\bar{z}) = \sum a_{ij}\bar{z}^i z^j$ of total degree at most n, and for $q \in \mathcal{P}_n$, let $\hat{q} \equiv (a_{ij})$ denote the coefficient vector of q with respect to the basis $\{\bar{z}^i z^j\}_{0\le i+j\le n}$ of $\mathcal{P}_n$ (ordered lexicographically: $1, z, \bar{z}, \ldots, z^n, \ldots, \bar{z}^n$). Note that $dim\ \mathcal{P}_n = m(n) \equiv (n+1)(n+2)/2$; for $v \in \mathbf{C}^{m(n)}$ and $0 \le k \le n$, let $[v]_k$ denote the truncation of v to components indexed by basis monomials of degree $\le k$. For $p \in \mathcal{P}_{2n}$, $p(z,\bar{z}) = \sum b_{ij}\bar{z}^i z^j$, let $\Lambda_\gamma(p) = \sum b_{ij}\gamma_{ij}$. The *moment matrix* $M(n) \equiv M(n)(\gamma)$ is the unique matrix (of size $m(n)$) such that

$$\langle M(n)\hat{f}, \hat{g}\rangle = \Lambda_\gamma(f\bar{g})\ (f, g \in \mathcal{P}_n).$$

If we label the rows and columns of $M(n)$ lexicographically, as $1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots, Z^n, \ldots, \bar{Z}^n$, it follows that the row $\bar{Z}^k Z^l$, column $\bar{Z}^i Z^j$ entry of $M(n)$ is equal to $\langle M(n)\widehat{\bar{z}^i z^j}, \widehat{\bar{z}^k z^l}\rangle = \Lambda_\gamma(\bar{z}^{i+l}z^{j+k}) = \gamma_{i+l,j+k}$. For example, with $n = 1$, the *quadratic moment problem* for $\gamma^{(2)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ corresponds to

$$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}.$$

If γ admits a representing measure μ, then for $f \in \mathcal{P}_n$, $\langle M(n)\hat{f}, \hat{f}\rangle = \Lambda_\gamma(|f|^2) = \int\ |f|^2 d\mu \ge 0$, whence $M(n) \ge 0$.

For an arbitrary matrix $A \in M_{m(n)}(\mathbf{C})$, we define a sesquilinear form $\langle\cdot,\cdot\rangle_A$ on $\mathcal{P}_n$ by $\langle p, q\rangle_A = \langle A\hat{p}, \hat{q}\rangle$; the following characterization of moment matrices will be used in the proof of Theorem 1.1.

Theorem 1.10. *([CF2, Theorem 2.1]) Let $n \ge 1$ and let $A \in M_{m(n)}(\mathbf{C})$. There exists a truncated moment sequence $\gamma \equiv \gamma^{(2n)}$ such that $A = M(n)(\gamma)$ if and only if the following properties hold:*
i) $\langle 1, 1\rangle_A > 0$;
ii) $A = A^*$;
iii) $\langle p, q\rangle_A = \langle \bar{q}, \bar{p}\rangle_A$ $(p, q \in \mathcal{P}_n)$;
iv) $\langle zp, q\rangle_A = \langle p, \bar{z}q\rangle_A$ $(p, q \in \mathcal{P}_{n-1})$.

For $p \in \mathcal{P}_n$, $p(z,\bar{z}) = \sum a_{ij}\bar{z}^i z^j$, we define $p(Z,\bar{Z}) \in \mathcal{C}_{M(n)}$ by $p(Z,\bar{Z}) = \sum a_{ij}\bar{Z}^i Z^j$ $(= M(n)\hat{p})$; from [CF2, Lemma 3.10], we have

$$For\ p \in \mathcal{P}_n,\ p(Z,\bar{Z}) = 0 \Longleftrightarrow \bar{p}(Z,\bar{Z}) = 0. \tag{1.2}$$

If μ is a representing measure for γ, then (from [CF2, Proposition 3.1])

$$\textit{For } p \in \mathcal{P}_n,\ p(Z,\bar{Z}) = 0 \Longleftrightarrow \textit{supp}\ \mu \subset \mathcal{Z}(p) \equiv \{z \in \mathbf{C} : p(z,\bar{z}) = 0\}. \tag{1.3}$$

It follows that if μ is a representing measure, $p(Z,\bar{Z}) = 0$, and $pq \in \mathcal{P}_n$, then $(pq)(Z,\bar{Z}) = 0$. Thus, a necessary condition for representing measures is that $M(n)$ be *recursively generated* in the following sense:

$$p,\ q,\ pq \in \mathcal{P}_n,\ p(Z,\bar{Z}) = 0 \Longrightarrow (pq)(Z,\bar{Z}) = 0.$$

Positivity and recursiveness are the basic necessary conditions for solubility of a truncated complex moment problem. For the analogous one dimensional real truncated moment problem, these conditions are also sufficient: a real sequence $\beta^{(2n)} : \beta_0, \ldots, \beta_{2n}$ has a representing measure supported in $\mathbf{R}$ if and only if the Hankel matrix $(\beta_{i+j})_{0 \le i,j \le n}$ is positive and recursively generated (with respect to the column labeling $1,\ t,\ ,\ldots, t^n$) [CF1, Theorem 3.9]. By contrast, Example 1.7 (above) illustrates $M(2)(\gamma)$ that is positive and recursivley generated, but for which γ has no representing measure. The following structure theorem for positive moment matrices provides a basic tool for constructing representing measures; it shows that a positive moment matrix is "almost" recursively generated.

Theorem 1.11. *([CF4, Theorem 1.6]) Let $M(n) \ge 0$. If $f,\ g,\ fg \in \mathcal{P}_{n-1}$ and $f(Z,\bar{Z}) = 0$, then $(fg)(Z,\bar{Z}) = 0$. Moreover, if $f,\ g,\ fg \in \mathcal{P}_n$ and $f(Z,\bar{Z}) = 0$, then $[(fg)(Z,\bar{Z})]_{n-1} = 0$.*

Let $\mathcal{V}(\gamma) = \bigcap_{\substack{p \in \mathcal{P}_n \\ p(Z,\bar{Z})=0}} \mathcal{Z}(p)$, the *variety* associated to $\gamma^{(2n)}$. One consequence of (1.3) is that if μ is a representing measure for γ, then *card* $\mathcal{V}(\gamma) \ge$ *card supp* $\mu \ge$ *rank* $M(n)$ [CF4, (1.7)]; in particular, if *rank* $M(n) >$ *card* $\mathcal{V}(\gamma)$, then γ admits no representing measure (cf. Example 1.7 above). The main result of [CF2], which follows, provides the equivalence of iv) and v) in Theorem 1.1.

Theorem 1.12. *([CF2, Theorem 5.13]) $\gamma^{(2n)}$ admits a rank $M(n)$-atomic representing measure if and only if $M(n) \ge 0$ and $M(n)$ admits a* flat extension*, i.e., $M(n)$ admits an extension to a moment matrix $M(n+1)$ satisfying rank $M(n+1) =$ rank $M(n)$.*

In [CF2] [CF3] we established the existence of flat extensions in the following cases:

Theorem 1.13. *([CF2, Corollary 5.14]) If $M(n) \ge 0$ is* flat*, i.e., rank $M(n) =$ rank $M(n-1)$, then $\gamma^{(2n)}$ admits a unique finitely atomic representing measure, which is rank $M(n)$-atomic.*

Theorem 1.14. *([CF3, Theorem 2.1]) If $M(n)$ is positive, recursively generated, and $\bar{Z} = A1 + BZ$ $(B \ne 0)$, then $M(n)$ admits a flat extension (corresponding to a rank $M(n)$-atomic representing measure supported on the line $\bar{z} = A + Bz$).*

Theorem 1.15. *([CF3, Theorem 3.1]) Suppose $M(n)$ is positive and recursively generated. If $1 \le k \le [n/2] + 1$ and $Z^k = p(Z,\bar{Z})$ for some $p \in \mathcal{P}_{k-1}$, then*

$M(n)$ admits a unique flat extension (corresponding to the unique finitely atomic representing measure for $\gamma^{(2n)}$).

In [CF2, Theorem 6.1] we proved that if $M(1) \geq 0$, then $M(1)$ admits a flat extension. In view of Theorems 1.14 and 1.15, if $n \geq 2$, $M(n)$ is positive and recursively generated, and $\{1, Z, \bar{Z}, Z^2\}$ is dependent in $\mathcal{C}_{M(n)}$, then $M(n)$ admits a flat extension. Further, in [CF7] we proved that a flat extension exists if $M(n) \geq 0$ is recursively generated, $\{1, Z, \bar{Z}\}$ is independent and $\bar{Z}Z \in \langle 1, Z, \bar{Z} \rangle$. The preceding results motivated our interest in (1.1) and the hypothesis of Theorem 1.1.

2. Moment matrices, extensions, and representing measures

In this section we recall some additional terminology and results concerning moment matrices and extensions. We use these results to present the first part of the proof of Theorem 1.1 and proofs of Theorem 1.3 and Proposition 1.6.

Given $\gamma \equiv \gamma^{(2n)}$, for $0 \leq i, j \leq n$ we define the $(i+1) \times (j+1)$ matrix B_{ij} whose entries are the moments of order $i+j$:

$$B_{ij} = \begin{pmatrix} \gamma_{ij} & \gamma_{i+1,j-1} & & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{ij} & \gamma_{i+1,j-1} & & \\ & \gamma_{i-1,j+1} & & & \\ \vdots & & & & \vdots \\ & & & & \\ \gamma_{0,j+i} & & & \cdots & \gamma_{ji} \end{pmatrix}. \tag{2.1}$$

It follows from the definition of $M(n)(\gamma)$ that it admits a block decomposition $M(n) = (B_{ij})_{0 \leq i,j \leq n}$.

We may also define auxiliary blocks $B_{0,n+1}, \ldots, B_{n-1,n+1}$ via (2.1). Given "new moments" of degree $2n+1$ for a prospective representing measure, let $B_{n,n+1}$ denote the corresponding moment matrix block given by (2.1), and let

$$B(n+1) = \begin{pmatrix} B_{0,n+1} \\ \vdots \\ B_{n-1,n+1} \\ B_{n,n+1} \end{pmatrix}. \tag{2.2}$$

Given a moment matrix block $C(n+1)$ of the form $B_{n+1,n+1}$ (corresponding to "new moments" of degree $2n+2$), we may define the moment matrix *extension* $M(n+1)$ via the block decomposition

$$M(n+1) = \begin{pmatrix} M(n) & B(n+1) \\ B(n+1)^* & C(n+1) \end{pmatrix}. \tag{2.3}$$

Note that $M(n+1)$ is completely determined once column Z^{n+1} is specified.

A result of Smul'jan [Smu] shows that a block matrix

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{2.4}$$

is positive semidefinite if and only if i) $A \geq 0$, ii) there exists a matrix W such that $B = AW$, and iii) $C \geq W^*AW$ (since $A = A^*$, W^*AW is independent of W satisfying $B = AW$). Note also that if $M \geq 0$, then $rank\ M = rank\ A$ if and only if $C = W^*AW$; conversely, if $A \geq 0$ and there exists W such that $B = AW$ and $C = W^*AW$, then $M \geq 0$ and $rank\ M = rank\ A$. A block matrix M as in (2.4) is an *extension* of A, and is a *flat extension* if $rank\ M = rank\ A$. A flat extension of a positive matrix A is completely determined by a choice of block B satisfying $B = AW$ for some matrix W, and by setting $C = W^*AW$; we denote such a flat extension by $[A; B]$.

For a moment matrix block $B_{n,n+1}$, representing "new moments" of order $2n+1$ for a prospective representing measure for $\gamma^{(2n)}$, let $B = B(n+1)$ (as in (2.2)). It follows that $M(n) \geq 0$ admits a (necessarily positive) flat extension

$$[M(n); B] = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

in the form of a moment matrix $M(n+1)$ if and only if

$$B = M(n)W \ for\ some\ W\ (i.e.,\ Ran\ B \subset Ran\ M(n));$$
$$C := W^*M(n)W\ is\ Toeplitz,\ i.e., has\ the\ form\ of\ a\ block\ B_{n+1,n+1}. \tag{2.5}$$

The following result is our main tool for constructing finitely atomic representing measures.

Theorem 2.1. *(Flat Extension Theorem, [CF2, Remark 3.15, Theorem 5.4, Corollary 5.12, Theorem 5.13, Corollary 5.15] [CF3, Lemma 1.9] [F1]) Suppose $M(n)(\gamma)$ is positive and admits a flat extension $M(n+1)$, so that $Z^{n+1} = p(Z, \bar{Z}) \in \mathcal{C}_{M(n+1)}$ for some $p \in \mathcal{P}_n$. Then there exist unique successive flat (positive) moment matrix extensions $M(n+2)$, $M(n+3)$,..., which are determined by the relations*

$$Z^{n+k} = (z^{k-1}p)(Z, \bar{Z}) \in \mathcal{C}_{M(n+k)}\ (k \geq 2). \tag{2.6}$$

Let $r = rank\ M(n)$. There exist unique scalars $a_0, \ldots, a_{r-1}$ such that in $\mathcal{C}_{M(r)}$,

$$Z^r = a_0 1 + \cdots + a_{r-1}Z^{r-1}.$$

The characteristic polynomial $g_\gamma(z) := z^r - (a_0 + \cdots + a_{r-1}z^{r-1})$ has r distinct roots, $z_0, \ldots, z_{r-1}$, and γ has a rank $M(n)$-atomic minimal representing measure of the form

$$\nu[M(n+1)] = \sum \rho_i \delta_{z_i},$$

where the densities ρ_i are determined by the Vandermonde equation

$$V(z_0, \ldots, z_{r-1})(\rho_0, \ldots, \rho_{r-1})^t = (\gamma_{00}, \ldots, \gamma_{0,r-1})^t. \tag{2.7}$$

The measure $\nu \equiv \nu[M(n+1)]$ is the unique finitely atomic representing measure for $\gamma^{(2n+2)}$, and is also the unique representing measure for $M(\infty)[\nu]$.

It is clear from (2.6) that under the hypothesis of Theorem 2.1, $M(n+1)$ has unique successive *recursively generated* extensions (which are also flat and positive). We next use the Flat Extension Theorem to explicitly construct a particular representing measure corresponding to the flat extensions of Example 1.9.

Example 2.2. *We previously used Theorem 1.1 to show that $M(3)$ in Example 1.9 has a flat extension $M(4)$ corresponding to each choice γ_{34} with Re $\gamma_{34} = (1-x)/2$. Let $\gamma_{34} = (1-x)/2$. Since $\mathcal{C}_{M(3)}$ has basis $\{1, Z, \bar{Z}, Z^2, Z^3\}$, to determine the characteristic polynomial g_γ in Theorem 2.1, we seek a purely analytic column relation in the flat extension $M(5)$ determined by (2.6). A calculation shows that in $\mathcal{C}_{M(4)}$ we have $Z^4 = a_0 1 + a_1 Z + b_1 \bar{Z} + a_2 Z^2 + a_3 Z^3$, where $a_0 = 1-r$, $a_1 = (3-r)/2$, $b_1 = (-3+2r-r^2+2x)/(2r-2)$, $a_2 = (1-x)/(r-1)$, $a_3 = -2$. Thus $M(5)$ is determined by $Z^5 = a_0 Z + a_1 Z^2 + b_1 \bar{Z} Z + a_2 Z^3 + a_3 Z^4 = a_0 Z + a_1 Z^2 + b_1 A 1 + b_1 B Z + C(Z^4 - a_0 1 - a_1 Z - a_2 Z^2 - a_3 Z^3) + b_1 D Z^2 + a_2 Z^3 + a_3 Z^4 \equiv r(Z)$. Let $r(z)$ denote the polynomial corresponding to $r(Z)$; a calculation shows that the characteristic polynomial, $g_\gamma(z) \equiv z^5 - r(z)$, factors as $g_\gamma(z) = \frac{-1}{2r-2}(z-1)q(z)$, where $q(z) = (1-2r+3r^2-2x) + 2(r-x)z + 2(1-x)z^2 + 4(1-r)z^3 + 2(1-r)z^4$. Let $\delta = (-2+6r-9r^2+6r^3+4x-6rx+x^2)^{1/2}$, $\alpha_1 = x-r-\delta$, $\alpha_2 = x-r+\delta$, $\rho_1 = ((-2+2r)^2 - 4(-2+2r)\alpha_1)^{1/2}$, $\rho_2 = ((-2+2r)^2 - 4(-2+2r)\alpha_2)^{1/2}$; the five distinct roots of g_γ (guaranteed by Theorem 2.1) are $z_0 = 1$, $z_1 = (2-2r-\rho_1)/(4r-4)$, $z_2 = (2-2r+\rho_1)/(4r-4)$, $z_3 = (2-2r-\rho_2)/(4r-4)$, $z_4 = (2-2r+\rho_2)/(4r-4)$. With $r=2$, $x=6$, we have $z_1 = \frac{1}{4}(-2-i\sqrt{-4+8(4-\sqrt{10})}) \approx 0.5-0.410927i$, $z_2 = \bar{z}_1$, $z_3 = \frac{1}{4}(-2-i\sqrt{-4+8(4+\sqrt{10})}) \approx -0.5-1.82514i$, $z_4 = \bar{z}_3$; the corresponding densities given by (2.7) are $\rho_0 = 1/3$, $\rho_1 = \rho_2 = \frac{1}{60}(10+2\sqrt{10}) \approx 0.272076$, $\rho_3 = \rho_4 = \frac{1}{60}(10-2\sqrt{10}) \approx 0.0612574$.* $\square$

We now turn to the proofs of the main results.

Proof of Theorem 1.1. Part 1. We will prove the implications $iv) \Leftrightarrow v) \Rightarrow i) \Rightarrow ii) \Rightarrow iii) \Rightarrow vi)$. We have $iv) \Leftrightarrow v)$ by Theorem 1.12. The implications $v) \Rightarrow i) \Rightarrow ii)$ are clear. If ii) holds, let ν be a representing measure for $\gamma^{(2n)}$ with finite moments up to order $2n+2$; then $M(n+1)[\nu]$ is a positive extension of $M(n)$, so iii) holds. For $iii) \Rightarrow vi)$, assume that $M(n+1)$ is a positive extension of $M(n)$. Since $\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2$, $D \neq 0$, Theorem 1.11 (applied to $M(n+1)$) implies $[\bar{Z}Z^n]_n = [AZ^{n-1}]_n + [BZ^n]_n + [C\bar{Z}Z^{n-1}]_n + [DZ^{n+1}]_n$, whence vi) follows by specialization to row Z^n. $\square$

To complete the proof of Theorem 1.1, it suffices to prove $vi) \Rightarrow iv)$; the proof is given in Section 3.

Proof of Proposition 1.6. We are assuming $|D| \neq 1$. The conclusion that γ has a unique finitely atomic representing measure, which is 4 atomic, is proved in [CF7, Theorem 1.3] for $n = 2$. We may thus assume $n > 2$ and we view $M(3)$ as an extension of $M(2)$, i.e.,

$$M(3) = \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}.$$

Now the relation

$$\bar{Z}Z^2 = AZ + BZ^2 + C\bar{Z}Z + DZ^3 \tag{2.8}$$

and moment matrix structure show that γ_{23} completely determines block B_{23}, i.e., $\gamma_{14} = (1/D)(\gamma_{23} - A\gamma_{12} - B\gamma_{13} - C\gamma_{22})$ and $\gamma_{05} = (1/D)(\gamma_{14} - A\gamma_{03} - B\gamma_{04} - C\gamma_{13})$. Moreover, since $|D| \neq 1$, the equation $\bar{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}$ admits a *unique* solution γ_{23}; thus $B(3)$ is uniquely determined. Again, since $|D| \neq 1$, it follows that γ_{33} and γ_{24} are uniquely determined by the equations $\gamma_{42} = A\gamma_{31} + B\gamma_{32} + C\gamma_{41} + D\gamma_{33}$ and $\gamma_{33} = A\gamma_{22} + B\gamma_{23} + C\gamma_{32} + D\bar{\gamma}_{42}$. Since γ_{33} and γ_{42} in turn uniquely determine $C(3)$ via (2.8), it follows that $M(3)$ is the unique recursively generated moment matrix extension of $M(2)$. Since [CF7, Corollary 3.4] shows that $M(2)$ has a unique *flat* extension $M(3)$, which is recursively generated by [CF3, Lemma 1.9], we conclude that $M(3)$ is a flat extension of $M(2)$. Since $M(n)$ is recursively generated, Theorem 2.1 and the remarks following it now imply that $M(n)$ is *flat*, i.e., $rank\ M(n) = rank\ M(n-1) = \cdots = rank\ M(2) = 4$, whence it follows from Theorem 1.12 that $M(n)$ has a unique finitely atomic representing measure, which is $rank\ M(n)$-atomic. Since $rank\ M(n) = rank\ M(2) = 4$, the result follows. □

We conclude this section with a discussion of Conjecture 1.2 and a proof of Theorem 1.3. In [CF3, Section 4], we exhibited a positive invertible $M(3)$ having no representing measure. Since this $M(3)$ is invertible, it is recursively generated, and $\mathcal{V}(\gamma) = \mathbf{C}$, whence $card\ \mathcal{V}(\gamma) > rank\ M(3)$. On the other hand, in all of the examples of [CF4], whenever $M(n)(\gamma)$ is positive, recursively generated and *singular*, it turns out that if γ admits no finitely atomic representing measure, then $card\ \mathcal{V}(\gamma) < rank\ M(n)$, so γ admits no representing measure whatsoever. These observations suggest the following questions related to Conjecture 1.2.

Question 2.3. If $M(n)$ is positive and recursively generated, and $card\ \mathcal{V}(\gamma) \geq rank\ M(n)$, does $\gamma^{(2n)}$ admit a representing measure?

Question 2.4. If $\gamma^{(2n)}$ admits a representing measure, does it admit a finitely atomic representing measure?

Positive answers to these questions would affirm Conjecture 1.2 and would reduce the truncated complex moment problem to standard issues in linear algebra and algebraic curve theory. Indeed, positivity, recursiveness, and calculation of $rank\ M(n)$ entail standard linear algebra, while $card\ \mathcal{V}(\gamma)$ can be estimated (at least in principle) by techniques from algebraic geometry. Concerning Question 2.4, a result of M. Putinar [P3] implies that if $\gamma^{(2n)}$ has a representing measure μ with finite moments up to order at least $2n+2$, then there exists a finitely atomic representing measure. (In [CF8] we show that the same conclusion holds if μ merely has finite moments up to order $2n+1$.)

We next prove Theorem 1.3 (which we restate for convenience); this appears to be the first result in the literature directly addressing Question 2.3 or Conjecture 1.2.

Theorem 2.5. *Suppose $M(n)$ is positive and recursively generated, $\{1,\ Z,\ \bar{Z},\ Z^2\}$ is independent, and there is a column relation $Z\bar{Z} = A1+BZ+C\bar{Z}+DZ^2$, $D \neq 0$. The following are equivalent for $\gamma^{(2n)}$.*
i) γ admits a rank $M(n)$-atomic representing measure;
ii) γ admits a representing measure;
iii) card $\mathcal{V}(\gamma) \geq$ rank $M(n)$.

Proof. The implications $i) \Rightarrow ii) \Rightarrow iii)$ are clear, so it suffices to prove $iii) \Rightarrow i)$. If $|D| \neq 1$, Proposition 1.6 (and its proof) imply that there exists a unique finitely atomic representing measure μ, and that *card* $\mathcal{V}(\gamma) \geq$ *card supp* $\mu = 4 =$ *rank* $M(n)$. In the sequel we may thus assume $|D| = 1$. We next reduce to the case $D = 1$. Let us write relation (1.1) in the form

$$Z\bar{Z} = \alpha_{00}1 + \alpha_{01}Z + \alpha_{10}\bar{Z} + \alpha_{02}Z^2, \tag{2.9}$$

where $\alpha_{02} = e^{i\psi}$, $0 \leq \psi < 2\pi$. Let $\theta = \psi/2$ and set $\lambda = e^{i\theta}$. Following [CF7], let $J_\lambda \in M_{m(n)}$ denote the invertible diagonal matrix whose entry in row $\bar{Z}^iZ^j$, column $\bar{Z}^iZ^j$ is $\bar{\lambda}^i\lambda^j$ $(0 \leq i+j \leq n)$; [CF7, Prop. 1.10-(i)] implies that $\tilde{M}(n) \equiv J_\lambda^* M(n) J_\lambda$ is the moment matrix corresponding to $\tilde{\gamma}^{(2n)}$, where $\tilde{\gamma}_{ij} = \bar{\lambda}^i\lambda^j\gamma_{ij}$ $(0 \leq i+j \leq 2n)$. [CF7, Prop. 1.7] shows that γ admits a representing measure μ if and only if $\tilde{\gamma}$ admits a representing measure $\tilde{\mu}$, where *supp* $\tilde{\mu} = \lambda$*supp* μ. In particular, $M(n)$ is positive if and only if $\tilde{M}(n)$ is positive; further, corresponding to a relation $\bar{Z}^iZ^j = \sum_{rs}\alpha_{rs}\bar{Z}^rZ^s$ in $\mathcal{C}_{M(n)}$, there is a relation in $\mathcal{C}_{\tilde{M}(n)}$ of the form

$$\bar{\tilde{Z}}^i\tilde{Z}^j = \sum_{rs}(\bar{\lambda}^{i-r}\lambda^{j-s}\alpha_{rs})\bar{\tilde{Z}}^r\tilde{Z}^s \tag{2.10}$$

[CF7, Prop. 1.10-(iii)]. Thus, corresponding to (2.9), there is a relation $\tilde{Z}\bar{\tilde{Z}} = \alpha_{00}\tilde{1} + \bar{\lambda}\alpha_{01}\tilde{Z} + \lambda\alpha_{10}\bar{\tilde{Z}} + \bar{\lambda}\lambda^{-1}\alpha_{02}\tilde{Z}^2$, and since $\bar{\lambda}\lambda^{-1}\alpha_{02} = 1$, in the sequel we may assume $D = 1$.

We now have $Z\bar{Z} = A1 + BZ + C\bar{Z} + Z^2$, so $\mathcal{V}(\gamma) \subset \mathcal{K} \equiv \{z : z\bar{z} = A1 + Bz + C\bar{z} + z^2\}$. Write $z = x+iy$, $A = A_1 + iA_2$, $B = B_1 + iB_2$, $C = C_1 + iC_2$. Then $\mathcal{K} = \{x+iy : re(x,y) = im(x,y) = 0\}$, where $re(x,y) = 2y^2 + (B_2 - C_2)y - (B_1 + C_1)x - A_1$ and $im(x,y) = 2xy + (B_2 + C_2)x + (B_1 - C_1)y + A_2$. If $B_1 + C_1 \neq 0$, then $re(x,y) = 0$ corresponds to a "horizontal" parabola of the form $x = p(y) \equiv Ey^2 + Fy + G$, and thus, substituting $x = p(y)$, $im(x,y)$ can be expressed as a cubic in y, whence *card* $\mathcal{K} \leq 3 <$ *rank* $M(n)$.

We may now assume $B_1 + C_1 = 0$, whence $re(x,y) = 2y^2 + (B_2 - C_2)y - A_1$; thus $re(x,y) = 0$ corresponds to 0, 1, or 2 horizontal lines, depending on the value of the discriminant $\delta \equiv (B_2 - C_2)^2 + 8A_1$. If $\delta < 0$, then clearly $\mathcal{K} = \emptyset$.

We next consider the case $\delta = 0$, which corresponds to $A_1 = -(B_2 - C_2)^2/8$. In this case, $re(x,y) = 0$ is the line $y = (C_2 - B_2)/4$, and thus, in $\mathcal{K}$, we have $im(x,y) = (3C_2 + B_2)x + 2A_2 + B_1(C_2 - B_2)$. If $B_2 + 3C_2 \neq 0$, then $im(x,y) = 0$ uniquely determines x, whence *card* $\mathcal{V}(\gamma) \leq$ *card* $\mathcal{K} = 1 <$ *rank* $M(n)$. If $B_2 + 3C_2 = 0$, then $2A_2 + B_1(C_2 - B_2) = (2/3)(3A_2 - 2B_1B_2)$; if, in addition, $3A_2 - 2B_1B_2 \neq 0$, then clearly $\mathcal{K} = \emptyset$.

To conclude with $\delta = 0$, we show that the case $B_2 + 3C_2 = 0$ and $3A_2 - 2B_1B_2 = 0$ cannot arise under the hypotheses that $M(n)$ is positive and $\{1, Z, \bar{Z}\}$ is independent, which hypotheses imply $det\ M(1)(\gamma) > 0$. Indeed, let $\gamma_{01} = a_1 + ia_2$, $\gamma_{11} = b$, $\gamma_{02} = c_1 + ic_2$. Since $\gamma_{11} = A\gamma_{00} + B\gamma_{01} + C\gamma_{10} + \gamma_{02}$, we have $c_1 = b - (A_1 + B_1a_1 - B_2a_2 + C_1a_1 + C_2a_2)$ and $c_2 = -A_2 - B_1a_2 - B_2a_1 - C_2a_1 + C_1a_2$. Using $C_1 = -B_1$, $A_1 = -(B_2 - C_2)^2/8$, $C_2 = -B_2/3$, $A_2 = (2/3)B_1B_2$, a calculation shows that $det\ M(1)(\gamma) = (-4/81)(3a_2 + B_2)^2\Delta$, where $\Delta = 9b + 18a_1B_1 + 9B_1^2 + 6a_2B_2 + B_2^2$. Now $M(n) \geq 0$ and $\{1, Z\}$ is independent, so $det \begin{pmatrix} 1 & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix} > 0$, whence $b > a_1^2 + a_2^2$. Thus $\Delta > 9(a_1 + B_1)^2 + (3a_2 + B_2)^2 \geq 0$, whence $det\ M(1)(\gamma) \leq 0$, a contradiction.

We next consider the case of $B_1 + C_1 = 0$ where $\delta > 0$, so $re(x, y) = 0$ corresponds to two horizontal lines, $y = y_1 \equiv (1/4)(C_2 - B_2 + \delta^{1/2})$ and $y = y_2 \equiv (1/4)(C_2 - B_2 - \delta^{1/2})$. Now $im(x, y) = 2xy + (B_2 + C_2)x + 2B_1y + A_2$, so $im(x, y) = 0$ represents either a proper hyperbola or a degenerate hyperbola consisting of intersecting lines. In the hyperbola case, for $y = y_1$, $im(x, y) = 0$ corresponds to $(2y_1 + B_2 + C_2)x = -A_2 + 2B_1y_1$, and since this relation comes from a proper hyperbola, $2y_1 + B_2 + C_2$ and $-A_2 + 2B_1y_1$ cannot both equal 0; thus there is at most one value of x such that $(x, y_1) \in \mathcal{K}$. A similar argument holds for $y = y_2$, so $card\ \mathcal{K} \leq 2$.

Finally, we consider the case when $im(x, y) = 0$ corresponds to a degenerate hyperbola (intersecting lines), which occurs precisely when $A_2 = B_1(B_2 + C_2)$. In this case, $im(x, y) = 0$ is equivalent to $(x + B_1)(y + (B_2 + C_2)/2) = 0$. Since $re(x, y) = 0$ consists of the distinct lines $y = y_1$ and $y = y_2$, it follows that $card\ \mathcal{K} \geq rank\ M(n) \geq 4$ if and only if $y + (B_2 + C_2)/2 = 0$ coincides with $y = y_1$ or with $y = y_2$. A calculation shows that this occurs if and only if $A_1 = C_2(B_2 + C_2)$ (in which case $\mathcal{K}$ includes a horizontal line).

To complete the proof, we will show that the conditions $B_1 + C_1 = 0$, $\delta > 0$, $A_2 = B_1(B_2 + C_2)$, $A_1 = C_2(B_2 + C_2)$ imply that Theorem 1.1-vi) is satisfied. We note for future reference that the preceding relations imply that $R \equiv A_1B_1 - C_2A_2$, $S \equiv A_1 + B_1^2 + B_1C_1 - B_2C_2 - C_2^2$, $T \equiv A_2 - B_1B_2 + C_1C_2$ satisfy $R = S = T = 0$.

Theorem 1.1-vi) is equivalent to the real system

$$Im\gamma_{n,n+1} = -(1/2)(-A_1 Im\gamma_{n-1,n} + A_2 Re\gamma_{n-1,n} + B_2\gamma_{n,n} + C_2 Re\gamma_{n-1,n+1} - C_1 Im\gamma_{n-1,n+1}) \quad (2.11)$$

and

$$0 = (A_1 Re\gamma_{n-1,n} + A_2 Im\gamma_{n-1,n} + B_1\gamma_{n,n} + C_1 Re\gamma_{n-1,n+1} + C_2 Im\gamma_{n-1,n+1}) \quad (2.12)$$

(2.11) merely shows how to define $Im\gamma_{n,n+1}$, so to complete the proof it remains to show that (2.12) holds (in which case a free choice for $Re\gamma_{n,n+1}$ determines infintely many distinct flat extensions $M(n+1)$).

From the relation $\bar{Z}Z = A1 + BZ + C\bar{Z} + Z^2$, recursiveness implies $\bar{Z}Z^{n-1} = AZ^{n-2} + BZ^{n-1} + C\bar{Z}Z^{n-2} + Z^n$. Thus $\langle \bar{Z}Z^{n-1}, Z^{n-1}\bar{Z}\rangle = A\langle Z^{n-2}, Z^{n-1}\bar{Z}\rangle +$

$B\langle Z^{n-1}, Z^{n-1}\bar{Z}\rangle + C\langle \bar{Z}Z^{n-2}, Z^{n-1}\bar{Z}\rangle + \langle Z^n, Z^{n-1}\bar{Z}\rangle$, or $\gamma_{n,n} = A\gamma_{n-1,n-1} + B\gamma_{n-1,n} + C\gamma_{n,n-1} + \gamma_{n-1,n+1}$. The last relation is equivalent to the real system

$$\gamma_{n,n} = A_1\gamma_{n-1,n-1} + B_1 Re\gamma_{n-1,n} - B_2 Im\gamma_{n-1,n} + C_1 Re\gamma_{n-1,n} + C_2 Im\gamma_{n-1,n} + Re\gamma_{n-1,n+1} \quad (2.13)$$

and

$$0 = A_2\gamma_{n-1,n-1} + B_2 Re\gamma_{n-1,n} + B_1 Im\gamma_{n-1,n} + C_2 Re\gamma_{n-1,n} - C_1 Im\gamma_{n-1,n} + Im\gamma_{n-1,n+1} \quad (2.14)$$

Substituting in (2.12) for $\gamma_{n,n}$ (from (2,13)) and for $Im\gamma_{n-1,n+1}$ (from (2.14)), we see that (2.12) is equivalent to $R\gamma_{n-1,n-1} + S\ Re\gamma_{n-1,n} + T\ Im\gamma_{n-1,n} = 0$, and since $R = S = T = 0$, the proof is now complete. $\square$.

3. Existence of representing measures

In this section we complete the proof of Theorem 1.1 by proving that vi) $\Rightarrow$ iv). In the sequel, $n \geq 2$, $M(n)$ is positive and recursively generated, $\{1, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(n)}$, and there is a dependence relation of the form $\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2$, $D \neq 0$. We will use the hypothesis of Theorem 1.1-vi) to prove that $M(n)$ admits a flat extension $M(n+1)$.

We begin with a proof of Proposition 1.8. Recall that $\{1, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(n)}$. Let q be the largest integer, $2 \leq q \leq n$, such that $\mathcal{B} \equiv \mathcal{B}_q = \{1, Z, \bar{Z}, Z^2, \ldots, Z^i, \ldots, Z^q\}$ is independent; if $q < n$, there exist unique scalars $\alpha_0, \alpha_1, \beta_1, \alpha_2, \ldots, \alpha_q$ such that $Z^{q+1} = \alpha_0 1 + \alpha_1 Z + \beta_1\bar{Z} + \alpha_2 Z^2 + \cdots + \alpha_q Z^q$. For $0 \leq j \leq n-1$, let $\langle\mathcal{B}\rangle_j$ denote the subspace of $\mathcal{C}_{M(n)}$ spanned by elements of $\mathcal{B}$ having degree $\leq j$.

Lemma 3.1. *If $0 \leq j \leq n-1$ and $V \in \langle\mathcal{B}\rangle_j$, then $ZV \in \langle\mathcal{B}\rangle_{j+1}$.*

Proof. The result is obvious for $j = 0$. If $V \in \langle\mathcal{B}\rangle_1$, then $V = a_0 1 + a_1 Z + b_1\bar{Z}$, so $ZV = a_0 Z + a_1 Z^2 + b_1 Z\bar{Z} = b_1 A1 + (a_0 + b_1 B)Z + b_1 C\bar{Z} + (a_1 + b_1 D)Z^2 \in \langle\mathcal{B}\rangle_2$. Let $V \in \langle\mathcal{B}\rangle_j$, $j \geq 2$, $V = a_0 1 + a_1 Z + b_1\bar{Z} + a_2 Z^2 + \cdots + a_j Z^j$ (where, if $j > q$, $a_{q+1} = \cdots = a_j = 0$). Then $ZV = a_0 Z + a_1 Z^2 + b_1(A1 + BZ + C\bar{Z} + DZ^2) + a_2 Z^3 + \cdots + a_j Z^{j+1}$. If $j+1 \leq q$, then clearly $ZV \in \langle\mathcal{B}\rangle_{j+1}$. If $q \leq j$, then $V = a_0 1 + a_1 Z + b_1\bar{Z} + a_2 Z^2 + \cdots + a_{q-1}Z^{q-1} + a_q Z^q$, so $ZV = a_0 Z + a_1 Z^2 + b_1(A + BZ + C\bar{Z} + DZ^2) + a_2 Z^3 + \cdots + a_{q-1}Z^q + a_q(\alpha_0 1 + \alpha_1 Z + \beta_1\bar{Z} + \alpha_2 Z^2 + \cdots \alpha_q Z^q) \in \langle\mathcal{B}\rangle_q \subset \langle\mathcal{B}\rangle_j$.$\square$

The following result implies Proposition 1.8.

Proposition 3.2. *$\mathcal{B}$ is a basis for $\mathcal{C}_{M(n)}$; each non-$\mathcal{B}$ column vector of degree p is in $\langle\mathcal{B}\rangle_p$, $0 \leq p \leq n$.*

Proof. The proof is by induction on p; the result is vacuous for $p = 0, 1$ and is true for $p = 2$ since $Z\bar{Z} = A1 + BZ + C\bar{Z} + DZ^2$ and $\bar{Z}^2 = (1/\bar{D})(A1 + BZ +$

$C\bar{Z}+DZ^2-(\bar{A}1+\bar{B}\bar{Z}+\bar{C}Z))$. Assume the result is true for $p=0,1,2,\ldots,k-1$, $2\le k-1\le n-1$, and let $V\equiv \bar{Z}^iZ^j$ be a non-$\mathcal{B}$ vector with $i+j=k$. Suppose first that $i>1$. Then $W\equiv \bar{Z}^{i-1}Z^j$ has degree $k-1$ (≥ 2), and since $i-1>0$, $W\notin\mathcal{B}$. By induction, $W\in\langle\mathcal{B}\rangle_{k-1}$, so W has the form $W=a_01+a_1Z+b_1\bar{Z}+a_2Z^2+\cdots+a_{k-1}Z^{k-1}$ (where, if $q<k-1$, then $a_{q+1}=\cdots=a_{k-1}=0$). Now $V=\bar{Z}^iZ^j=\bar{Z}W=a_0\bar{Z}+a_1\bar{Z}Z+b_1\bar{Z}^2+a_2\bar{Z}Z^2+\cdots+a_{k-1}\bar{Z}Z^{k-1}=X+a_2Z(\bar{Z}Z)+\cdots+a_{k-1}Z(\bar{Z}Z^{k-2})$, where, clearly, $X\equiv a_0\bar{Z}+a_1\bar{Z}Z+b_1\bar{Z}^2\in\langle\mathcal{B}\rangle_2\subset\langle\mathcal{B}\rangle_k$. For $1\le r\le k-2$, $\bar{Z}Z^r$ is a non-$\mathcal{B}$ vector of degree $r+1$ $(\le k-1)$, so by induction, $\bar{Z}Z^r\in\langle\mathcal{B}\rangle_{r+1}$. Since $r+1\le k-1\le n-1$, Lemma 3.1 implies $\bar{Z}Z^{r+1}=Z(\bar{Z}Z^r)\in\langle\mathcal{B}\rangle_{r+2}\subset\langle\mathcal{B}\rangle_k$, $1\le r\le k-2$, so it follows that $V\in\langle\mathcal{B}\rangle_k$. For the case when $i=1$, we have $j\ge 2$ and $V=\bar{Z}Z^j=Z(\bar{Z}Z^{j-1})$. Since $\bar{Z}Z^{j-1}$ is a non-$\mathcal{B}$ vector of degree j $(=k-1)$, the result follows by induction and by an application of Lemma 3.1. □

As discussed in Section 1, if $q<n$ in Proposition 1.8, then $M(n)$ is flat, so the existence of a flat extension follows from Theorem 1.13. In the sequel we may thus assume that $q=n$ and that $\mathcal{J}\equiv\{1,Z,\bar{Z},Z^2,...,Z^i,...,Z^n\}$ is a basis for $\mathcal{C}_{M(n)}$; this considerably simplifies the proof at one point that we note below.

To prove Theorem 1.1 $vi\Rightarrow iv$), our first goal is to define columns

$$Z^{n+1}\equiv(\gamma_{0,n+1},\gamma_{1,n+1},\gamma_{0,n+2},\ldots,\gamma_{i,n+1},\ldots,\gamma_{0,n+i+1},\ldots,\gamma_{n,n+1},\ldots,\gamma_{0,2n+1})^t,$$

$$\bar{Z}Z^n\equiv(\gamma_{1,n},\gamma_{2,n},\gamma_{1,n+1},\ldots,\gamma_{i+1,n},\ldots,\gamma_{1,n+i},\ldots,\gamma_{n+1,n},\ldots,\gamma_{1,2n})^t$$

for block $B\equiv B(n+1)$ of a recursively generated extension $M(n+1)$ of $M(n)$. Since $M(n+1)$ is to be recursively generated, (1.1) implies that in $\mathcal{C}_{M(n+1)}$, the column space of $M(n+1)$, we must have

$$\bar{Z}Z^n=AZ^{n-1}+BZ^n+C\bar{Z}Z^{n-1}+DZ^{n+1}. \tag{3.1}$$

In particular, in the column space of B we require

$$[\bar{Z}Z^n]_n=A[Z^{n-1}]_n+B[Z^n]_n+C[\bar{Z}Z^{n-1}]_n+D[Z^{n+1}]_n. \tag{3.2}$$

Now $M(n)$ is recursively generated, so we already have

$$[\bar{Z}Z^n]_{n-1}=A[Z^{n-1}]_{n-1}+B[Z^n]_{n-1}+C[\bar{Z}Z^{n-1}]_{n-1}+D[Z^{n+1}]_{n-1}$$

(i.e., the moment relations implicit in this equation can be established from the relation $\bar{Z}Z^{n-1}=AZ^{n-2}+BZ^{n-1}+C\bar{Z}Z^{n-2}+DZ^n$ in $\mathcal{C}_{M(n)}$). To establish (3.2) we must thus define certain new moments of degree $2n+1$,

$$\gamma_{n+1,n},\gamma_{n,n+1},\ \gamma_{n-1,n+2},...,\gamma_{i,2n+1-i},...,\gamma_{0,2n+1},$$

with $\gamma_{n+1,n}\equiv\bar{\gamma}_{n,n+1}$, such that

$$\gamma_{i+1,2n-i}=A\gamma_{i,2n-i-1}+B\gamma_{i,2n-i}+C\gamma_{i+1,2n-i-1}+D\gamma_{i,2n-i+1}\ (0\le i\le n). \tag{3.3}$$

The hypothesis of Theorem 1.1-vi) is that (3.3) holds for $i=n$, i.e., there exists $\gamma_{n,n+1}\in\mathbf{C}$ such that

$$\gamma_{n+1,n}\equiv\bar{\gamma}_{n,n+1}=A\gamma_{n,n-1}+B\gamma_{n,n}+C\gamma_{n+1,n-1}+D\gamma_{n,n+1}. \tag{3.4}$$

We now use (3.3) and (3.4) to define the remaining $\gamma_{i,2n-i+1}$ successively:

$$\gamma_{i,2n-i+1} = (1/D)(\gamma_{i+1,2n-i} - A\gamma_{i,2n-i-1} - B\gamma_{i,2n-i} - C\gamma_{i+1,2n-i-1}), \quad i = n-1,\ n-2, ..., 0. \tag{3.5}$$

The new moments (together with certain "old" moment data from $M(n)$) define B-block columns Z^{n+1} and $\bar{Z}Z^n$ which satisfy (3.2). Our next goal is to show that these columns belong to *Ran* $M(n)$, as required for columns in the B-block of a positive extension $M(n+1)$ (cf. (2.5)). Note that the compression of $M(n)$ to rows and columns indexed by the elements of $\mathcal{J}$ is positive and invertible. Let $[v]_{\mathcal{J}}$ denote the compression of a column of $M(n)$ or a column of B to components indexed by the elements of $\mathcal{J}$; $[v]_{\mathcal{J}}$ consists of the components of v in rows $1, Z, \bar{Z}, Z^2, ..., Z^i, ..., Z^n$. It follows that there exist unique scalars, $a_0, a_1, b_1, a_2, a_3, ..., a_n$, such that

$$[\bar{Z}Z^n]_{\mathcal{J}} = a_0[1]_{\mathcal{J}} + a_1[Z]_{\mathcal{J}} + b_1[\bar{Z}]_{\mathcal{J}} + a_2[Z^2]_{\mathcal{J}} + \cdots + a_n[Z^n]_{\mathcal{J}}, \tag{3.6}$$

or, equivalently,

$$\textit{for each } F \in \mathcal{J},$$
$$\langle \bar{Z}Z^n, F\rangle = a_0\langle 1, F\rangle + a_1\langle Z, F\rangle + b_1\langle \bar{Z}, F\rangle + a_2\langle Z^2, F\rangle + \cdots + a_n\langle Z^n, F\rangle. \tag{3.7}$$

We next show that $\bar{Z}Z^n$ (as defined above) satisfies $\bar{Z}Z^n \in$ *Ran* $M(n)$.

Lemma 3.3. *In the column space of* $\begin{pmatrix} M(n) & B \end{pmatrix}$, $\bar{Z}Z^n = a_0 1 + a_1 Z + b_1 \bar{Z} + a_2 Z^2 + \cdots + a_n Z^n$.

Proof. In terms of inner products, what we seek to prove may be expressed as follows:

$$\textit{For } 0 \le i+j \le n, \langle \bar{Z}Z^n, \bar{Z}^i Z^j\rangle = a_0\langle 1, \bar{Z}^i Z^j\rangle + a_1\langle Z, \bar{Z}^i Z^j\rangle + b_1\langle \bar{Z}, \bar{Z}^i Z^j\rangle + a_2\langle Z^2, \bar{Z}^i Z^j\rangle + \cdots + a_n\langle Z^n, \bar{Z}^i Z^j\rangle. \tag{3.8}$$

The proof of (3.8) is by induction on the *level number* $p \equiv i+j$, $0 \le p \le n$. Since $1, Z, \bar{Z} \in \mathcal{J}$, it follows from (3.6) that (3.8) holds for $p = 0$ and $p = 1$. We assume that $1 \le k-1 \le n-1$ and that (3.8) holds for $p = 0, 1, \dots k-1$, and we next prove that (3.8) holds for $p = k$; this is equivalent to the following:

$$\textit{For each } r,\ 0 \le r \le k,\ \gamma_{k-r+1,n+r}\ (= \langle \bar{Z}Z^n, \bar{Z}^r Z^{k-r}\rangle) = a_0\langle 1, \bar{Z}^r Z^{k-r}\rangle + a_1\langle Z, \bar{Z}^r Z^{k-r}\rangle + b_1\langle \bar{Z}, \bar{Z}^r Z^{k-r}\rangle + a_2\langle Z^2, \bar{Z}^r Z^{k-r}\rangle + \cdots + a_n\langle Z^n, \bar{Z}^r Z^{k-r}\rangle. \tag{3.9}$$

To prove (3.9), we introduce some notation; for $0 \le r \le k$, let $F(k,r) = a_0\gamma_{k-r,r} + a_1\gamma_{k-r,r+1} + b_1\gamma_{k-r+1,r} + a_2\gamma_{k-r,r+2} + \cdots + a_n\gamma_{k-r,r+n}$; $F(k,r)$ coincides with the right hand side of the equation in (3.9); thus (3.9) is equivalent to showing that $F(k,r) = \gamma_{k-r+1,n+r}$, $0 \le r \le k$. The moments appearing in $F(k,r)$ are components of column $Z^r \bar{Z}^{k-r}$; indeed, we have

$$F(k,r) = a_0\langle Z^r \bar{Z}^{k-r}, 1\rangle + a_1\langle Z^r \bar{Z}^{k-r}, \bar{Z}\rangle + b_1\langle Z^r \bar{Z}^{k-r}, Z\rangle + a_2\langle Z^r \bar{Z}^{k-r}, \bar{Z}^2\rangle + \cdots + a_n\langle Z^r \bar{Z}^{k-r}, \bar{Z}^n\rangle. \quad (3.10)$$

We also require an identity which follows from recursiveness:

For $1 \le r < k$,

$$\begin{aligned} Z^r \bar{Z}^{k-r} &= Z^{r-1}\bar{Z}^{k-r-1}(Z\bar{Z}) = Z^{r-1}\bar{Z}^{k-r-1}(A1 + BZ + C\bar{Z} + DZ^2) \\ &= AZ^{r-1}\bar{Z}^{k-r-1} + BZ^r\bar{Z}^{k-r-1} + CZ^{r-1}\bar{Z}^{k-r} + DZ^{r+1}\bar{Z}^{k-r-1} \quad (3.11) \end{aligned}$$

Further, since $\bar{Z}Z^n = AZ^{n-1} + BZ^n + C\bar{Z}Z^{n+1} + DZ^{n+1}$, then, for $0 \le r \le k$,

$$\begin{aligned} \gamma_{k-r+1,n+r} &= \langle \bar{Z}Z^n, Z^{k-r}\bar{Z}^r\rangle \\ &= A\gamma_{k-r,n+r-1} + B\gamma_{k-r,n+r} + C\gamma_{k-r+1,n+r-1} + D\gamma_{k-r,n+r+1} \quad (3.12) \end{aligned}$$

The proof of (3.9) is by induction on r, $0 \le r \le k$. It follows from (3.7) (with $F = Z^k$) that (3.9) holds for $r = 0$. (It is at this point that we are using the fact that $q = n$ in Proposition 1.7, which guarantees that $Z^k \in \mathcal{J}$; in the case $q < n$, we would require a separate argument for the base case $r = 0$.) The induction on r is organized as follows. We first show that

$$\textit{For } 1 \le r \le k-1,\ F(k,r+1) - \gamma_{k-r,n+r+1} = (1/D)(F(k,r) - \gamma_{k-r+1,n+r}). \quad (3.13)$$

Thus, (3.13) reduces the induction to the case $r = 1$, i.e., to showing that $F(k,1) = \gamma_{k,n+1}$; this utilizes the base case $r = 0$ and will be the last step of the proof.

We now procede to prove (3.13). From (3.10) and (3.11) we have

$$\begin{aligned} F(k,r) &= a_0\langle Z^r \bar{Z}^{k-r}, 1\rangle + a_1\langle Z^r \bar{Z}^{k-r}, \bar{Z}\rangle + b_1\langle Z^r \bar{Z}^{k-r}, Z\rangle \\ &\quad + a_2\langle Z^r \bar{Z}^{k-r}, \bar{Z}^2\rangle + \cdots + a_n\langle Z^r \bar{Z}^{k-r}, \bar{Z}^n\rangle \\ &= a_0\langle AZ^{r-1}\bar{Z}^{k-r-1} + BZ^r\bar{Z}^{k-r-1} + CZ^{r-1}\bar{Z}^{k-r} + DZ^{r+1}\bar{Z}^{k-r-1}, 1\rangle \\ &\quad + a_1\langle AZ^{r-1}\bar{Z}^{k-r-1} + BZ^r\bar{Z}^{k-r-1} + CZ^{r-1}\bar{Z}^{k-r} + DZ^{r+1}\bar{Z}^{k-r-1}, \bar{Z}\rangle \\ &\quad + b_1\langle AZ^{r-1}\bar{Z}^{k-r-1} + BZ^r\bar{Z}^{k-r-1} + CZ^{r-1}\bar{Z}^{k-r} + DZ^{r+1}\bar{Z}^{k-r-1}, Z\rangle \\ &\quad + a_2\langle AZ^{r-1}\bar{Z}^{k-r-1} + BZ^r\bar{Z}^{k-r-1} + CZ^{r-1}\bar{Z}^{k-r} + DZ^{r+1}\bar{Z}^{k-r-1}, \bar{Z}^2\rangle \\ &\quad + \cdots + a_n\langle AZ^{r-1}\bar{Z}^{k-r-1} + BZ^r\bar{Z}^{k-r-1} + CZ^{r-1}\bar{Z}^{k-r} + DZ^{r+1}\bar{Z}^{k-r-1}, \bar{Z}^n\rangle \\ &= A(a_0\gamma_{k-r-1,r-1} + a_1\gamma_{k-r-1,r} + b_1\gamma_{k-r,r-1} + a_2\gamma_{k-r-1,r+1} + \cdots + a_n\gamma_{k-r-1,r+n-1}) \\ &\quad + B(a_0\gamma_{k-r-1,r} + a_1\gamma_{k-r-1,r+1} + b_1\gamma_{k-r,r} + a_2\gamma_{k-r-1,r+2} + \cdots + a_n\gamma_{k-r-1,r+n}) \\ &\quad + C(a_0\gamma_{k-r,r-1} + a_1\gamma_{k-r,r} + b_1\gamma_{k-r+1,r-1} + a_2\gamma_{k-r,r+1} + \cdots + a_n\gamma_{k-r,r+n-1}) \\ &\quad + D(a_0\gamma_{k-r-1,r+1} + a_1\gamma_{k-r-1,r+2} + b_1\gamma_{k-r,r+1} \\ &\quad + a_2\gamma_{k-r-1,r+3} + \cdots + a_n\gamma_{k-r-1,r+n+1}). \end{aligned}$$

Thus, $F(k,r) = AF(k-2,r-1) + BF(k-1,r) + CF(k-1,r-1) + DF(k,r+1)$. By induction on k, and using (3.12), it follows that $F(k,r+1) - \gamma_{k-r,r+n+1} = (1/D)(F(k,r) - (A\gamma_{k-r,n+r-1} + B\gamma_{k-r,n+r} + C\gamma_{k-r+1,n+r-1} + D\gamma_{k-r,n+r+1})) = (1/D)(F(k,r) - \gamma_{k-r+1,n+r})$; thus (3.13) holds.

To complete the proof of (3.9), it now remains to prove that (3.9) holds for $r = 1$, i.e., $F(k,1) = \gamma_{k,n+1}$. We have $F(k,1) = a_0\gamma_{k-1,1} + a_1\gamma_{k-1,2} + b_1\gamma_{k,1} + a_2\gamma_{k-1,3} + \cdots + a_n\gamma_{k-1,n+1}$, so the moments in $F(k,1)$ all appear in $Z\bar{Z}^{k-1}$. By recursiveness, this column may be expressed as $Z\bar{Z}^{k-1} = \bar{Z}^{k-2}Z\bar{Z} = \bar{Z}^{k-2}(\bar{A}1 + \bar{B}\bar{Z} + \bar{C}Z + \bar{D}\bar{Z}^2) = \bar{A}\bar{Z}^{k-2} + \bar{B}\bar{Z}^{k-1} + \bar{C}Z\bar{Z}^{k-2} + \bar{D}\bar{Z}^k$. Thus,

$$
\begin{aligned}
F(k,1) = a_0\langle Z\bar{Z}^{k-1},1\rangle &+ a_1\langle Z\bar{Z}^{k-1},\bar{Z}\rangle + b_1\langle Z\bar{Z}^{k-1},Z\rangle \\
&+ a_2\langle Z\bar{Z}^{k-1},\bar{Z}^2\rangle + \cdots + a_n\langle Z\bar{Z}^{k-1},\bar{Z}^n\rangle \\
&= a_0\langle \bar{A}\bar{Z}^{k-2} + \bar{B}\bar{Z}^{k-1} + \bar{C}Z\bar{Z}^{k-2} + \bar{D}\bar{Z}^k, 1\rangle \\
&+ a_1\langle \bar{A}\bar{Z}^{k-2} + \bar{B}\bar{Z}^{k-1} + \bar{C}Z\bar{Z}^{k-2} + \bar{D}\bar{Z}^k, \bar{Z}\rangle \\
&+ b_1\langle \bar{A}\bar{Z}^{k-2} + \bar{B}\bar{Z}^{k-1} + \bar{C}Z\bar{Z}^{k-2} + \bar{D}\bar{Z}^k, Z\rangle \\
&+ a_2\langle \bar{A}\bar{Z}^{k-2} + \bar{B}\bar{Z}^{k-1} + \bar{C}Z\bar{Z}^{k-2} + \bar{D}\bar{Z}^k, \bar{Z}^2\rangle \\
&+ \cdots + a_n\langle \bar{A}\bar{Z}^{k-2} + \bar{B}\bar{Z}^{k-1} + \bar{C}Z\bar{Z}^{k-2} + \bar{D}\bar{Z}^k, \bar{Z}^n\rangle \\
&= a_0(\bar{A}\gamma_{k-2,0} + \bar{B}\gamma_{k-1,0} + \bar{C}\gamma_{k-2,1} + \bar{D}\gamma_{k,0}) \\
&+ a_1(\bar{A}\gamma_{k-2,1} + \bar{B}\gamma_{k-1,1} + \bar{C}\gamma_{k-2,2} + \bar{D}\gamma_{k,1}) \\
&+ b_1(\bar{A}\gamma_{k-1,0} + \bar{B}\gamma_{k,0} + \bar{C}\gamma_{k-1,1} + \bar{D}\gamma_{k+1,0}) \\
&+ a_2(\bar{A}\gamma_{k-2,2} + \bar{B}\gamma_{k-1,2} + \bar{C}\gamma_{k-2,3} + \bar{D}\gamma_{k,2}) \\
&+ \cdots + a_n(\bar{A}\gamma_{k-2,n} + \bar{B}\gamma_{k-1,n} + \bar{C}\gamma_{k-2,n+1} + \bar{D}\gamma_{k,n}) \\
&= \bar{A}(a_0\gamma_{k-2,0} + a_1\gamma_{k-2,1} + b_1\gamma_{k-1,0} + a_2\gamma_{k-2,2} + \cdots + a_n\gamma_{k-2,n}) \\
&+ \bar{B}(a_0\gamma_{k-1,0} + a_1\gamma_{k-1,1} + b_1\gamma_{k,0} + a_2\gamma_{k-1,2} + \cdots + a_n\gamma_{k-1,n}) \\
&+ \bar{C}(a_0\gamma_{k-2,1} + a_1\gamma_{k-2,2} + b_1\gamma_{k-1,1} + a_2\gamma_{k-2,3} + \cdots + a_n\gamma_{k-2,n+1}) \\
&+ \bar{D}(a_0\gamma_{k,0} + a_1\gamma_{k,1} + b_1\gamma_{k+1,0} + a_2\gamma_{k,2} + \cdots + a_n\gamma_{k,n}) \\
&= \bar{A}F(k-2,0) + \bar{B}F(k-1,0) + \bar{C}F(k-1,1) + \bar{D}F(k,0).
\end{aligned}
$$

By induction on k, and using the base case of k when $r = 0$, the last expression coincides with $G \equiv \bar{A}\gamma_{k-1,n} + \bar{B}\gamma_{k,n} + \bar{C}\gamma_{k-1,n+1} + \bar{D}\gamma_{k+1,n}$.

To show that $G = \gamma_{k,n+1}$, we first consider the case $k < n$. By recursiveness, $Z^2\bar{Z}^{k-1} = Z\bar{Z}^{k-2}(Z\bar{Z}) = Z\bar{Z}^{k-2}(\bar{A}1 + \bar{B}\bar{Z} + \bar{C}Z + \bar{D}\bar{Z}^2) = \bar{A}Z\bar{Z}^{k-2} + \bar{B}Z\bar{Z}^{k-1} + \bar{C}Z^2\bar{Z}^{k-2} + \bar{D}Z\bar{Z}^k$, so $\gamma_{k,n+1} = \langle Z^2\bar{Z}^{k-1}, Z\bar{Z}^{n-1}\rangle = \langle \bar{A}Z\bar{Z}^{k-2} + \bar{B}Z\bar{Z}^{k-1} + \bar{C}Z^2\bar{Z}^{k-2} + \bar{D}Z\bar{Z}^k, Z\bar{Z}^{n-1}\rangle = G$. Thus $F(k,1) = \gamma_{k,n+1}$ when $k < n$. In the case $k = n$, we have $F(n,1) = G = \bar{A}\gamma_{n-1,n} + \bar{B}\gamma_{n,n} + \bar{C}\gamma_{n-1,n+1} + \bar{D}\gamma_{n+1,n}$, so (3.4) immediately implies that $F(n,1) = \gamma_{n,n+1}$. The proof of Lemma 3.3 is now complete. □

In view of Lemma 3.3 and (3.1), we now have columns Z^{n+1}, $\bar{Z}Z^n \in Ran\ M(n)$. We may thus successively define the remaining columns for block B by utilizing (1.1) and recursiveness:

$$\bar{Z}^2Z^{n-1} = A\bar{Z}Z^{n-2} + B\bar{Z}Z^{n-1} + C\bar{Z}^2Z^{n-2} + D\bar{Z}Z^n,$$

$$\bar{Z}^3Z^{n-2} = A\bar{Z}^2Z^{n-3} + B\bar{Z}^2Z^{n-2} + C\bar{Z}^3Z^{n-3} + D\bar{Z}^2Z^{n-1}, \cdots$$

$$\cdots,\ \bar{Z}^nZ = A\bar{Z}^{n-1} + B\bar{Z}^{n-1}Z + C\bar{Z}^n + D\bar{Z}^{n-1}Z^2,$$

and

$$\bar{Z}^{n+1} = (1/\bar{D})(\bar{Z}^nZ - \bar{A}\bar{Z}^{n-1} - \bar{B}\bar{Z}^n - \bar{C}\bar{Z}^{n-1}Z),$$

Since Z^{n+1}, $\bar{Z}Z^n \in Ran\ M(n)$, it follows that $\bar{Z}^kZ^l \in Ran\ M(n)$, $(k+l = n+1)$.

Having defined columns of order $n+1$ as above, which define a block B, we must show that B has the structure of a moment matrix block $B(n+1)$. Since $M(n)$ is recursively generated, the defining column relations for B given above readily imply that B has the form

$$B = \begin{pmatrix} B_{0,n+1} \\ B_{1,n+1} \\ \vdots \\ B_{n-1,n+1} \\ B[n,n+1] \end{pmatrix},$$

where, for $0 \le j \le n-1$, $B_{j,n+1}$ is a moment matrix block consisting of "old data" of order $j+n+1$. To show that B is of the form $B(n+1)$, it thus suffices to verify that $B[n,n+1]$ has the following form of a moment matrix block $B_{n,n+1}$:

$$\begin{array}{ccccc} Z^{n+1} & \bar{Z}Z^n & \bar{Z}^2Z^{n-1} & \dots & \bar{Z}^{n+1} \end{array}$$

$$\begin{pmatrix} \gamma_{n,n+1} & \gamma_{n+1,n} & \gamma_{n+2,n-1} & \cdots & \gamma_{2n+1,0} \\ \gamma_{n-1,n+2} & \gamma_{n,n+1} & \gamma_{n+1,n} & \cdots & \gamma_{2n,1} \\ \gamma_{n-2,n+3} & \gamma_{n-1,n+2} & \gamma_{n,n+1} & \cdots & \gamma_{2n-1,2} \\ \vdots & & & & \vdots \\ \gamma_{0,2n+1} & \gamma_{1,2n} & \gamma_{2,2n-1} & \cdots & \gamma_{n+1,n} \end{pmatrix},$$

where $\gamma_{ji} = \bar{\gamma}_{ji}$ To establish that B has the required form, we will prove the following two properties:

$$\gamma_{n+i,n+1-i} \equiv \langle Z^{n+1-i}\bar{Z}^i, Z^n\rangle = \langle Z^{n+1}, Z^{n-i+1}\bar{Z}^{i-1}\rangle^-,$$
$$i.e.,\ \gamma_{n+i,n+1-i} = \bar{\gamma}_{n+1-i,n+i},\ 1 \le i \le n+1 \quad (3.14)$$

$$B[n,n+1]\ is\ constant\ on\ diagonals. \quad (3.15)$$

Lemma 3.4. $\langle Z^{n+1-i}\bar{Z}^i, Z^n\rangle = \langle Z^{n+1}, Z^{n-i+1}\bar{Z}^{i-1}\rangle^-$, *i.e.*, $\gamma_{n+i,n+1-i} = \bar{\gamma}_{n+1-i,n+i}$, $1 \le i \le n+1$.

Proof. The proof is by induction on i. The identity holds for $i = 1$ by (3.4). We assume the identity holds for $i = 1, \ldots, j-1 < n$ and we next establish (3.14) for $i = j$. (We treat the case $i = n+1$ separately at the end.) We have

$$\bar{Z}^j Z^{n+1-j} = A\bar{Z}^{j-1}Z^{n-j} + B\bar{Z}^{j-1}Z^{n-j+1} + C\bar{Z}^j Z^{n-j} + D\bar{Z}^{j-1}Z^{n-j+2},$$

whence (by considering level Z^n)

$$\gamma_{j+n,n+1-j} = A\gamma_{n+j-1,n-j} + B\gamma_{n+j-1,n-j+1} + C\gamma_{n+j,n-j} + D\gamma_{n+j-1,n-j+2}. \tag{3.16}$$

Our first goal is to provide identities for the terms on the right hand side of (3.16). By recursiveness and conjugation in $\mathcal{C}_{M(n)}$ (cf. (1.2)),

$$\bar{Z}^n = (1/\bar{D})(\bar{Z}^{n-1}Z - \bar{A}\bar{Z}^{n-2} - \bar{B}\bar{Z}^{n-1} - \bar{C}\bar{Z}^{n-2}Z,$$

whence

$$\begin{aligned}\gamma_{n+j-1,n-j} &= \langle \bar{Z}^n, \bar{Z}^{n-j}Z^{j-1}\rangle \\ &= (1/\bar{D})(\gamma_{n+j-2,n-j+1} - \bar{A}\gamma_{n+j-3,n-j} - \bar{B}\gamma_{n+j-2,n-j} - \bar{C}\gamma_{n+j-3,n-j+1}),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\gamma_{n+j-1,n-j+1} &= \langle \bar{Z}^n, \bar{Z}^{n-j+1}Z^{j-1}\rangle \\ &= (1/\bar{D})(\gamma_{n+j-2,n-j+2} - \bar{A}\gamma_{n+j-3,n-j+1} - \bar{B}\gamma_{n+j-2,n-j+1} - \bar{C}\gamma_{n+j-3,n-j+2}),\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\gamma_{n+j,n-j} &= \langle \bar{Z}^n, \bar{Z}^{n-j}Z^{j}\rangle \\ &= (1/\bar{D})(\gamma_{n+j-1,n-j+1} - \bar{A}\gamma_{n+j-2,n-j} - \bar{B}\gamma_{n+j-1,n-j} - \bar{C}\gamma_{n+j-2,n-j+1}).\end{aligned} \tag{3.19}$$

Substituting (3.17)-(3.19) into (3.16) yields

$$\begin{aligned}\gamma_{j+n,n+1-j} &= D\gamma_{n+j-1,n-j+2} \\ &\quad + (A/\bar{D})(\gamma_{n+j-2,n-j+1} - \bar{A}\gamma_{n+j-3,n-j} - \bar{B}\gamma_{n+j-2,n-j} - \bar{C}\gamma_{n+j-3,n-j+1}) \\ &\quad + (B/\bar{D})(\gamma_{n+j-2,n-j+2} - \bar{A}\gamma_{n+j-3,n-j+1} - \bar{B}\gamma_{n+j-2,n-j+1} - \bar{C}\gamma_{n+j-3,n-j+2}) \\ &\quad + (C/\bar{D})(\gamma_{n+j-1,n-j+1} - \bar{A}\gamma_{n+j-2,n-j} - \bar{B}\gamma_{n+j-1,n-j} - \bar{C}\gamma_{n+j-2,n-j+1}) \\ &= D\gamma_{n+j-1,n-j+2} + (1/\bar{D})(A\gamma_{n+j-2,n-j+1} + B\gamma_{n+j-2,n-j+2} + C\gamma_{n+j-1,n-j+1}) \\ &\qquad - (\bar{A}/\bar{D})(A\gamma_{n+j-3,n-j} + B\gamma_{n+j-3,n-j+1} + C\gamma_{n+j-2,n-j}) \\ &\qquad - (\bar{B}/\bar{D})(A\gamma_{n+j-2,n-j} + B\gamma_{n+j-2,n-j+1} + C\gamma_{n+j-1,n-j}) \\ &\qquad - (\bar{C}/\bar{D})(A\gamma_{n+j-3,n-j+1} + B\gamma_{n+j-3,n-j+2} + C\gamma_{n+j-2,n-j+1}).\end{aligned}$$

In order to simplify the preceding expression, we require several further identities. Note that, by recursiveness,

$$\begin{aligned} &for\ 2 \le j \le n,\ \bar{Z}^{j-1}Z^{n-j+1} = \bar{Z}^{j-2}Z^{n-j}(\bar{Z}Z) \\ &\quad = A\bar{Z}^{j-2}Z^{n-j} + B\bar{Z}^{j-2}Z^{n-j+1} + C\bar{Z}^{j-1}Z^{n-j} + D\bar{Z}^{j-2}Z^{n-j+2}, \end{aligned} \tag{3.20}$$

so, for $2 \le j \le n$,

$$\begin{aligned} \gamma_{n+j-2,n+1-j} &= \langle \bar{Z}^{j-1}Z^{n-j+1}, Z^{n-1} \rangle \\ &= A\gamma_{n+j-3,n-j} + B\gamma_{n+j-3,n-j+1} + C\gamma_{n+j-2,n-j} + D\gamma_{n+j-3,n-j+2}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \gamma_{n+j-1,n+1-j} &= \langle \bar{Z}^{j-1}Z^{n-j+1}, Z^{n} \rangle \\ &= A\gamma_{n+j-2,n-j} + B\gamma_{n+j-2,n-j+1} + C\gamma_{n+j-1,n-j} + D\gamma_{n+j-2,n-j+2}) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \gamma_{n+j-2,n-j+2} &= \langle \bar{Z}^{j-1}Z^{n-j+1}, \bar{Z}Z^{n-1} \rangle \\ &= A\gamma_{n+j-3,n-j+1} + B\gamma_{n+j-3,n-j+2} + C\gamma_{n+j-2,n-j+1} + D\gamma_{n+j-3,n-j+3}). \end{aligned} \tag{3.23}$$

Further, since

$$\begin{aligned} \bar{Z}^{j}Z^{n+1-j} &= (\bar{Z}Z)(\bar{Z}^{j-1}Z^{n-j}) \\ &= A\bar{Z}^{j-1}Z^{n-j} + B\bar{Z}^{j-1}Z^{n-j+1} + C\bar{Z}^{j}Z^{n-j} + D\bar{Z}^{j-1}Z^{n-j+2}, \end{aligned}$$

then

$$\begin{aligned} &\gamma_{n+j-1,n-j+2} = \langle \bar{Z}^{j}Z^{n+1-j}, \bar{Z}Z^{n-1} \rangle \\ &= A\gamma_{n+j-2,n-j+1} + B\gamma_{n+j-2,n-j+2} + C\gamma_{n+j-1,n-j+1} + D\gamma_{n+j-2,n-j+3}. \end{aligned} \tag{3.24}$$

It follows from (3.21)-(3.24) that the expression for $\gamma_{j+n,n+1-j}$ that we derived above (following (3.19)) can be re-expressed as

$$\begin{aligned} \gamma_{j+n,n+1-j} &= D\gamma_{n+j-1,n-j+2} + (1/\bar{D})(\gamma_{n+j-1,n+2-j} - D\gamma_{n+j-2,n+3-j}) \\ &-(\bar{A}/\bar{D})(\gamma_{n+j-2,n+1-j} - D\gamma_{n+j-3,n+2-j}) - (\bar{B}/\bar{D})(\gamma_{n+j-1,n+1-j} - D\gamma_{n+j-2,n+2-j}) \\ &- (\bar{C}/\bar{D})(\gamma_{n+j-2,n+2-j} - D\gamma_{n+j-3,n+3-j}) \\ &= D\gamma_{n+j-1,n-j+2} + (1/\bar{D})(\gamma_{n+j-1,n-j+2} - \bar{A}\gamma_{n+j-2,n-j+1} \\ &\quad - \bar{B}\gamma_{n+j-1,n-j+1} - \bar{C}\gamma_{n+j-2,n-j+2}) \\ &- D(1/\bar{D})(\gamma_{n+j-2,n-j+3} - \bar{A}\gamma_{n+j-3,n-j+2} - \bar{B}\gamma_{n+j-2,n-j+2} - \bar{C}\gamma_{n+j-3,n-j+3}). \end{aligned}$$

To further simplify the last expression, we employ recursiveness again:

$$\begin{aligned} &For\ 1 \le j \le n,\ \bar{Z}^{n-j+1}Z^{j} = \\ &\bar{Z}^{n-j}Z^{j-1}(\bar{Z}Z) = A\bar{Z}^{n-j}Z^{j-1} + B\bar{Z}^{n-j}Z^{j} + C\bar{Z}^{n-j+1}Z^{j-1} + D\bar{Z}^{n-j}Z^{j+1}. \end{aligned} \tag{3.25}$$

Thus

$$\gamma_{n-j+2,n+j-1} = \langle \bar{Z}^{n+1-j}Z^j, \bar{Z}^{n-1}Z\rangle$$
$$= A\gamma_{n-j+1,n+j-2} + B\gamma_{n-j+1,n+j-1} + C\gamma_{n-j+2,n+j-2} + D\gamma_{n-j+1,n+j}), \quad (3.26)$$

whence, by induction (the case of $j-1$),

$$\gamma_{n+j-1,n+2-j} = \bar{\gamma}_{n-j+2,n+j-1}$$
$$= \bar{A}\gamma_{n+j-2,n-j+1} + \bar{B}\gamma_{n+j-1,n-j+1} + \bar{C}\gamma_{n+j-2,n-j+2} + \bar{D}\bar{\gamma}_{n+1-j,n+j}. \quad (3.27)$$

Similarly,

$$\gamma_{n-j+3,n+j-2} = \langle \bar{Z}^{n+1-j}Z^j, \bar{Z}^{n-2}Z^2\rangle$$
$$= A\gamma_{n-j+2,n+j-3} + B\gamma_{n-j+2,n+j-2} + C\gamma_{n-j+3,n+j-3} + D\gamma_{n-j+2,n+j-1}), \quad (3.28)$$

whence, by induction with $j-2$ and using (3.28),

$$\gamma_{n+j-2,n-j+3} = \bar{\gamma}_{n-j+3,n+j-2}$$
$$= \bar{A}\gamma_{n+j-3,n-j+2} + \bar{B}\gamma_{n+j-2,n-j+2} + \bar{C}\gamma_{n+j-3,n-j+3} + \bar{D}\gamma_{n+j-1,n-j+2}. \quad (3.29)$$

Substituting (3.27) and (3.29) into our last equation for $\gamma_{j+n,n+1-j}$, we obtain the desired conclusion, $\gamma_{j+n,n+1-j} = \bar{\gamma}_{n+1-j,n+j}$.

This completes the proof of Lemma 3.4 for $1 \le i \le n$. Using this result, we next establish the result for $i = n+1$, i.e., $\bar{\gamma}_{0,2n+1} = \gamma_{2n+1,0}$. Since

$$\bar{Z}^{n+1} = (1/\bar{D})(\bar{Z}^n Z - \bar{A}\bar{Z}^{n-1} - \bar{B}\bar{Z}^n - \bar{C}\bar{Z}^{n-1}Z),$$

it follows that

$$\gamma_{2n+1,0} \equiv \langle \bar{Z}^{n+1}, Z^n\rangle = (1/\bar{D})(\gamma_{2n,1} - \bar{A}\gamma_{2n-1,0} - \bar{B}\gamma_{2n,0} - \bar{C}\gamma_{2n-1,1}). \quad (3.30)$$

Further, since $\bar{Z}Z^n = AZ^{n-1} + BZ^n + C\bar{Z}Z^{n-1} + DZ^{n+1}$, then $\gamma_{1,2n} = \langle \bar{Z}Z^n, \bar{Z}^n\rangle$ $= A\gamma_{0,2n-1} + B\gamma_{0,2n} + C\gamma_{1,2n-1} + D\gamma_{0,2n+1}$, whence $\gamma_{0,2n+1} = (1/D)(\gamma_{1,2n} - (A\gamma_{0,2n-1} + B\gamma_{0,2n} + C\gamma_{1,2n-1}))$. The case $i = n$ implies $\bar{\gamma}_{1,2n} = \gamma_{2n,1}$, so it follows that $\bar{\gamma}_{0,2n+1} = (1/\bar{D})(\gamma_{2n,1} - \bar{A}\gamma_{2n-1,0} - \bar{B}\gamma_{2n,0} - \bar{C}\gamma_{2n-1,1}) = \gamma_{2n+1,0}$. The proof of Lemma 3.4 is now complete. □

To show that block B has the structure of a moment matrix block B_{2n+1}, it now suffices to show that B is constant on diagonals.

Lemma 3.5. *B is constant on diagonals.*

Proof. We first show that B is constant on diagonals formed by excluding elements from column $\bar{Z}^{n+1}$, i.e.,

$$For\ i+j = n+1,\ k+l = n,\ l \ge 1,\ j \ge 2,\ \langle \bar{Z}^i Z^j, \bar{Z}^k Z^l\rangle = \langle \bar{Z}^{i+1}Z^{j-1}, \bar{Z}^{k+1}Z^{l-1}\rangle \quad (3.31)$$

We number the $2n+1$ diagonals of B of this type from the lower left toward the upper right; the corresponding diagonal numbers d are $d = n, n-1, \ldots, 0, -1, \ldots, -n$. The proof of (3.31) is by (downward) induction on d; (3.31) holds trivially for $d = n$ and $d = -n$ and holds for $d = n-1$ by the definitions of Z^{n+1} and $\bar{Z}Z^n$

(cf. (3.5)). We first assume (3.31) holds for some diagonal $d > -n+1$ and show that it also holds for diagonal $d-1$. For $i+j = n-1$, $j \geq 1$, $l+k = n$, $l \geq 1$, let $\langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^k Z^l\rangle$ denote an element of diagonal $d-1$ of block B, not on the bottom row or in the rightmost 2 columns; we seek to show that

$$\langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^k Z^l\rangle = \langle \bar{Z}^{i+2}Z^{j}, \bar{Z}^{k+1} Z^{l-1}\rangle \tag{3.32}$$

Now, from (1.1) and recursiveness,

$$\bar{Z}^{i+1}Z^{j+1} = A\bar{Z}^iZ^j + B\bar{Z}^iZ^{j+1} + C\bar{Z}^{i+1}Z^j + D\bar{Z}^iZ^{j+2},$$

so

$$\begin{aligned}\langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^k Z^l\rangle &= A\langle \bar{Z}^iZ^j, \bar{Z}^kZ^l\rangle + B\langle \bar{Z}^iZ^{j+1}, \bar{Z}^kZ^l\rangle \\ &\quad + C\langle \bar{Z}^{i+1}Z^j, \bar{Z}^kZ^l\rangle + D\langle \bar{Z}^iZ^{j+2}, \bar{Z}^kZ^l\rangle \\ &= A\langle \bar{Z}^{i+1}Z^{j-1}, \bar{Z}^{k+1}Z^{l-1}\rangle + B\langle \bar{Z}^{i+1}Z^{j}, \bar{Z}^{k+1}Z^{l-1}\rangle \\ &\quad + C\langle \bar{Z}^{i+2}Z^{j-1}, \bar{Z}^{k+1}Z^{l-1}\rangle + D\langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^{k+1}Z^{l-1}\rangle\end{aligned}$$

(applying moment matrix structure in the first three terms (which refer to $M(n)$), and applying induction to the fourth term, since $\langle \bar{Z}^iZ^{j+2}, \bar{Z}^kZ^l\rangle$ belongs to diagonal d, for which (3.31) holds.) Now, by the recursive definition of $\bar{Z}^{i+2}Z^j$, the last sum coincides with $\langle \bar{Z}^{i+2}Z^j, \bar{Z}^{k+1}Z^{l-1}\rangle$. This completes the proof of (3.31). To complete the proof that B is constant on diagonals, it suffices to prove that this is true in moving from column $\bar{Z}^nZ$ to column $\bar{Z}^{n+1}$, i.e.,

$$for\ k+l=n,\ with\ n>k\geq 0,\ \langle \bar{Z}^nZ, \bar{Z}^kZ^l\rangle = \langle \bar{Z}^{n+1}, \bar{Z}^{k+1}Z^{l-1}\rangle. \tag{3.33}$$

Now $\langle \bar{Z}^{n+1}, \bar{Z}^{k+1}Z^{l-1}\rangle = (1/\bar{D})\langle \bar{Z}^nZ - \bar{A}\bar{Z}^{n-1} - \bar{B}\bar{Z}^n - \bar{C}\bar{Z}^{n-1}Z, \bar{Z}^{k+1}Z^{l-1}\rangle$, $= (1/\bar{D})\langle \bar{Z}^{n-1}Z^2 - \bar{A}\bar{Z}^{n-2}Z - \bar{B}\bar{Z}^{n-1}Z - \bar{C}\bar{Z}^{n-2}Z^2, \bar{Z}^kZ^l\rangle$, (by (3.32) (for the first term on the left hand side of the inner product), and by moment matrix structure in $M(n)$ (for the last three terms)). It now suffices to prove that

$$(1/\bar{D})(\bar{Z}^{n-1}Z^2 - \bar{A}\bar{Z}^{n-2}Z - \bar{B}\bar{Z}^{n-1}Z - \bar{C}\bar{Z}^{n-2}Z^2) = \bar{Z}^n \tag{3.34}$$

Recall that

$$\bar{Z}^2Z^{n-1} = A\bar{Z}Z^{n-2} + B\bar{Z}Z^{n-1} + C\bar{Z}^2Z^{n-2} + D\bar{Z}Z^n.$$

From Lemma 3.2 and what we have proved above, the sub-block $\tilde{B}$ of block B formed by deleting column $\bar{Z}^{n+1}$ obeys moment matrix structure. Since the preceding column relation does not involve Z^{n+1} or $\bar{Z}^{n+1}$, it follows exactly as in the proof of [CF2, Lemma 3.10] that this relation may be conjugated in the column space of $M(n)\ \tilde{B})$, yielding (3.34), whence (3.33) follows. The proof of Lemma 3.5 is complete. □

From Lemmas 3.3-3.5, block $B[n, n+1]$ is of the form $B_{n,n+1}$, block B is of the form $B(n+1)$, and and $Ran\ B(n+1) \subset Ran\ M(n)$. Thus, to complete the proof of Theorem 1.1, it remains to prove that the C-block of $M \equiv [M(n); B(n+1)]$ is Toeplitz, i.e., constant on diagonals.

Lemma 3.6. *The C block of $[M(n); B(n+1)]$ is constant on diagonals.*

Proof. Since $C \equiv (C_{i,j})_{0\le i,j\le n+1}$ is self-adjoint, it suffices to consider the main diagonal and the diagonals below it. These we index by $d = n, n-1, \ldots, 0$, beginning with the 2-element diagonal $c_{n,1},\ c_{n+1,2}$. That $c_{n,1} = c_{n+1,2}$ follows from the structure of $[M(n); B(n+1)]$ [CF2, Proposition 2.3]: each diagonal of C is symmetric with respect to its midpoint. Assume, by induction, that each of the diagonals indexed by $n,\ n-1, \ldots, d\ (>0)$ is constant. We seek to prove that diagonal $d-1$ is also constant. We first consider an element of this diagonal that is not in the leftmost column, the rightmost two columns, or in the bottom row, and we denote this element by $\lambda \equiv \langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^k Z^l\rangle$, where $i+j+2 = k+l = n+1$ and $l, j > 0$. We must prove that

$$\langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^k Z^l\rangle = \langle \bar{Z}^{i+2}Z^{j}, \bar{Z}^{k+1} Z^{l-1}\rangle.$$

Now [CF7, Lemma 3.15] shows that in $M \equiv [M(n); B(n+1)]$, dependence relations which define the columns of block $B \equiv B(n+1)$ extend to the full columns of M (and hence define the columns of block C). Thus, in the column space of $(B(n+1)^*\ C)$ we have

$$\bar{Z}^{i+1}Z^{j+1} = A\bar{Z}^iZ^j + B\bar{Z}^iZ^{j+1} + C\bar{Z}^{i+1}Z^j + D\bar{Z}^iZ^{j+2}\ (i+j=n-1), \quad (3.35)$$

whence, by moment matrix structure in $B(n+1)^*$,

$$\begin{aligned}\lambda &= \langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^k Z^l\rangle \\ &= A\langle \bar{Z}^iZ^j, \bar{Z}^kZ^l\rangle + B\langle \bar{Z}^iZ^{j+1}, \bar{Z}^kZ^l\rangle + C\langle \bar{Z}^{i+1}Z^j, \bar{Z}^kZ^l\rangle + D\langle \bar{Z}^iZ^{j+2}, \bar{Z}^kZ^l\rangle \\ &= A\langle \bar{Z}^{i+1}Z^{j-1}, \bar{Z}^{k+1}Z^{l-1}\rangle + B\langle \bar{Z}^{i+1}Z^{j}, \bar{Z}^{k+1}Z^{l-1}\rangle \\ &\quad + C\langle \bar{Z}^{i+2}Z^{j-1}, \bar{Z}^{k+1}Z^{l-1}\rangle + D\langle \bar{Z}^iZ^{j+2}, \bar{Z}^kZ^l\rangle.\end{aligned}$$

Now $\langle \bar{Z}^iZ^{j+2}, \bar{Z}^kZ^l\rangle$ is on diagonal d, so by induction,

$$\langle \bar{Z}^iZ^{j+2}, \bar{Z}^kZ^l\rangle = \langle \bar{Z}^{i+1}Z^{j+1}, \bar{Z}^{k+1}Z^{l-1}\rangle.$$

Thus, by (3.35),

$$\begin{aligned}\lambda &= \langle A\bar{Z}^{i+1}Z^{j-1} + B\bar{Z}^{i+1}Z^j + C\bar{Z}^{i+2}Z^{j-1} + D\bar{Z}^{i+1}Z^{j+1}, \bar{Z}^{k+1}Z^{l-1}\rangle \\ &= \langle \bar{Z}^{i+2}Z^j, \bar{Z}^{k+1}Z^{l-1}\rangle.\end{aligned}$$

Since we are working on or below the main diagonal, and the diagonal is symmetric with respect to its midpoint, at this point we may conclude that the entire diagonal is constant, except perhaps in the case of $d = 0$, where it suffices to prove that $\langle \bar{Z}^nZ, \bar{Z}^nZ\rangle = \langle \bar{Z}^{n+1}, \bar{Z}^{n+1}\rangle$, or, equivalently, $\langle Z^{n+1}, Z^{n+1}\rangle = \langle \bar{Z}Z^n, \bar{Z}Z^n\rangle$. Now

$$Z^{n+1} = (1/D)(\bar{Z}Z^n - AZ^{n-1} - BZ^n - C\bar{Z}Z^{n-1}),$$

so

$$
\begin{aligned}
\langle Z^{n+1}, Z^{n+1}\rangle & \\
&= (1/D)(\langle \bar{Z}Z^n, Z^{n+1}\rangle - A\langle Z^{n-1}, Z^{n+1}\rangle \\
&\quad - B\langle Z^n, Z^{n+1}\rangle - C\langle \bar{Z}Z^{n-1}, Z^{n+1}\rangle) \\
&= (1/D)(\langle \bar{Z}Z^n, Z^{n+1}\rangle - A\langle \bar{Z}Z^{n-2}, \bar{Z}Z^n\rangle \\
&\quad - B\langle \bar{Z}Z^{n-1}, \bar{Z}Z^n\rangle - C\langle \bar{Z}^2Z^{n-2}, \bar{Z}Z^n\rangle) \quad (3.36)
\end{aligned}
$$

(by the moment matrix structure of block $B(n+1)^*$). Since $C = C^*$,

$$\langle \bar{Z}Z^n, Z^{n+1}\rangle = \langle Z^{n+1}, \bar{Z}Z^n\rangle^- = \langle \bar{Z}Z^n, \bar{Z}^2Z^{n-1}\rangle^-$$

(by induction, since $\langle Z^{n+1}, \bar{Z}Z^n\rangle$ is on the first subdiagonal). Thus $\langle \bar{Z}Z^n, Z^{n+1}\rangle = \langle \bar{Z}^2Z^{n-1}, \bar{Z}Z^n\rangle$, whence (3.36) and recursiveness imply

$$
\begin{aligned}
&\langle Z^{n+1}, Z^{n+1}\rangle \\
&= (1/D)\langle \bar{Z}^2Z^{n-1} - A\bar{Z}Z^{n-2} - B\bar{Z}Z^{n-1} - C\bar{Z}^2Z^{n-2}, \bar{Z}Z^n\rangle) = \langle \bar{Z}Z^n, \bar{Z}Z^n\rangle.
\end{aligned}
$$

The proof of Lemma 3.6 is complete. □

Lemmas 3.3-3.6 together complete the proof of Theorem 1.1 $vi) \Rightarrow iv)$.

4. Solving full moment problems via truncated moment problems

In this section we show how to apply a recent theorem of J. Stochel [St2] which provides a link between the full and truncated multidimensional moment problems. Although Stochel's result applies to moment problems in any number of real or complex variables, we paraphrase it here only for one complex variable.

Theorem 4.1. *(cf. Stochel [St2]) Let K be a closed subset of* $\mathbf{C}$. *A full sequence $\gamma^{(\infty)} \equiv (\gamma_{ij})_{i,j\geq 0}$ has a representing measure supported in K if and only if, for each $n \geq 1$, $\gamma^{(2n)}$ has a representing measure supported in K.*

The following result permits us to implement Stochel's theorem in concrete situations.

Proposition 4.2. *If $M(\infty) \geq 0$, then $M(n)$ is positive and recursively generated for each $n \geq 1$. In this case, for $p \in \mathbf{C}[z, \bar{z}]$, $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(\infty)}$ if and only if $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$ for some (respectively, for all) $n \geq deg\ p$.*

Proof. Fix $n \geq 1$ and let $f,\ g,\ fg \in \mathcal{P}_n$, with $f(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$. We seek to show that $(fg)(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$. Since $M(n+1) \geq 0$, the Extension Principle for positive matrices [F1] implies that $f(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(n+1)}$. Theorem 1.11, applied to $M(n+1)$, now implies the desired conclusion that $[(fg)(Z, \bar{Z})]_n = 0$.

Suppose $M(\infty)$ is positive and $p \in \mathbf{C}[z, \bar{z}]$ satisfies $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$ for some $n \geq deg\ p$. Since $M(k) \geq 0$ for all $k \geq n$, the Extension Principle implies $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(k)}$ for all $k \geq n$, whence $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(\infty)}$. □

Before proceding to applications of Theorem 4.1, we consider connections between Theorem 4.1 and Conjecture 1.2. In the full moment problem for $\gamma^{(\infty)}$, with $M(\infty)$ singular, we may define the variety $\mathcal{V}(\gamma^{(\infty)})$ by analogy with $\mathcal{V}(\gamma^{(2n)})$. It follows exactly as in the trucated moment problem (using [CF2,Proposition 3.1 and Lemma 4.1]) that if $\gamma^{(\infty)}$ admits a representing measure, then $M(\infty) \geq 0$ and *card* $\mathcal{V}(\gamma^{(\infty)}) \geq$ *rank* $M(\infty)$. If Conjecture 1.2 is true, it would follow that these conditions are also *sufficient* for the existence of a representing measure. Indeed, if $M(\infty) \geq 0$, then Proposition 4.2 implies that for each n, $M(n)$ is positive and recursively generated. Moreover, the variety hypothesis implies that *card* $\mathcal{V}(\gamma^{(2n)}) \geq$ *card* $\mathcal{V}(\gamma^{(\infty)}) \geq$ *rank* $M(\infty) \geq$ *rank* $M(n)$. If Conjecture 1.2 is true, it would then follow that there exists a representing measure μ_n for $\gamma^{(2n)}$, whence the existence of a representing measure for $\gamma^{(\infty)}$ would follow from Theorem 4.1.

Question 4.3. Suppose $M(\infty)$ is singular. If $M(\infty) \geq 0$ and *card* $\mathcal{V}(\gamma^{(\infty)}) \geq$ *rank* $M(\infty)$, does $\gamma^{(\infty)}$ admit a representing measure?

We noted in Section 1 results of Stochel [St1] concerning the existence of a sequence $\gamma^{(\infty)}$ and a polynomial p of degree 3, with $M(\infty) \geq 0$ and $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(\infty)}$, such that $\gamma^{(\infty)}$ has no representing measure. As a test for Question 4.3 (and Conjecture 1.2), it would be helpful to be able to construct a concrete example of such a sequence $\gamma^{(\infty)}$ and to compute *card* $\mathcal{V}(\gamma^{(\infty)})$ and *rank* $M(\infty)$.

We next convert some terminolgy of [St1] [SS2] for the 2-dimensional full real moment problem into the language of the full complex moment problem. Let $\gamma \equiv \gamma^{(\infty)}$ and let Λ_γ denote the Riesz functional on $\mathbf{C}[z, \bar{z}]$ defined by $\Lambda_\gamma(\sum a_{ij} \bar{z}^i z^j) = \sum a_{ij} \gamma_{ij}$; thus $\Lambda_\gamma(p\bar{q}) = \langle M(\infty)(\gamma)\hat{p}, \hat{q} \rangle$ $(p, q \in \mathbf{C}[z, \bar{z}])$. Let $\sum^2$ denote the set of finite sums of complex squares $|p|^2$ $(p \in \mathbf{C}[z, \bar{z}])$. Λ_γ is said to be *positive definite* if $\Lambda_\gamma(q) \geq 0$ for each $q \in \sum^2$. Positive definiteness is a necessary (but in general, not sufficient) condition for the existence of a representing measure for γ; note that Λ_γ is positive definite if and only if $M(\gamma)(\infty)$ is positive semidefinite. A necessary condition for the existence of a representing measure supported in $\mathcal{Z}(p)$ is the following property $(A_{\mathbf{C}})$: $\Lambda_\gamma(p\bar{q}) = 0$ for each $q \in \mathcal{C}[z, \bar{z}]$. (This is just a complexified version of Stochel's property (A) for the full real moment problem.) Thus, in order for there to exist a representing measure for γ supported in $\mathcal{Z}(p)$, it is necessary that Λ_γ be positive definite and satisfy property $(A_{\mathbf{C}})$. If these conditions are *sufficient* to imply the existence of a measure, p is said to be of *type* $A_{\mathbf{C}}$ (p satisfies the complexified version of Stochel's type A for $p(x, y)$). Now $\Lambda_\gamma(p\bar{q}) = \langle M(\infty)\hat{p}, \hat{q} \rangle = \langle p(Z, \bar{Z}), \hat{q} \rangle$. Thus Proposition 4.2 shows that Λ_γ is positive definite and satisfies $(A_{\mathbf{C}}) \iff M(\infty) \geq 0$ and $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(\infty)}$.

As a first application of Proposition 4.2, we solve the full complex moment problem subordinate to a column relation in $\mathcal{C}_{M(\infty)}$ of the form $Z\bar{Z} = A1 + BZ + C\bar{Z} + DZ^2$. (Note that we do not assume $D \neq 0$ as in (1.1)).

Proposition 4.4. *$\gamma^{(\infty)}$ has a representing measure supported in $K \equiv \{z : z\bar{z} = A + Bz + C\bar{z} + Dz^2\}$ if and only if $M(\infty) \geq 0$ and $Z\bar{Z} = A1 + BZ + C\bar{Z} + DZ^2$ in $\mathcal{C}_{M(\infty)}$.*

Proof. The necessity of the conditions is clear. For sufficiency, since $M(\infty) \geq 0$, Proposition 4.2 implies that for each n, $M(n)$ is positive and recursively generated. For $n \geq 2$, we have $Z\bar{Z} = A1 + BZ + C\bar{Z} + DZ^2$ in $\mathcal{C}_{M(n)}$. Since $M(\infty) \geq 0$, the Extension Principle [F1] implies that $\{1,\ Z,\ \bar{Z},\ Z^2\}$ is dependent in $\mathcal{C}_{M(\infty)}$ if and only if $\{1,\ Z,\ \bar{Z},\ Z^2\}$ is dependent in $\mathcal{C}_{M(n)}$ for some $n \geq 2$ (equivalently, for each $n \geq 2$). If $\{1,\ Z,\ \bar{Z},\ Z^2\}$ is dependent, then [CF3, Theorems 2.1 and 3.1] imply that there exists a representing measure μ_n for $\gamma^{(2n)}$ with *supp* $\mu_n \subset \mathcal{V}(\gamma^{(2n)}) \subset K$. If $\{1,\ Z,\ \bar{Z},\ Z^2\}$ is independent, then the existence of a representing measure μ_n for $\gamma^{(2n)}$ (necessarily supported in K) follows from [CF7, Theorem 1.1] if $D = 0$, and from Theorem 1.1-iii) if $D \neq 0$ (since $M(n+1) \geq 0$). The result now follows from Theorem 4.1. □

Remark 4.5. As we noted in the Introduction, Proposition 4.4 also follows from [St1, Theorem 5.4]. In the case when $\{1,\ Z,\ \bar{Z},\ Z^2\}$ is independent and $D = 0$, K is a circle, and Proposition 4.4 is equivalent to the solution of the classical full trigonometric moment problem (cf. [Akh] [CF7]).

The preceding results show that each polynomial of the form $A + Bz + C\bar{z} + Dz^2 + Ez\bar{z}$ satisfies $(A_{\mathbf{C}})$. We next identify a class of type $(A_{\mathbf{C}})$ polynomials of arbitarily large degree; the following result proves Proposition 1.5.

Proposition 4.6. *If $p(z,\bar{z}) = z^k - q(z,\bar{z})$, with deg $q < k$, then p is of type $(A_{\mathbf{C}})$.*

Proof. Suppose $M(\infty) \geq 0$ and $p(Z,\bar{Z}) = 0$ in $C_{M(\infty)}$. For $n > 2k$, $M(n)$ is positive and recursively generated (Proposition 4.2), and $p(Z,\bar{Z}) = 0$ in $C_{M(n)}$. Since $k \leq [n/2]$, it follows from Theorem 1.15 that $\gamma^{(2n)}$ has a representing measure supported in $\mathcal{Z}(p)$. The result now follows from Theorem 4.1. □

Remark 4.7. The referee has kindly pointed out that Proposition 4.6 also follows directly from [SS1, Theorem 4] (since $p(z,\bar{z})$ has a dominating coefficient), or, indirectly, from [Cas, Theorem 6]. The referee further notes that [SS1, Remark 2] implies that $\mathcal{Z}(p)$ is bounded; indeed, in [CF8] we showed that *card* $\mathcal{Z}(p) \leq k^2$. Since $\mathcal{Z}(p)$ is compact, Proposition 4.6 can also be deduced from Schmüdgen's solution to the K-moment problem for compact semi-algebraic sets [Sch].

As we noted earlier, Stochel's convergence theorem applies to moment problems in any number of real or complex variables. We conclude by using the convergence theorem to re-prove two classical theorems concerning the one-dimensional real full moment problem. For a real sequence $\beta^{(2n)} : \beta_0, \ldots, \beta_{2n}$, let $H(n)$ denote the Hankel matrix $(\beta_{i+j})_{0 \leq i,j \leq n}$; with columns labelled $1,\ t, \ldots, t^n$, $H(n)$ is the analogue of $M(n)$ appropriate for truncated moment problems on $\mathbf{R}$. Similarly, to a full sequence $\beta^{(\infty)}$, we associate $H(\infty)$. We also consider $L(n) = (\beta_{i+j+1})_{0 \leq i,j \leq n}$ and $L(\infty)$.

Proposition 4.8. *(Stieltjes (cf. [Akh]))* $\beta^{(\infty)}$ *has a representing measure supported in* $[0, +\infty)$ *if and only if* $H \equiv H(\infty) \geq 0$ *and* $L \equiv L(\infty) \geq 0$.

Proof. Necessity of the conditions is straightforward: for a representing measure μ supported in $[0, +\infty)$, and for $f \in \mathbf{C}[t]$, $\langle H\hat{f}, \hat{f}\rangle = \int |f|^2 \, d\mu(t) \geq 0$ and $\langle L\hat{f}, \hat{f}\rangle = \int t|f(t)|^2 \, d\mu(t) \geq 0$. For sufficiency, the hypothesis implies that for each n, $H(n) \geq 0$ and $L(n) \geq 0$. In particular, [Smu] implies that $L(n-1) \geq 0$ and $(\beta_{n+1}, \ldots, \beta_{2n})^t \in Ran\ L(n-1)$ (cf. the remarks following (2.4)). [CF1, Theorem 5.3] now implies that there exists a representing measure μ_n for $\beta^{(2n)}$ supported in $[0, +\infty)$, so the result follows from the analogue of Theorem 4.1 for the $\mathbf{R}$ moment problem. □

Proposition 4.9. *(Hamburger (cf. [Akh]))* $\beta^{(\infty)}$ *has a representing measure supported in* $\mathbf{R}$ *if and only if* $H \equiv H(\infty) \geq 0$.

Proof. As in the previous result, necessity is clear. For sufficiency, since $H \geq 0$, Proposition 4.2 (or, more precisely, its analogue for Hankel matrices) implies that for each n, $H(n)$ is positive and recursively generated. [CF1, Theorem 3.9] thus implies that $\beta^{(2n)}$ admits a representing measure supported in $\mathbf{R}$, so the result follows from the $\mathbf{R}$ analogue of Theorem 4.1. □

Remark 4.10. For the case when $H > 0$, a similar proof of Hamburger's Theorem is sketched in [Lan], although apparently without a proof of the convergence argument that Stochel has recently formalized in [St2].

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