

## LIMITS OF POSITIVE FLAT BIVARIATE MOMENT MATRICES

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ABSTRACT. The bivariate moment problem for a sequence  $\beta \equiv \beta^{(6)}$  of degree 6 remains unsolved, but we prove that if the associated  $10 \times 10$  moment matrix  $M_3(\beta)$  satisfies  $M_3 \succeq 0$  and  $\text{rank } M_3 \leq 6$ , then  $\beta$  admits a sequence of *approximate* representing measures, and  $\beta^{(5)}$  has a representing measure. More generally, let  $\overline{\mathcal{F}}_d$  denote the closure of the positive flat moment matrices of degree  $2d$  in  $n$  variables. Each matrix in  $\overline{\mathcal{F}}_d$  admits computable approximate representing measures, and in 2013, Jiawang Nie and the author began to study concrete conditions for membership in this class. Let  $\beta \equiv \beta^{(2d)} = \{\beta_i\}_{i \in \mathbb{Z}_+^n, |i| \leq 2d}$ ,  $\beta_0 > 0$ , denote a real  $n$ -dimensional sequence of degree  $2d$ . If the corresponding moment matrix  $M_d \equiv M_d(\beta)$  is the limit of a sequence of positive *flat* moment matrices  $\{M_d^{(k)}\}$ , i.e.,  $M_d^{(k)} \succeq 0$  and  $\text{rank } M_d^{(k)} = \text{rank } M_{d-1}^{(k)}$ , then i)  $M_d \succeq 0$ , ii)  $\text{rank } M_d \leq \rho_{d-1} \equiv \dim \mathbb{R}[x_1, \dots, x_n]_{d-1}$ , and iii)  $\beta^{(2d-1)}$  admits a representing measure. We extend our earlier results by proving, conversely, that for  $n = 2$ , if  $M_d$  satisfies certain positivity and rank conditions related to i)-iii), then  $M_d$  is the limit of positive flat moment matrices.

### 1. INTRODUCTION

Let  $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i \in \mathbb{Z}_+^n, |i| \leq m}$  denote a real  $n$ -dimensional sequence of degree  $m$ ,  $\beta_0 > 0$ , and let  $K \subseteq \mathbb{R}^n$  be a closed set. The Truncated  $K$ -Moment Problem (TKMP) concerns conditions for the existence of a positive Borel measure  $\mu$ , supported in  $K$ , such that

$$(1.1) \quad \beta_i = \int x^i d\mu \quad (|i| \leq m)$$

(where  $i \equiv (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ ,  $|i| = i_1 + \dots + i_n$ ,  $x \equiv (x_1, \dots, x_n) \in \mathbb{R}^n$ , and  $x^i := x_1^{i_1} \dots x_n^{i_n}$ ).  $\beta$  as above is a *truncated moment sequence*, and we refer to  $\mu$  as a  *$K$ -representing measure* for  $\beta$ . In the case  $K = \mathbb{R}^n$ , we refer to TKMP as the Truncated Moment Problem (TMP) and to  $\mu$  as a *representing measure*. Although several abstract criteria for the existence of representing measures are known [CF9], [CF7] (cf. Theorems 1.2 and 1.4 below), the only concrete condition available is *flatness* of the moment data, i.e., the moment matrix  $M_d$  associated to  $\beta^{(2d)}$  satisfies  $M_d \succeq 0$  (positive semidefinite) and  $\text{rank } M_d = \text{rank } M_{d-1}$  [CF7] (cf. Theorem 1.4). With the aid of this condition, TKMP has been solved concretely for  $K$  a planar curve of degree 1 or 2 [CF3], [CF5], [CF6], [CF8], and for  $y = x^3$  [F]. In particular, for  $n = 2$ , TMP has been solved concretely for degree 2 ( $d = 1$ ) and degree 4 ( $d = 2$ ) [CF3], [CF5]. Nevertheless, the degree 6 problem ( $d = 3$ )

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remains largely unsolved, and similarly for  $\beta^{(m)}$  with  $m \geq 5$ . In view of this difficulty, our focus here is on membership of  $M_d$  in  $\overline{\mathcal{F}_d}$ , the closure of the positive flat moment matrices. For if  $M_d(\beta)$  belongs to  $\overline{\mathcal{F}_d}$ , then the moment problem for  $\beta$  is “almost” solved in two respects: first, there exists a sequence of *approximate* representing measures for  $\beta$  and, second, as described by Theorem 1.2 (cf. [CF9]),  $\beta^{(2d-1)}$  has a representing measure. Moreover, as we discuss below, the existence of approximate representing measures provides a criterion for finite convergence in the polynomial optimization program of J.-B. Lasserre [Las]. Our main result, Theorem 1.9, provides a sufficient condition for membership in  $\overline{\mathcal{F}_d}$ . The main application of Theorem 1.9 is the following simple criterion for approximation by positive flat moment matrices in the degree 6 problem.

**Theorem 1.1.** *Let  $n = 2$  and let  $\beta \equiv \beta^{(6)} = \{\beta_{ij}\}_{i,j \geq 0, i+j \leq 6}$ . The associated  $10 \times 10$  moment matrix  $M_3 \equiv M_3(\beta)$  belongs to  $\overline{\mathcal{F}_3}$  if and only if  $M_3 \succeq 0$  and  $\rho \equiv \text{rank } M_3 \leq 6$ . In this case, given  $\tau$ ,  $\rho \leq \tau \leq 6$ ,  $M_3$  is in the closure of the rank- $\tau$  positive flat moment matrices, and there exists a sequence of (computable)  $\tau$ -atomic positive measures  $\{\mu_k\}$  such that  $\beta_{ij} = \lim_{k \rightarrow \infty} \int x^i y^j d\mu_k$  ( $i, j \geq 0, i+j \leq 6$ ). Moreover,  $\beta^{(5)}$  has a representing measure.*

Let  $\mathbb{R}[x]_m \equiv \mathbb{R}[x_1, \dots, x_n]_m := \{p \in \mathbb{R}[x_1, \dots, x_n] : \deg p \leq m\}$ . We associate to  $\beta$  the Riesz functional  $L_\beta : \mathbb{R}[x]_m \mapsto \mathbb{R}$  defined by  $L_\beta(\sum_{|i| \leq m} a_i x^i) = \sum a_i \beta_i$ . If

$\beta$  has a  $K$ -representing measure  $\mu$ , then  $L_\beta$  is  $K$ -positive, i.e.,  $p \in \mathbb{R}[x]_m, p|_K \geq 0 \implies L_\beta(p) \geq 0$ ; indeed, in this case,  $L_\beta(p) = \int_K p d\mu \geq 0$ . For  $K = \mathbb{R}^n$ , we say that  $L_\beta$  is *positive*. In the classical Full Multidimensional  $K$ -Moment Problem for  $\beta \equiv \beta^{(\infty)}$ , the Riesz-Haviland Theorem [R], [H] states that  $\beta$  admits a  $K$ -representing measure if and only if  $L_\beta$  is  $K$ -positive. Such is not the case in TKMP. The proof of Tchakaloff's Theorem [T] shows that if  $K$  is compact, then  $K$ -positivity of  $L_\beta$  does imply that  $\beta$  has a (finitely atomic)  $K$ -representing measure. However, for  $K$  noncompact, this implication fails; for example, with  $n = 1, K = \mathbb{R}, m = 4$ , the sequence 1, 1, 1, 1, 2 has a positive functional but no representing measure (cf. Example 1.5). The following result of [CF9] illustrates the role of  $K$ -positivity in TKMP.

**Theorem 1.2** ([CF9, Theorem 1.2]).  *$\beta \equiv \beta^{(2d)}$  (or  $\beta \equiv \beta^{(2d+1)}$ ) admits a  $K$ -representing measure if and only if  $\beta$  can be extended to a sequence  $\tilde{\beta} \equiv \tilde{\beta}^{(2d+2)}$  for which  $L_{\tilde{\beta}}$  is  $K$ -positive.*

(The result in [CF9] is stated only for  $\beta^{(2d)}$ , but it is clear from the proof that it applies as well to  $\beta^{(2d+1)}$ .)

$K$ -positivity also solves TKMP in the sense of approximation. Let  $\mathcal{M}_{n,m}$  denote the set of all real  $n$ -dimensional sequences of degree  $m$ , viewed as a subset of  $\mathbb{R}^\eta$  (endowed with the Euclidean norm), where  $\eta := \dim \mathbb{R}[x]_m (= \binom{n+m}{m})$ . Let  $\mathcal{R}_{n,m}(K) := \{\beta \in \mathcal{M}_{n,m} : \beta \text{ has a } K\text{-representing measure}\}$ , and let  $\mathcal{P}_{n,m}(K) := \{\beta \in \mathcal{M}_{n,m} : L_\beta \text{ is } K\text{-positive}\}$ . Note that  $\mathcal{R}_{n,m}(K) \subset \mathcal{P}_{n,m}(K) \subset \mathcal{M}_{n,m}$  is an inclusion of convex cones, with  $\mathcal{M}_{n,m}$  and  $\mathcal{P}_{n,m}(K)$  closed; it will become clear in the sequel that in general  $\mathcal{R}_{n,m}(K)$  is not closed.

**Theorem 1.3** ([FN1, Theorem 2.2]).  *$\mathcal{P}_{n,m}(K) = \overline{\mathcal{R}_{n,m}(K)}$ . If  $L_{\beta^{(m)}}$  is  $K$ -positive, then there exists a sequence of positive Borel measures  $\{\mu_k\}$ , each supported in*

$K$ , having finite moments up to degree  $m$ , such that for each  $i$ ,  $|i| \leq m$ ,  $\beta_i = \lim_{k \rightarrow \infty} \int x^i d\mu_k$ .

In view of the preceding two results, it would be desirable to have a concrete test for  $K$ -positivity, but at present there is no such test that is applicable to a general multisequence. Similarly, in the Full  $K$ -Moment Problem, concrete conditions for  $K$ -positivity are known only in certain cases, e.g., in Hamburger's solution for  $K = \mathbb{R}$  (cf. [A], [KN]) and in K. Schmüdgen's solution for  $K$  a basic compact semialgebraic set [S2]. Remarkably, in the compact case treated by Tchakaloff [S2], there is no known concrete test for  $K$ -positivity in TKMP, even in the case when  $K$  is a basic compact semialgebraic set. In [FN2] we began to study conditions for  $\beta$  to be a limit of "flat" multisequences (as defined below), in which case positivity of  $L_\beta$  is obvious. Moreover, if the flat approximants  $\beta^{[k]}$  are known, then corresponding approximate representing measures  $\mu^{(k)}$  for  $\beta$  can be explicitly computed (cf. Example 1.5). In [FN2] we obtained concrete necessary and sufficient conditions for flat approximation when  $n = 1$ ,  $d \geq 1$  and when  $n = d = 2$  (cf. Theorem 1.6 below). In the present note, we extend the results of [FN2] to the bivariate truncated moment problem, i.e.,  $n = 2$ ,  $d \geq 1$ . The conditions that we present in Theorem 1.9 apply in certain cases in which the truncated moment problem is still unsolved, including certain cases of the bivariate degree 6 problem, so we can at least test for approximate representing measures in these cases; Theorem 1.1 provides one such test.

To describe our results in detail, we require some additional terminology. Following [CF7], we associate to  $\beta \equiv \beta^{(2d)}$  the *moment matrix*  $M_d(\beta)$ . For  $p \in \mathbb{R}[x]_d$ ,  $p = \sum_{|i| \leq d} a_i x^i$ , let  $\hat{p} \equiv (a_i)$  denote the vector of coefficients of  $p$  with respect to the basis for  $\mathbb{R}[x]_d$  consisting of the monomials in degree-lexicographic order. Let  $\rho_d = \dim \mathbb{R}[x]_d$ . Then the moment matrix  $M_d \equiv M_d(\beta)$  is the  $\rho_d \times \rho_d$  matrix defined by

$$\langle M_d(\beta) \hat{p}, \hat{q} \rangle = L_\beta(pq) \quad (p, q \in \mathbb{R}[x]_d).$$

As noted above, if  $\beta$  has a representing measure, then  $L_\beta$  is positive. Further, if  $L_\beta$  is positive, then  $M_d(\beta)$  is positive semidefinite ( $M_d(\beta) \succeq 0$ ), since  $\langle M_d(\beta) \hat{p}, \hat{p} \rangle = L_\beta(p^2) \geq 0$  ( $p \in \mathbb{R}[x]_d$ ). In general, the preceding implication cannot be reversed, but there is one situation where positivity of  $M_d(\beta)$  is equivalent to positivity of  $L_\beta$ . Recall that  $p \in \mathbb{R}[x]_{2d}$  is *nonnegative* or *positive semidefinite (psd)* if  $p|_{\mathbb{R}^n} \geq 0$ , and  $p$  is a *sum of squares (SOS)* if there exist  $p_1, \dots, p_k \in \mathbb{R}[x]_d$  such that

$$p = \sum_{j=1}^k p_j^2. \quad \text{Now, if every nonnegative polynomial } p \text{ in } \mathbb{R}[x]_{2d} \text{ is SOS, then positive semidefiniteness of } M_d(\beta) \text{ implies positivity of } L_\beta, \text{ since, for } p \text{ nonnegative, } L_\beta(p) = L_\beta\left(\sum_{j=1}^k p_j^2\right) = \sum \langle M_d(\beta) \hat{p}_j, \hat{p}_j \rangle \geq 0. \quad \text{A well-known theorem of Hilbert (cf. [Rez])}$$

shows that each psd polynomial is SOS if and only if  $n = 1$ ,  $d \geq 1$ ;  $n = d = 2$ ; or  $n \geq 1$ ,  $d = 1$ . In each of these cases, checking that  $L_\beta$  is positive reduces to simply verifying that  $M_d(\beta)$  is positive semidefinite. Now, for  $n = 2$ ,  $d = 3$ , there exists  $M_3(\beta) \succ 0$  (positive definite) for which  $L_\beta$  is not positive [CF3], [S1]; more generally, [FN2, Proposition 1.6] shows that *except* in the cases of Hilbert's

theorem, positive semidefiniteness of a moment matrix does not imply positivity of its Riesz functional.

In the sequel we seek to identify cases of Riesz functional positivity that are beyond the scope of Hilbert's theorem and are not due to the existence of representing measures. The first such examples appear in [EF] for certain bivariate  $M_3$  with  $\text{rank } M_3 = 9$ . In the present note we focus on positivity arising from approximation by positive flat moment matrices. The following result of [CF7] illustrates the central role of positive flat moment matrices in TMP.

**Theorem 1.4** ([CF7]). *Let  $\beta \equiv \beta^{(2d)}$ . If  $M_d(\beta) \succeq 0$  is flat, i.e.,  $\text{rank } M_d = \text{rank } M_{d-1}$ , then  $\beta$  admits a unique representing measure which is rank  $M_d(\beta)$ -atomic. More generally,  $\beta$  has a representing measure if and only if  $\beta$  can be extended to a sequence  $\tilde{\beta} \equiv \tilde{\beta}^{(2(d+k))}$  (for some  $k$ ,  $0 \leq k \leq \rho_{2d} - \text{rank } M_d(\beta) + 1$ ) for which  $M_{d+k}(\tilde{\beta})$  is positive semidefinite and flat.*

The result in [CF7] is stated in terms of finitely atomic representing measures, but in [BT] Bayer and Teichmann proved that the existence of a representing measure implies the existence of a finitely atomic representing measure. Theorem 1.4 is the basis for a concrete solution to TKMP when  $K$  is the real line  $\mathbb{R}$  [CF1], a planar line or curve of degree 2 [CF3], [CF5], [CF6], [CF8], or the planar curve  $y = x^3$  [F]. Theorem 1.4 also leads to an algorithmic solution to TMP for the class of *recursively determinate* moment matrices [CF10] and is used by Helton and Nie [HN] in a numerical solution to TKMP based on semidefinite programming (see also [FN3]).

Let  $\mathcal{F}_d := \{M_d(\beta) : M_d(\beta) \succeq 0, \text{rank } M_d(\beta) = \text{rank } M_{d-1}(\beta)\}$ , the set of positive flat moment matrices with data of degree  $2d$ . For  $M_d(\beta) \in \mathcal{F}_d$ , we refer to  $\beta$  as a *flat* multisequence. Now if  $M_d(\beta)$  belongs to  $\overline{\mathcal{F}_d}$ , the closure of  $\mathcal{F}_d$  in any of the (equivalent) norms on the  $\rho_d \times \rho_d$  matrices, then, by Theorems 1.3 and 1.4,  $L_\beta$  is positive. Moreover, if  $M_d(\beta) = \lim_{k \rightarrow \infty} M_d^{(k)}$ , where each  $M_d^{(k)}$  is positive semidefinite and flat, then a unique representing measure for  $M_d^{(k)}$  can be explicitly computed using [CF7] (see Section 2), and this serves as an approximate representing measure for  $M_d$ . Although a concrete characterization of positivity for  $L_\beta$  seems unlikely, in view of the ease of detecting flatness (by simply checking the positivity and rank conditions), we are motivated to seek a concrete characterization of membership in  $\overline{\mathcal{F}_d}$  (cf. Question 1.7 below). To illustrate membership in  $\overline{\mathcal{F}_d}$ , consider the example mentioned just before Theorem 1.2.

**Example 1.5.** We have  $n = 1$  and

$$M \equiv M_2(\beta^{(4)}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

$M$  is positive semidefinite, but not *recursively generated*, so  $\beta$  has no measure (see Section 2). However, with

$$M_2^{(k)} := \begin{pmatrix} 1 & 1 & 1 + \frac{1}{k} \\ 1 & 1 + \frac{1}{k} & 1 + \epsilon_k \\ 1 + \frac{1}{k} & 1 + \epsilon_k & 2 \end{pmatrix} \quad (\epsilon_k = \frac{k^2 + \sqrt{k^5 - 2k^4 - k^3}}{k^3}),$$

we see that  $M_2^{(k)}$  is positive and flat, with  $\text{rank } M_2^{(k)} = \text{rank } M_1^{(k)} = 2$ , and clearly  $\lim_{k \rightarrow \infty} M_2^{(k)} = M$ , so  $M_2(\beta) \in \overline{\mathcal{F}_2}$  and  $L_\beta$  is positive. (Of course, since  $n = 1$  in this example, positivity of  $L_\beta$  also follows from  $M \succeq 0$  via sums of squares.) Let  $\gamma = \sqrt{k^3(k^2 - 2k - 1)}$  and let  $\psi = \sqrt{k^2(3k + 2k^2 + k^3 - 4\gamma)}$ . Then (as described in Section 2) the unique representing measure for  $M_2^{(k)}$ , which serves as an approximate representing measure for  $M$ , is given by  $\mu^{(k)} := \alpha_1^{(k)} \delta_{x_1^{(k)}} + \alpha_2^{(k)} \delta_{x_2^{(k)}}$ , where  $x_1^{(k)} = \frac{1}{2k^2}(\gamma - \psi) \rightarrow 1$ ,  $x_2^{(k)} = \frac{1}{2k^2}(\gamma + \psi) \rightarrow +\infty$ ,  $\alpha_1^{(k)} = \frac{-2k^2 + \gamma + \psi}{2\psi} \rightarrow 1$ ,  $\alpha_2^{(k)} = \frac{-2k^2 + \gamma - \psi}{2\psi} \rightarrow 0$ . The fact that  $\{x_2^{(k)}\}$  diverges reflects the fact that  $\beta$  has no representing measure.

Now suppose  $n \geq 1$  and  $M_d(\beta) = \lim_{k \rightarrow \infty} M_d(\beta^{[k]})$ , where each  $\beta^{[k]}$  is a sequence of degree  $2d$  with  $M_d(\beta^{[k]})$  positive and flat. It follows that

$$(1.2) \quad M_d(\beta) \succeq 0,$$

and lower semicontinuity of rank ([Her, Proposition 1.12(i)]) implies that

$$\text{rank } M_d(\beta) \leq \liminf_{k \rightarrow \infty} \text{rank } M_d(\beta^{[k]}) = \liminf_{k \rightarrow \infty} \text{rank } M_{d-1}(\beta^{[k]}) \leq \rho_{d-1},$$

whence

$$(1.3) \quad \text{rank } M_d(\beta) \leq \rho_{d-1}.$$

For  $n = 1$ ,  $d \geq 1$  or  $n = d = 2$ , the main result of [FN2], which follows, characterizes membership in  $\overline{\mathcal{F}_d}$  in terms of (1.2) and (1.3).

**Theorem 1.6** ([FN2]). *Let  $n = 1, d \geq 1$  or  $n = d = 2$ .  $M_d \equiv M_d(\beta)$  belongs to  $\overline{\mathcal{F}_d}$  if and only if  $M_d \succeq 0$  and  $\text{rank } M_d \leq \rho_{d-1}$ . In this case, there exist moment matrices  $M_d^{(k)} \equiv M_d(\beta^{[k]})$  ( $k \geq 1$ ) such that  $\lim_{k \rightarrow \infty} M_d^{(k)} = M_d$  and for each  $k$ ,  $\text{rank } M_d^{(k)} = \text{rank } M_{d-1}^{(k)} = \text{rank } M_d$ .*

Note also that for  $n \geq 1$ ,  $d = 1$ , (1.2) and (1.3) imply that  $M_1(\beta)$  is flat (with rank 1), so Theorem 1.6 holds in all the cases of Hilbert's theorem; these results motivate the following question.

**Question 1.7.** Let  $n, d \geq 1$ . If  $M_d(\beta) \succeq 0$  and  $\text{rank } M_d \leq \rho_{d-1}$ , does  $M_d(\beta)$  belong to  $\overline{\mathcal{F}_d}$ ?

Note that in each case in which Question 1.7 has an affirmative answer, (1.2) and (1.3) provide a simple test for the existence of a representing measure for  $\beta^{(2d-1)}$  (via Theorem 1.2). Theorem 1.1 provides a positive answer to Question 1.7 for  $n = 2$ ,  $d = 3$ ; this is perhaps surprising, because several moment theorems which hold within the framework of Hilbert's theorem, e.g., the solution to TMP for  $n = d = 2$  [CF5], break down in the bivariate degree 6 case (cf. [F]). To further address membership in  $\overline{\mathcal{F}_d}$ , observe the necessary condition arising from Theorem 1.2 that positivity of  $L_\beta$ , including membership of  $M_d(\beta)$  in  $\overline{\mathcal{F}_d}$ , entails

$$(1.4) \quad \beta^{(2d-1)} \in \mathcal{R}_{n, 2d-1}.$$

Of course, (1.4) is equivalent to TMP for  $\beta^{(2n-1)}$  and is therefore difficult to check in general. Nevertheless, the results of this paper provide some positive evidence concerning the following weaker version of Question 1.7.

**Question 1.8.** Let  $n, d \geq 1$ . If  $M_d \succeq 0$ ,  $\text{rank } M_d \leq \rho_{d-1}$ , and  $\beta^{(2d-1)}$  has a representing measure, is  $M_d$  a limit of positive flat moment matrices?

Let  $M_d \equiv M_d(\beta)$  denote a bivariate moment matrix of degree  $2d$ , and consider the block matrix decomposition

$$M_d \equiv \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d \end{pmatrix};$$

thus  $M_{d-1}$  and  $B_d$  together contain the data in  $\beta^{(2d-1)}$ , and  $C_d$  is a Hankel matrix comprised of the data of degree  $2d$ . As we discuss in Section 2,  $M_d$  is positive semidefinite if and only if i)  $M_{d-1} \succeq 0$ , ii) there exists a matrix  $W$  such that  $B_d = M_{d-1}W$ , and iii)  $C_d \succeq C_d^b \equiv C_d^b := W^T M_{d-1} W$ , i.e.,  $\Delta^b \equiv C_d - C_d^b \succeq 0$ . Let  $\rho_d = \dim \mathbb{R}[x, y]_d (= \frac{(d+1)(d+2)}{2})$ , which coincides with the size of  $M_d$ ; thus,  $r \equiv \text{rank } M_{d-1} \leq \rho_{d-1}$ . Now suppose  $\beta^{(2d-1)}$  has a  $\kappa$ -atomic representing measure  $\mu$ ; then  $M_d[\mu]$  is of the form

$$M_d[\mu] = \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d[\mu] \end{pmatrix}.$$

Let  $\Delta \equiv \Delta[\mu] := C_d - C_d[\mu]$ ,  $s = \text{rank } \Delta[\mu]$ ,  $\rho[\mu] := s + \kappa$ , and

$$M_\Delta := \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}.$$

We now state our main result.

**Theorem 1.9.** Let  $n = 2$ ,  $d \geq 1$ . Suppose that  $M_d(\beta) \succeq 0$  and that  $\beta^{(2d-1)}$  has a  $\kappa$ -atomic representing measure  $\mu$ . If  $\Delta \equiv \Delta[\mu] \succeq 0$  and  $\rho[\mu] \leq \rho_{d-1}$ , then  $M_d \in \overline{\mathcal{F}}_d$ . Moreover, if  $\rho[\mu] \leq \tau \leq \rho_{d-1}$ , then there exists a sequence of positive flat moment matrices,  $\{M_d^{(k)}\}_{k=1}^\infty$ , such that  $M_d(\beta) = \lim_{k \rightarrow \infty} M_d^{(k)}$  and, for each  $k$ ,  $\text{rank } M_d^{(k)} = \text{rank } M_{d-1}^{(k)} = \tau$ .

We note two basic cases where Theorem 1.9 applies. For  $M_d \succeq 0$ , let

$$M^b := \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C^b \end{pmatrix}$$

and

$$M_{\Delta^b} := \begin{pmatrix} 0 & 0 \\ 0 & \Delta^b \end{pmatrix},$$

so that

$$(1.5) \quad M_d = M^b + M_{\Delta^b}.$$

Let  $r = \text{rank } M_{d-1}$  and  $s^b = \text{rank } \Delta^b$ . Then  $\rho \equiv \text{rank } M_d = r + s^b$  and  $\text{rank } M^b = r$ . For the cases covered by Theorem 1.6, we showed in [FN2] that  $C^b$  is always Hankel; thus, Theorem 1.6 can be derived from the following result.

**Corollary 1.10.** The conclusions of Theorem 1.9 hold if  $M_d \succeq 0$ ,  $\rho \equiv \text{rank } M_d \leq \rho_{d-1}$ , and  $C^b$  is Hankel.

*Proof.* Suppose  $M_d \succeq 0$  and  $\text{rank } M_d \leq \rho_{d-1}$ . If  $C^b$  is a Hankel matrix, then  $M^b$  is a flat moment matrix extension of  $M_{d-1}$  using the data in  $B_d$ . Theorem 1.4 thus shows that  $M^b$  has a unique representing measure  $\mu$ , which is  $r$ -atomic ( $r \equiv \text{rank } M_{d-1}$ ). Thus, the hypotheses of Theorem 1.9 hold with  $\kappa = r$ ,  $\Delta = \Delta^b$ ,  $s = s^b$ ,  $\rho[\mu] = r + s = \rho \leq \rho_{d-1}$ ; note from iii) above that  $\Delta \succeq 0$  since  $M_d \succeq 0$ .  $\square$

In Example 2.1, (1.2) and (1.3) hold, but  $C^b$  is not Hankel; however, we show in Section 4 that this example falls within the scope of the next result.

**Corollary 1.11.** *The conclusions of Theorem 1.9 hold if  $M_d$  has a  $\kappa$ -atomic representing measure  $\mu$  with  $\kappa \leq \rho_{d-1}$ ; in particular, this applies if  $M_d \succeq 0$ ,  $\text{rank } M_d \leq \rho_{d-1}$ , and  $M_d$  admits a flat extension  $M_{d+1}$ .*

*Proof.* For  $\mu$  satisfying the hypothesis,  $\Delta[\mu] = 0$ ,  $s = 0$ ,  $\rho[\mu] = \kappa \leq \rho_{d-1}$ , so Theorem 1.9 applies. In the flat extension case, Theorem 1.4 implies the existence of  $\mu$  with  $\kappa \equiv \text{card supp } \mu = \text{rank } M_d \leq \rho_{d-1}$ .  $\square$

If we ignore the calculations of ranks in the proof of Theorem 1.9, we arrive at the following criterion for positivity of  $L_\beta$ .

**Proposition 1.12.** *If  $M_d(\beta) \succeq 0$  and  $\beta^{(2d-1)}$  has a finitely atomic representing measure  $\mu$  such that  $\Delta[\mu] \succeq 0$ , then  $L_\beta$  is positive.*

Section 2 concerns positive moment matrices, including a geometric characterization of flatness (Theorem 2.7) that we require in the sequel. Section 3 concerns *determining sequences*, which provide a tool for relating the rank of a moment matrix to the geometry of the support of a representing measure. Using Proposition 3.1 and particular determining sequences that we describe in Section 3, we prove Theorem 1.9 and Proposition 1.12 in Section 4. The proof of Theorem 1.1 is a synthesis of these and other results and appears in Section 5. Theorem 1.1 also yields a new solution to the singular quartic moment problem (Proposition 5.13). The proofs of Theorem 1.1 and of several examples that we present below depend in part on calculations using the computer algebra system in *Mathematica*; in the sequel we refer to these as *symbolic calculations*.

We conclude this section by noting an application of Theorem 1.9 to polynomial optimization on  $\mathbb{R}^2$ . For  $p \in \mathbb{R}[x, y]$ , the Optimization Problem entails estimating

$$(1.6) \quad p_* := \inf_{(x,y) \in \mathbb{R}^2} p(x, y).$$

We recall the *moment relaxations* for (1.6) introduced by J.-B. Lasserre [Las]. For  $t \geq 1$ , we define the  $t$ -th Lasserre relaxation by

$$(1.7) \quad p_t := \inf\{L_y(p) : y \equiv y^{(2t)}, y_{00} = 1, M_t(y) \succeq 0\}.$$

It is not difficult to check that  $p_t \leq p_*$  and that for  $t' \geq t$ ,  $p_{t'} \geq p_t$ ; thus,  $\{p_t\}$  is convergent, and  $p^{mom} \equiv \lim_{t \rightarrow \infty} p_t \leq p_*$ . For fixed  $t$ , the infimum in (1.6) is not necessarily attained. Assuming that the infimum is attained, at some optimal sequence  $y_* \equiv y_*^{(2t)}$ , we seek criteria which imply that  $L_{y_*}(p) = p_*$ , so that we have finite convergence of  $\{p_s\}$  to  $p_*$  at stage  $s = t$ . A basic result of [Las] shows that this is the case if  $M_t(y_*)$  is flat, and this concrete condition is used as the stopping criterion in the optimization program in [HeLa]. In [FN2, Theorem 1.5] we showed that convergence at stage  $t$  occurs, more generally, if  $L_{y_*}$  is positive. Thus, Corollary 1.10 provides a broader concrete condition for finite convergence than flatness, namely,  $\text{rank } M_t(y_*) \leq \rho_{t-1}$  and  $C_t^b(y_*)$  Hankel.

## 2. POSITIVE MOMENT MATRICES

In this section we present some results concerning positive moment matrices that will be used in the sequel. We begin, more generally, with a real symmetric block



matrix of the form

$$(2.1) \quad M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}.$$

It is well known that  $M$  is positive semidefinite if and only if  $A \succeq 0$ ,  $B = AW$  for some matrix  $W$ , and  $C \succeq W^T AW (= B^T W)$  (cf. [CF1]). In this case, let  $\Delta \equiv C - W^T AW$  denote the Schur complement, so that  $\text{rank } M = \text{rank } A + \text{rank } \Delta$ . We have  $\text{rank } M = \text{rank } A$ , and we say that  $M$  is a *flat extension* of  $A$  if and only if  $C = W^T AW$ . Flat extensions are uniquely determined by  $A$  and  $B$ , for if there are matrices  $W$  and  $V$  such that  $AW = B = AV$ , then  $W^T AW - V^T AV = W^T AV - V^T AV = (AW - AV)^T V = 0$ . Given  $A \succeq 0$  and  $B = AW$  as above, if we set  $C^b = W^T AW$  and

$$(2.2) \quad M^b = \begin{pmatrix} A & B \\ B^T & C^b \end{pmatrix},$$

then clearly  $M^b$  is a positive, flat extension of  $A$ .

Let us denote the positive  $n$ -variable moment matrix  $M_d \equiv M_d(\beta)$  by

$$(2.3) \quad M_d(\beta) = \begin{pmatrix} M_{d-1}(\beta) & B_d \\ B_d^T & C_d \end{pmatrix},$$

with  $B_d = M_{d-1}W$ . In this setting, we sometimes denote  $C^b$  by  $C_d^b$ . Following [CF2], [CF7], we say that  $M_d$  is *flat* if  $\text{rank } M_d = \text{rank } M_{d-1}$ , i.e.,  $C_d = C_d^b$ . Theorem 1.4 implies that in this case  $M_d$  has a unique representing measure which is  $(\text{rank } M_d)$ -atomic. For  $M_d \succeq 0$ , a rank-preserving extension  $M_{d+1}$  is said to be a *flat extension*; in this case,  $M_{d+1} \succeq 0$  and  $M_{d+1}$  is itself flat. In the sequel, if  $\mu$  is a positive Borel measure with convergent moments  $\beta \equiv \beta^{(2d)}$ , we sometimes denote  $M_d(\beta)$  by  $M_d[\mu]$ ; moreover, for a moment matrix  $M_d(\beta)$ , we sometimes refer to a representing measure for  $\beta$  as a representing measure for  $M_d(\beta)$ .

Let  $n = 2$  and suppose  $M_d(\beta) \succeq 0$ , so that  $B_d = M_{d-1}(\beta)W$  for some matrix  $W$  (as above). Since  $n = 2$ ,  $C_d$  is a *Hankel* matrix; i.e.,  $C_d$  is constant on cross-diagonals. We note that in general  $C^b \equiv W^T M_{d-1} W$  need not be Hankel. We illustrate this in the following example of [FN2], which we will analyze further in Section 4.

**Example 2.1.** Consider the positive moment matrix  $M_3(\beta)$  defined by

$$M_3(\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}.$$



Then  $B_3 = M_2 W$ , where

$$W := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{so } C^b \equiv W^T M_2 W = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix} \text{ and } \Delta \equiv C_3 - C^b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In [FN2] we used ad hoc methods to show that  $M_3 \in \overline{\mathcal{F}}_3$ . In Section 4 we will establish membership in  $\overline{\mathcal{F}}_3$  based on Corollary 1.11.  $\square$

In the sequel we denote the successive columns of  $M_d(\beta)$  by  $X^i$  ( $|i| \leq d$ ) in degree-lexicographic order. Let  $p \in \mathbb{R}[x]_d$ ,  $p = \sum a_i x^i$ . We define a corresponding element of  $\text{Col } M_d(\beta)$ , the column space of  $M_d(\beta)$ , by  $p(X) := \sum a_i X^i (= M_d \widehat{p})$ . Thus, each column dependence relation in  $M_d(\beta)$  can be expressed as  $p(X) = 0$  for some  $p \in \mathbb{R}[x]_d$ ,  $p \neq 0$ . Following [CF2], [CF7], we say that  $M_d(\beta)$  is *recursively generated* if  $p, q, pq \in \mathbb{R}[x]_d$ ,  $p(X) = 0 \implies (pq)(X) = 0$ . Positivity and recursiveness are necessary conditions for  $\beta$  to have a representing measure [CF7], and for  $n = 1$  these conditions are also sufficient [CF1] (cf. Theorem 2.3). In the sequel, to construct examples, we will employ the following result without further reference.

**Theorem 2.2** (Structure Theorem [CF2], [CF7]). *If  $M_d(\beta)$  is positive semidefinite, then the following properties hold:*

- i)  $M_{d-1}(\beta)$  is recursively generated.
- ii) If  $p \in \mathcal{P}_{d-1}$  and  $p(X) = 0$  in  $\text{Col } M_{d-1}$ , then  $p(X) = 0$  in  $\text{Col } M_d$ .
- iii) If  $p \in \mathcal{P}_d$  satisfies  $p(X) = 0$  in  $\text{Col } M_d$  and  $q \in \mathcal{P}_d$  satisfies  $\deg pq \leq d$ , then  $(pq)(X) = 0$  in  $\text{Col } \begin{pmatrix} M_{d-1}(y) & B_d \end{pmatrix}$ . Further, if  $M_d$  is flat, then  $(pq)(X) = 0$  in  $\text{Col } M_d$ .

For  $n = 1$ , we will denote a truncated moment sequence of degree  $2d$  by  $y^{(2d)} = \{y_0, \dots, y_{2d}\}$ . Then  $M_d(y)$  is a Hankel matrix, which we henceforth denote by  $H_d \equiv H_d(y) := (y_{a+b})_{0 \leq a, b \leq d}$ . In this case, Theorem 2.2 may be expressed as follows.

**Theorem 2.3** ([CF1]). *Suppose  $H_d \equiv H_d(y)$  is a positive Hankel matrix,  $y_0 > 0$ . If  $H_d \succ 0$  (positive definite), then  $H_d$  admits infinitely many distinct flat extensions and corresponding  $(d+1)$ -atomic representing measures. If  $H_d$  is singular, let  $r := \min\{s : 1 \leq s \leq d : H_s \text{ is singular}\}$ . Let  $v = (y_r, \dots, y_{2r-1})^T$  and let  $c \equiv (c_0, \dots, c_{r-1})^T = M_{r-1}^{-1} v$ . Then (a)  $y_j = c_0 y_{j-r} + \dots + c_{r-1} y_{j-1}$  ( $r \leq j \leq 2d-1$ ) and (b)  $y_{2d} \geq c_0 y_{2d-r} + \dots + c_{r-1} y_{2d-1}$ .*

*Further, the following are equivalent: i)  $y$  has a representing measure; ii) equality holds in (b) (above); iii)  $H_d$  is recursively generated; iv)  $\text{rank } H_d = r$ ; v)  $H_d$  is a flat extension of  $H_{r-1}$ . There is strict inequality in (b) (above)  $\iff \text{rank } H_d = r + 1$ .*

Recall the *variety* associated with  $\beta$  and  $M_d(\beta)$ , defined by

$$\mathcal{V} \equiv \mathcal{V}(\beta) := \bigcap_{p \in \mathbb{R}[x]_d, p(X)=0} \mathcal{Z}(p),$$

where  $\mathcal{Z}(p) := \{x \in \mathbb{R}^n : p(x) = 0\}$ . We will repeatedly use the following result.

**Proposition 2.4** ([CF2], [CF7, Prop. 2.10, Cor. 2.12]). *If  $\beta$  has a representing measure  $\mu$ , then  $\text{supp } \mu \subseteq \mathcal{V}(\beta)$ . Furthermore,*

- i)  $\text{rank } M_d \leq \text{card } \text{supp } \mu \leq \text{card } \mathcal{V}(\beta)$ ;
- ii) for  $p \in \mathbb{R}[x]_d$ ,  $p(X) = 0$  in Col  $M_d$  (equivalently,  $M_d \hat{p} = 0$ )  $\iff p|_{\text{supp } \mu} \equiv 0$ .

The next result shows how the variety can be used to construct the unique representing measure corresponding to a flat extension.

**Theorem 2.5** ([CF7, Theorem 1.2]). *Suppose  $M_d \succeq 0$  admits a flat extension  $M_{d+1}(\tilde{\beta})$ , i.e.,  $\text{rank } M_{d+1}(\tilde{\beta}) = \text{rank } M_d$ . Then  $\text{card } \mathcal{V}(\tilde{\beta}) = \rho \equiv \text{rank } M_d$ , say  $\mathcal{V}(\tilde{\beta}) = \{w_1, \dots, w_\rho\}$ . Let  $X^{i_1}, \dots, X^{i_\rho}$  denote a basis for the column space of  $M_d$ , define  $p_j(x) := x^{i_j} \in \mathbb{R}[x]_d$ , and let  $\omega = (\beta_{i_1}, \dots, \beta_{i_\rho})$ . Then the Vandermonde matrix  $V \equiv (p_j(w_k))_{1 \leq j, k \leq \rho}$  is invertible, and we define  $\alpha \equiv (\alpha_1, \dots, \alpha_\rho)$  by  $\alpha^T := V^{-1}\omega^T$ . The unique representing measure for  $\tilde{\beta}$  is given by*

$$(2.4) \quad \mu := \sum_{i=1}^{\rho} \alpha_i \delta_{w_i}.$$

We now consider the moment matrix determined by a finitely atomic measure. For  $w \equiv (x_1, \dots, x_n) \in \mathbb{R}^n$ , let  $\zeta \equiv \zeta[w] = (1, x_1, x_2, \dots, x_1^d, x_1^{d-1}x_2, \dots, x_{n-1}x_n^{d-1}, x_n^d)$ . In the sequel,  $\delta_{\{w\}}$  denotes the atomic measure with support  $\{w\}$ , whose moment sequence  $\beta \equiv \beta^{(2d)}$  satisfies  $\beta_i = x_1^{i_1} \cdots x_n^{i_n}$  ( $i \equiv (i_1, \dots, i_n)$ ,  $|i| \leq 2d$ ). As noted in [Lau],  $M_d[\delta_{\{w\}}] \equiv M_d(\beta) = \zeta^T \zeta$ , so  $\text{rank } M_d[\delta_{\{w\}}] = 1$ . It follows from subadditivity of rank that if  $\mu$  is a  $t$ -atomic measure,  $\mu \equiv \sum_{i=1}^t \alpha_i \delta_{\{w_i\}}$  with each  $\alpha_i > 0$ , then

$$(2.5) \quad \text{rank } M_d[\mu] = \text{rank } \sum_{i=1}^t \alpha_i M_d[\delta_{\{w_i\}}] \leq \sum_{i=1}^t \text{rank } \alpha_i M_d[\delta_{\{w_i\}}] = t.$$

In Theorem 2.7, we present a geometric criterion for flatness that we will use in the proof of Theorem 1.9. We begin with a lower estimate for  $\text{rank } M_d$  which complements the upper estimate in Proposition 2.4 i). Given  $\mathcal{P} \equiv \{p_1, \dots, p_t\} \subset \mathbb{R}[x]_d$  and points  $w \equiv \{w_i\}_{i=1}^t \subset \mathbb{R}^n$ , let  $V \equiv V[\mathcal{P}; w]$  denote the Vandermonde-type matrix

$$(2.6) \quad \begin{pmatrix} p_1(w_1) & \cdots & p_t(w_1) \\ \vdots & \vdots & \vdots \\ p_1(w_t) & \cdots & p_t(w_t) \end{pmatrix}.$$

**Proposition 2.6.** *If  $w \equiv \{w_1, \dots, w_t\} \subseteq \mathcal{V}(\beta)$ ,  $\mathcal{P} \equiv \{p_1, \dots, p_t\} \subset \mathbb{R}[x]_d$ , and  $V \equiv V[\mathcal{P}; w]$  is invertible, then  $\text{rank } M_d(\beta) \geq t$ . If  $M_d$  has a  $t$ -atomic representing measure  $\mu$ ,  $\text{supp } \mu = w \equiv \{w_1, \dots, w_t\}$ , then  $\text{rank } M_d = t \iff$  there exist polynomials  $\mathcal{P} \equiv \{p_1, \dots, p_t\} \subset \mathbb{R}[x]_d$  such that  $V[\mathcal{P}; w]$  is invertible.*

*Proof.* Let  $v := \rho_d$ ; since  $V$  is invertible,  $\mathcal{P}$  is independent, so we may extend  $\mathcal{P}$  to a basis  $\{p_j\}_{j=1}^v$  for  $\mathbb{R}[x]_d$ . Suppose  $p \equiv \sum_{i=1}^v c_i p_i \in \mathbb{R}[x]_d$  and  $\hat{p} \in \ker M_d$ ; then  $p|_{\mathcal{V}(\beta)} \equiv 0$ , so  $p(w_j) = 0$  ( $1 \leq j \leq t$ ).

We first consider the case when  $t < v$ . Setting

$$\widehat{V} = \begin{pmatrix} p_1(w_1) & \cdots & p_t(w_1) & p_{t+1}(w_1) & \cdots & p_v(w_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1(w_t) & \cdots & p_t(w_t) & p_{t+1}(w_t) & \cdots & p_v(w_t) \end{pmatrix} \equiv \begin{pmatrix} V & \widetilde{V} \end{pmatrix}$$

and  $c \equiv (c_1, \dots, c_v)^T$ , we have

$$(2.7) \quad \widehat{V}c = 0.$$

Now  $V$  is invertible, so multiplying (2.7) by  $V^{-1}$  on the left, we obtain  $Kc = 0$ , with  $K$  of the form  $\begin{pmatrix} I_t & D \end{pmatrix}$ , where  $I_t$  denotes the  $t \times t$  identity matrix and  $D := V^{-1}\widetilde{V}$  is of the form  $D_{t \times (v-t)} \equiv (d_{ij})_{1 \leq i \leq t, 1 \leq j \leq v-t}$ . We thus see that

$$(2.8) \quad c_i + d_{i1}c_{t+1} + \cdots + d_{i,v-t}c_v = 0 \quad (1 \leq i \leq t).$$

Now,

$$\begin{aligned} p &= (-d_{1,1}c_{t+1} - \cdots - d_{1,v-t}c_v)p_1 + \cdots + (-d_{t,1}c_{t+1} - \cdots - d_{t,v-t}c_v)p_t \\ &\quad + c_{t+1}p_{t+1} + \cdots + c_vp_v \\ &= c_{t+1}(-d_{1,1}p_1 - \cdots - d_{t,1}p_t + p_{t+1}) + c_{t+2}(-d_{1,2}p_1 - \cdots - d_{t,2}p_t + p_{t+2}) \\ &\quad + \cdots + c_v(-d_{1,v-t}p_1 - \cdots - d_{t,v-t}p_t + p_v). \end{aligned}$$

The preceding calculation shows that  $\ker M_d$  is spanned by the  $v - t$  vectors  $\widehat{\eta}_j$  ( $1 \leq j \leq v - t$ ), where

$$(2.9) \quad \eta_j := -d_{1,j}p_1 - \cdots - d_{t,j}p_t + p_{t+j}.$$

Thus,  $\dim \ker M_d \leq v - t$ , and it follows that  $\text{rank } M_d = \rho_d - \dim \ker M_d \geq \rho_d - (v - t) = t$ .

In the case when  $t = v$ , we set  $\widehat{V} = V$  and conclude from (2.7) that  $\widehat{c} = 0$ , whence  $\ker M_d = \{0\}$ . Thus,  $\text{rank } M_d = \rho_d = v = t$ .

Next, suppose  $\mu$  is a representing measure,  $\text{supp } \mu = w \equiv \{w_i\}_{i=1}^t$ . From Proposition 2.4,  $w \subseteq \mathcal{V}(\beta)$  and  $\text{rank } M_d \leq t$ . If there exists  $\mathcal{P} \equiv \{p_i\}_{i=1}^t \subset \mathbb{R}[x]_d$  such that  $V[\mathcal{P}; w]$  is invertible, then the above argument implies that  $\text{rank } M_d = t$ . Conversely, suppose  $\text{rank } M_d = t$ , and let  $\mathcal{B} \equiv \{X^{i_1}, \dots, X^{i_t}\}$  denote a basis

for  $\text{Col } M_d$ . Let  $\mathcal{P} = \{x^{i_j}\}_{j=1}^t \subset \mathbb{R}[x]_d$ . Suppose  $p := \sum_{j=1}^t c_j x^{i_j}$ ,  $p \neq 0$ , and

$V[\mathcal{P}; w]\widehat{p} = 0$ , i.e.,  $p|_{\text{supp } \mu} \equiv 0$ . Since  $\mu$  is a representing measure for  $M_d$ , Proposition 2.4 ii) implies  $p(X) = 0$  in  $\text{Col } M_d$ , contradicting the independence of  $\mathcal{B}$ ; thus  $V[\mathcal{P}; w]$  is invertible.  $\square$

**Theorem 2.7.**  $M_d \succeq 0$  is flat if and only if there exist a  $t$ -atomic representing measure  $\mu$  for  $M_d$ ,  $\text{supp } \mu = w \equiv \{w_1, \dots, w_t\} \subset \mathbb{R}^n$ , and polynomials  $\mathcal{P} \equiv \{p_1, \dots, p_t\} \subset \mathbb{R}[x]_{d-1}$ , such that  $V[\mathcal{P}; w]$  is invertible.

*Proof.* The “only if” direction follows immediately from Theorem 2.5 applied to the flat extension  $M_d$  of  $M_{d-1}$ . For the converse, it follows from Proposition 2.6 (applied to  $M_{d-1}$ ), and from (2.5), that  $t = \text{rank } M_{d-1} \leq \text{rank } M_d \leq t$ , so  $M_d$  is flat.  $\square$

We conclude this section with a change of variables result for  $n = 2$  that we will use in Section 5 to simplify certain moment matrices. An analogous change of variables for the truncated complex moment problem appears in [CF5, Proposition

1.7], but here we require a stronger version. Consider the *degree-one map*  $\mathcal{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$  defined by  $\mathcal{T}(x, y) = (\tilde{x}, \tilde{y}) \equiv (a + \alpha x + \gamma y, b + \delta x + \lambda y)$ , where  $\alpha\lambda - \gamma\delta \neq 0$  (which insures that  $\mathcal{T}$  is a bijection). Given  $\beta \equiv \beta^{(2d)}$ , let  $L_\beta$  denote its Riesz functional. We define a multisequence  $\tilde{\beta} \equiv \tilde{\beta}^{(2d)}$  by  $\tilde{\beta}_{ij} = L_\beta(\tilde{x}^i \tilde{y}^j)$  ( $i, j \geq 0$ ,  $i + j \leq 2d$ ). Let  $\tilde{M}_d \equiv M_d(\tilde{\beta})$  denote the moment matrix for  $\tilde{\beta}$  and let  $L_{\tilde{\beta}}$  denote its Riesz functional. For  $p \in \mathbb{R}[x, y]_{2d}$ ,  $p(x, y) \equiv \sum a_{ij} x^i y^j$ , we have  $L_{\tilde{\beta}}(p) = L_\beta(p(\tilde{x}, \tilde{y})) = L_\beta(\sum a_{ij} (a + \alpha x + \gamma y)^i (b + \delta x + \lambda y)^j)$ . Let  $\mathcal{P}_d$  denote the space of polynomial vectors of degree at most  $d$ , i.e.,  $\mathcal{P}_d = \{\hat{p} : p \in \mathbb{R}[x, y]_d\}$ , and define a matrix  $J \equiv J_d : \mathcal{P}_d \mapsto \mathcal{P}_d$  by  $J\hat{p} := \widehat{p(\tilde{x}, \tilde{y})}$  ( $p \in \mathbb{R}[x, y]_d$ ); for example,  $J\hat{x} \equiv J(0, 1, 0, 0, \dots, 0)^T = (a, \alpha, \gamma, 0, \dots, 0)^T$ .

**Proposition 2.8.** i)  $\tilde{M}_d = J^T M_d J$ ;

ii)  $J$  is invertible;

iii)  $\tilde{M}_d \succeq 0 \iff M_d \succeq 0$ ;

iv)  $\text{rank } \tilde{M}_d = \text{rank } M_d$ .

v) The formula  $\mu = \tilde{\mu} \circ \mathcal{T}$  is a one-to-one correspondence between representing measures for  $\beta$  and  $\tilde{\beta}$ , which preserves measure class and cardinality of support; moreover,  $\mathcal{T}(\text{supp } \mu) = \text{supp } \tilde{\mu}$ .

vi)  $\tilde{M}_d (\succeq 0)$  admits a flat extension if and only if  $M_d (\succeq 0)$  admits a flat extension.

vii) For  $p \in \mathbb{R}[x, y]_d$ , define  $q(\tilde{x}, \tilde{y}) := p \circ \mathcal{T}^{-1}(\tilde{x}, \tilde{y}) \in \mathbb{R}[\tilde{x}, \tilde{y}]_d$ . Then in  $\text{Col } \tilde{M}_d$ ,  $q(\tilde{X}, \tilde{Y}) = J^T p(X, Y)$ . In particular,  $p(X, Y) = 0$  if and only if  $q(\tilde{X}, \tilde{Y}) = 0$ , and  $\mathcal{V}(\tilde{\beta}) = \mathcal{T}(\mathcal{V}(\beta))$ ; further,  $\tilde{M}_d$  is recursively generated if and only if  $M_d$  is recursively generated.

viii)  $\tilde{M}_d \in \mathcal{F}_d$  if and only if  $M_d \in \mathcal{F}_d$ .

ix) Let  $\tau$  satisfy  $1 \leq \tau \leq \rho_{d-1}$  and let  $\mathcal{F}_{d,\tau} = \{M_d \in \mathcal{F}_d : \text{rank } M_d = \tau\}$ . Then  $\tilde{M}_d \in \mathcal{F}_{d,\tau}$  if and only if  $M_d \in \mathcal{F}_{d,\tau}$ .

x) For  $M_d \succeq 0$ ,  $M_d^b \equiv J^T M_d^b J = (\tilde{M}_d)^b$ ; in particular,  $C_d^b$  is Hankel in  $M_d^b$  if and only if the  $C_d^b$  block of  $(\tilde{M}_d)^b$  is Hankel.

*Proof.* The proofs of i)-vi) are direct analogues of those of the corresponding parts of [CF5, Proposition 1.7], so we omit the details. For vii), we have  $q(\tilde{X}, \tilde{Y}) = \tilde{M}_d \hat{q} = \tilde{M}_d p \circ \mathcal{T}^{-1} = J^T M_d ((Jp \circ \mathcal{T}^{-1})) = J^T M_d (p \circ \mathcal{T}^{-1} \circ \mathcal{T}) = J^T M_d \hat{p} = J^T p(X, Y)$ . Since  $J$  is invertible, it follows that  $q(\tilde{X}, \tilde{Y}) = 0 \iff p(X, Y) = 0$ ; the other conclusions follow similarly. For viii), first note that, relative to  $\tilde{\beta}$ , for  $1 \leq k \leq d$ , we also have the subsequence  $\tilde{\beta}^{(2k)}$  and its moment matrix  $\tilde{M}_k$  (a submatrix of  $\tilde{M}_d$ ). For  $p \in \mathbb{R}[x, y]_d$ ,  $\deg p(x, y) = \deg p(\tilde{x}, \tilde{y})$ , so  $J$  maps  $\mathcal{P}_k$  onto itself and the map  $J_k$  associated to  $\tilde{\beta}^{(2k)}$  satisfies  $J_k = J|_{\mathcal{P}_k}$ . Each  $J_k$  is invertible, and  $\tilde{M}_k = J_k^T M_k J_k$ . In particular,  $\text{rank } \tilde{M}_{d-1} = \text{rank } M_{d-1}$ . It now follows from iii) and iv) that  $\tilde{M}_d$  is positive and flat if  $M_d$  is positive and flat. The converse follows by symmetry (using the transform  $\mathcal{T}^{-1}$ ). For ix), suppose  $M_d = \lim_{k \rightarrow \infty} M_d^{[k]}$ , where each  $M_d^{[k]}$ , corresponding to  $\beta^{[k]}$  (a sequence of degree  $2d$ ), is positive and flat, with  $\text{rank } M_d^{[k]} = \tau$ . Then  $\tilde{M}_d \equiv J^T M_d J = \lim_{k \rightarrow \infty} J^T M_d^{[k]} J$ , and iii)-iv) and viii) imply that each  $J^T M_d^{[k]} J$  is positive and flat, with  $\text{rank } J^T M_d^{[k]} J = \tau$ . Thus

$M_d \in \overline{\mathcal{F}_{d,\tau}} \implies \widetilde{M}_d \in \overline{\mathcal{F}_{d,\tau}}$ , and the converse follows by symmetry (again using the transform  $\mathcal{T}^{-1}$ ).

For x), since  $M_d \succeq 0$ , there is a matrix  $W$  such that

$$M_d = \begin{pmatrix} M_{d-1} & M_{d-1}W \\ W^T M_{d-1} & C_d \end{pmatrix}, \quad C_d \succeq C_d^b \equiv W^T M_{d-1} W,$$

and

$$M_d^b = \begin{pmatrix} M_{d-1} & M_{d-1}W \\ W^T M_{d-1} & W^T M_{d-1} W \end{pmatrix}.$$

Denoting

$$J \equiv J_d = \begin{pmatrix} J_{d-1} & K \\ 0 & L \end{pmatrix},$$

we have

$$\widetilde{M}_d \equiv J^T M_d J = \begin{pmatrix} \widetilde{M}_{d-1} & \widetilde{B}_d \\ \widetilde{B}_d^T & \widetilde{C}_d \end{pmatrix},$$

where  $\widetilde{M}_{d-1} = J_{d-1}^T M_{d-1} J_{d-1}$  and  $\widetilde{B}_d = \widetilde{M}_{d-1} \widetilde{W}$ , with  $\widetilde{W} = J_{d-1}^{-1}(K + WL)$ . Thus

$$(\widetilde{M}_d)^b = \begin{pmatrix} \widetilde{M}_{d-1} & \widetilde{B}_d \\ \widetilde{B}_d^T & \widetilde{C}_d^b \end{pmatrix},$$

with  $\widetilde{C}_d^b \equiv \widetilde{W}^T \widetilde{M}_{d-1} \widetilde{W} = (K^T + L^T W^T)(J_{d-1}^{-1})^T (J_{d-1}^T M_{d-1} J_{d-1}) J_{d-1}^{-1} (K + WL) = (K + WL)^T M_{d-1} (K + WL)$ . A block matrix calculation now shows that the last expression coincides with the  $C$ -block of  $J^T M_d^b J$ , so  $(\widetilde{M}_d)^b$  coincides with  $\widetilde{M}_d^b \equiv J^T M_d^b J$ . Now, if  $C_d^b$  is Hankel in  $M_d^b$ , then  $M_d^b$  is a moment matrix, and therefore so also is  $\widetilde{M}_d^b$ . Thus  $(\widetilde{M}_d)^b (= \widetilde{M}_d^b)$  is a moment matrix, whence its  $C_d^b$  block is Hankel. The converse follows by symmetry.  $\square$

### 3. DETERMINING SEQUENCES FOR BIVARIATE POLYNOMIALS

In this section we examine certain determining sequences for bivariate polynomials that will be used in proving Theorem 1.9. By a *determining sequence* for  $\mathbb{R}[x, y]_d$  we mean a sequence of distinct points  $\Gamma \equiv \{(x_k, y_k)\}_{k=1}^\infty \subset \mathbb{R}^2$  with the following property: for  $p \in \mathbb{R}[x, y]_d$ , if there exists  $k_p \in \mathbb{N}$  such that  $p(x_k, y_k) = 0$  for every  $k \geq k_p$ , then  $p \equiv 0$ . In particular,  $\Gamma$  is determining if each nonzero polynomial in  $\mathbb{R}[x, y]_d$  has at most a finite number of zeros in  $\Gamma$ .

The following result will prove useful in relating the geometry of the support of a representing measure to the rank of a moment matrix.

**Proposition 3.1.** *Suppose  $\mathcal{P} \equiv \{p_1(x, y), \dots, p_t(x, y)\}$  is a set of independent polynomials in  $\mathbb{R}[x, y]_d$ . If  $w^{(j)} \equiv \{w_i^{(j)}\}_{i=1}^\infty$  is a determining sequence for  $\mathbb{R}[x, y]_d$  ( $1 \leq j \leq t$ ), then given  $k > 0$ , there exist integers  $k_1, \dots, k_t \geq k$  such that  $V[\mathcal{P}; w_{k_1}^{(1)}, \dots, w_{k_t}^{(t)}]$  is invertible.*

Note that we are not assuming that the determining sequences in Proposition 3.1 are distinct; however, in the proof of Theorem 1.9 in Section 4, the sequences will be distinct. Furthermore, the integers  $k_1, \dots, k_t$  need not be distinct, but clearly the points  $w_{k_1}^{(1)}, \dots, w_{k_t}^{(t)}$  are necessarily distinct. We also note that although we have formulated Proposition 3.1 only for 2 variables, the concept of determining sequence and the proof that we present below are valid as well for  $n$  variables.

*Proof of Proposition 3.1.* The proof is by induction on  $t \geq 1$ . Since  $p_1(x, y) \not\equiv 0$  and  $w^{(1)}$  is a determining sequence, there exists  $k_1 \geq k$  such that  $p_1(w_{k_1}^{(1)}) = \frac{1}{k_1^m} Q_d^{(m)}(w_1) \neq 0$ , so the result holds for  $t = 1$ . Assume by induction that the result holds up to  $t - 1$ . Since  $\mathcal{P}_{t-1} \equiv \{p_1, \dots, p_{t-1}\}$  is independent, there exist  $k_1, \dots, k_{t-1} \geq k$  such that  $V_{t-1} \equiv V[\mathcal{P}_{t-1}; w_{k_1}^{(1)}, \dots, w_{k_{t-1}}^{(t-1)}]$  is invertible. Let  $V(x, y)$  be defined by

$$(3.1) \quad V(x, y) := \begin{pmatrix} p_1(w_{k_1}^{(1)}) & \cdots & p_{t-1}(w_{k_1}^{(1)}) & p_t(w_{k_1}^{(1)}) \\ \vdots & \vdots & \vdots & \vdots \\ p_1(w_{k_{t-1}}^{(t-1)}) & \cdots & p_{t-1}(w_{k_{t-1}}^{(t-1)}) & p_t(w_{k_{t-1}}^{(t-1)}) \\ p_1(x, y) & \cdots & p_{t-1}(x, y) & p_t(x, y) \end{pmatrix},$$

and set  $p(x, y) := \det V(x, y) (\in \mathbb{R}[x]_d)$ . Expanding  $\det V(x, y)$  with respect to the bottom row, we have

$$p(x, y) = D_1 p_1(x, y) - D_2 p_2(x, y) + \cdots + (-1)^{t+1} D_t p_t(x, y),$$

where each  $D_i$  is a minor of  $p(x, y)$  and  $D_t = \det V_{t-1} (\neq 0, \text{ by induction})$ . Since  $\{p_1, \dots, p_t\}$  is independent in  $\mathbb{R}[x, y]_d$  and  $D_t \neq 0$ , then  $p(x, y) \not\equiv 0$  in  $\mathbb{R}[x, y]_d$ . Now, since  $\{w_i^{(t)}\}_{i=1}^\infty$  is a determining sequence, there exists  $k_t \geq k$  such that  $p(w_{k_t}^{(t)}) \neq 0$ , whence  $V[\mathcal{P}; w_{k_1}^{(1)}, \dots, w_{k_t}^{(t)}]$  is invertible.  $\square$

We require a determining sequence for  $\mathbb{R}[x, y]_d$  that is compatible with a moment matrix construction in the proof of Theorem 1.9, and to this end we will focus on the sequence  $\Gamma \equiv \{(k + \frac{1}{k^m}, \alpha k)\}_{k=1}^\infty$ , where  $m \in \mathbb{N}$  satisfies  $m \geq d$  and  $\alpha \in \mathbb{R}$  is nonzero.

**Proposition 3.2.**  $\Gamma$  is a determining sequence for  $\mathbb{R}[x, y]_d$ . If  $p \in \mathbb{R}[x, y]_d$  and  $p \not\equiv 0$ , then  $p$  has at most  $md + d$  zeros of the form  $(k + \frac{1}{k^m}, \alpha k)$ , where  $k \in \mathbb{N}$ .

Let  $p \in \mathbb{R}[x, y]_d$ ,  $p(x, y) = \sum_{i,j \geq 0, i+j \leq d} a_{ij} x^i y^j$ . For  $\alpha \neq 0$  and  $m \in \mathbb{N}$ ,  $m \geq d$ , let  $q_d^{(m)}(x) := p(x + \frac{1}{x^m}, \alpha x)$ , and define  $Q_d^{(m)} \in \mathbb{R}[x]_{md+d}$  by

$$(3.2) \quad Q_d^{(m)}(x) := x^{md} q_d^{(m)}(x) = x^{md} \sum_{i,j \geq 0, i+j \leq d} a_{ij} (x + \frac{1}{x^m})^i (\alpha x)^j.$$

It follows from (3.2) that to prove Proposition 3.2, it suffices to show that  $Q_d^{(m)}(x) \neq 0$ ; i.e.,  $Q_d^{(m)}$  has some nonzero coefficient. The coefficients of  $Q_d^{(m)}$  as a polynomial in  $\mathbb{R}[x]_{md+d}$  are not the coefficients  $a_{ij}$  of  $p(x, y)$ , but, rather, certain linear combinations of the  $a_{ij}$ . To show that at least one such coefficient is nonzero, we require some auxiliary results. In the sequel we say that a power monomial  $x^k$  appearing in  $Q_d^{(m)}$  is *affiliated* with coefficient  $a_{ij}$  of  $p(x, y)$  if the formal expansion of (3.2) into a sum of monomials contains one or more terms of the form  $\gamma a_{ij} x^k$ , say  $\gamma_r a_{ij} x^k$  ( $1 \leq r \leq s$ ) for some  $s \geq 1$ , where  $\gamma_r \equiv \alpha^j \delta_r$  is a nonzero absolute constant (i.e., independent of all  $a_{uv}$ ); we sometimes write this as  $x^k \rightsquigarrow a_{ij}$ . It is clear from (3.2) that for fixed  $i, j, k$ ,  $\sum_{r=1}^s \gamma_r a_{ij} x^k$  is a term of the form  $\Gamma a_{ij} x^k$ , where  $\Gamma = \alpha^j \beta$  and  $\beta$  is some binomial coefficient. For

our purposes, it is necessary to identify exactly which monomial powers  $x^k$  appear in  $Q_d^{(m)}$  and with which coefficients of  $p(x, y)$  each monomial power is affiliated. For example, if  $p(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$ , then  $Q_2^{(m)}(x) = a_{00}x^{2m} + a_{10}x^{2m+1} + a_{10}x^m + a_{01}\alpha x^{2m+1} + a_{20}x^{2m+2} + 2a_{20}x^{2m+1} + a_{20} + a_{11}\alpha x^{2m+2} + a_{11}\alpha x^{m+1} + a_{02}\alpha^2 x^{2m+2}$ , so the sequences of coefficients with which each power monomial of  $Q_2^{(m)}$  is affiliated are as follows:

$$x^0 \rightsquigarrow a_{20}; \quad x^m \rightsquigarrow a_{10}; \quad x^{m+1} \rightsquigarrow a_{20}, a_{11}; \quad x^{2m} \rightsquigarrow a_{00}; \quad x^{2m+1} \rightsquigarrow a_{10}, a_{01}; \quad x^{2m+2} \rightsquigarrow a_{20}, a_{11}, a_{02}.$$

To identify the powers and affiliations in  $Q_d^{(m)}$ , consider the index set for the monomials in  $p(x, y)$ ,  $\mathbb{Z}_d^2 := \{(i, j) | i, j \geq 0, i + j \leq d\}$ , and for  $m \geq d$ , let  $\mathcal{S}_{m,d} = \{rm + s : r, m \in \mathbb{Z}, 0 \leq r \leq d, 0 \leq s \leq r\}$ . The requirement  $m \geq d$  is necessary to insure that  $\mathcal{S}_{m,d}$  contains  $\frac{(d+1)(d+2)}{2}$  distinct elements. It is now easy to check that the map  $\phi \equiv \phi_d : \mathbb{Z}_d^2 \rightarrow \mathcal{S}_{m,d}$ , defined by  $\phi(i, j) = (d-i)m + j$  is a bijection, with inverse  $\psi : \mathcal{S}_{m,d} \rightarrow \mathbb{Z}_d^2$  given by  $\psi(rm + s) = (d-r, s)$ . (Indeed, if  $(i, j) \in \mathbb{Z}_d^2$ , then  $r \equiv d-i$  and  $s \equiv j$  satisfy  $0 \leq r \leq d$  and  $s = j \leq d-i = r$ , so  $\phi$  is well-defined, and similarly for  $\psi$ .) Note also that

$$(3.3) \quad \phi_{d-1}(i, j) + m = \phi_d(i, j).$$

Now let  $r_d(x) = \sum_{i,j \geq 0, i+j \leq d-1} a_{ij}(x + \frac{1}{x^m})^i (\alpha x)^j$  (which we can interpret as  $q_{d-1}^{(m)}(x)$

relative to  $\sum_{i,j \geq 0, i+j \leq d-1} a_{ij}x^i y^j$ ), and set  $s_d(x) = \sum_{i,j \geq 0, i+j=d} a_{ij}(x + \frac{1}{x^m})^i (\alpha x)^j$ , so

that  $q_d^{(m)}(x) = r_d(x) + s_d(x)$ . Thus,  $Q_d^{(m)}(x) = x^m(x^{(d-1)m}r_d(x)) + x^{md}s_d(x)$ , or

$$(3.4) \quad Q_d^{(m)}(x) = x^m Q_{d-1}^{(m)}(x) + x^{md} s_d(x).$$

**Lemma 3.3.** *Let  $d \geq 1$ ,  $m \geq d$ . For  $(i, j) \in \mathbb{Z}_d^2$ , the power monomials  $x^k$  in  $Q_d^{(m)}$  that are affiliated with coefficient  $a_{ij}$  of  $p(x, y)$  are  $x^{\phi(i,j)}$ ,  $x^{\phi(i,j)+m+1}$ ,  $\dots$ ,  $x^{\phi(i,j)+i(m+1)}$ .*

*Proof.* The proof is by induction on  $d \geq 1$ . For  $d = 1$ ,  $m \geq 1$ ,  $Q_1^{(m)}(x) = a_{00}x^m + a_{10}x^{m+1} + a_{10}x^0 + a_{01}\alpha x^{m+1}$ . The unique power affiliated with  $a_{00}$  is  $x^m$ , and  $\phi(0, 0) = m$ ,  $i = 0$ . The powers affiliated with  $a_{10}$  are  $x^0$  and  $x^{m+1}$ , and  $\phi(1, 0) = 0$ ,  $i = 1$ . The unique power affiliated with  $a_{01}$  is  $x^{m+1}$ , and  $\phi(0, 1) = m + 1$ ,  $i = 0$ .

Assume the result is true up to  $d-1$  and consider  $Q_d^{(m)}$ . For  $i, j \geq 0$  with  $i + j \leq d-1$ , it follows by induction that the powers affiliated with  $a_{ij}$  in  $Q_{d-1}^{(m)}$  are precisely  $x^{\phi_{d-1}(i,j)}$ ,  $x^{\phi_{d-1}(i,j)+m+1}$ ,  $\dots$ ,  $x^{\phi_{d-1}(i,j)+i(m+1)}$ . Thus, (3.4) implies that the powers affiliated with  $a_{ij}$  in  $Q_d^{(m)}$  are precisely  $x^{\phi_{d-1}(i,j)+m}$ ,  $x^{\phi_{d-1}(i,j)+2m+1}$ ,  $\dots$ ,  $x^{\phi_{d-1}(i,j)+(i+1)m+i}$ , and (3.3) implies that these powers coincide with  $x^{\phi_d(i,j)}$ ,  $x^{\phi_d(i,j)+m+1}$ ,  $\dots$ ,  $x^{\phi_d(i,j)+i(m+1)}$ .

Now consider the case  $i, j \geq 0$ ,  $i + j = d$ . It follows from (3.4) that the powers affiliated with  $a_{ij}$  in  $Q_d^{(m)}$  are those affiliated with  $a_{ij}$  in the formal expansion of  $x^{md}s_d(x)$ , i.e., those powers which appear in the complete expansion of  $x^{md}(x + \frac{1}{x^m})^i (\alpha x)^{d-i}$ . The monomial powers in this expansion are  $x^{md}x^i x^{d-i}$ ,  $x^{md}x^{i-1}x^{-m}x^{d-i}$ ,  $x^{md}x^{i-2}x^{-2m}x^{d-i}$ ,  $\dots$ ,  $x^{md}x^{-im}x^{d-i}$ , which simplify to  $x^{md+d}$ ,  $x^{md-m+d-1}$ ,  $x^{md-2m+d-2}$ ,  $\dots$ ,  $x^{md-mi+d-i}$ , and this sequence coincides with  $x^{\phi(i,d-i)+i(m+1)}$ ,  $x^{\phi(i,d-i)+(i-1)(m+1)}$ ,  $\dots$ ,  $x^{\phi(i,d-i)}$ .  $\square$



**Corollary 3.4.** For  $(i, j) \in \mathbb{Z}_d^2$ , the powers affiliated with  $a_{ij}$  in  $Q_d^{(m)}$  are  $x^{\phi(i,j)}$ ,  $x^{\phi(i-1,j+1)}$ ,  $\dots$ ,  $x^{\phi(0,j+i)}$ .

*Proof.* For  $0 \leq a \leq i$ ,  $\phi(i, j) + a(m+1) = (d-i)m + j + am + a = (d-i+a)m + j + a = (d-(i-a))m + j + a = \phi(i-a, j+a)$ . The result now follows from Lemma 3.3.  $\square$

Note that every power of  $x$  in the expansion of the right hand side of (3.2) is affiliated with some  $a_{ij}$ . The preceding result shows that each such power of  $x$  is of the form  $x^{\phi(i',j')}$  for some  $(i', j') \in \mathbb{Z}_d^2$ , and it also shows that every such power is affiliated with at least one coefficient in  $Q_d^{(m)}$ , namely,  $a_{i'j'}$ . Since  $\phi: \mathbb{Z}_d^2 \rightarrow \mathcal{S}_{m,d}$  is a bijection, it also follows that the powers in  $Q_d^{(m)}$  can be expressed as  $x^{rm+s}$  with  $0 \leq r \leq d$ ,  $0 \leq s \leq r$ .

*Proof of Proposition 3.2.* It suffices to show that  $Q_d^{(m)}$  has at most  $md + m$  distinct real roots. Suppose to the contrary that  $Q_d^{(m)}$  has more than  $md + d$  roots. Since  $\deg Q_d^{(m)} \leq md + d$ , then  $Q_d^{(m)} \equiv 0$  in  $\mathbb{R}[x]$ , and we will show that this implies  $p(x, y) \equiv 0$  in  $\mathbb{R}[x, y]$ , contradicting the hypothesis. We may list the powers corresponding to  $\mathcal{S}_{m,d}$  by increasing degree as

$$(3.5) \quad x^0, x^m, x^{m+1}, x^{2m}, x^{2m+1}, x^{2m+2}, \dots, x^{dm}, x^{dm+1}, \dots, x^{dm+d}.$$

We will prove by induction on the position,  $\tau$ , of a power in this list ( $1 \leq \tau \leq \frac{(d+1)(d+2)}{2}$ ) that for the  $\tau$ -th power  $x^{rm+s}$  ( $0 \leq r \leq d$ ,  $0 \leq s \leq r$ ), every coefficient  $a_{ij}$  of  $p(x, y)$  with which  $x^{rm+s}$  is affiliated satisfies  $a_{ij} = 0$ .

Note from Corollary 3.4 that if  $x^0$  is affiliated with coefficient  $a_{ij}$  of  $p(x, y)$ , then  $0 = \phi(i - \rho, j + \rho)$  for some  $\rho$ ,  $0 \leq \rho \leq i$ . Now  $0 = (d - (i - \rho))m + j + \rho = (d - i + \rho)m + j + \rho$ , and thus  $\rho = 0$ ,  $d = i$ ,  $j = 0$ . Conversely,  $\phi(d, 0) = 0$ , so  $x^0$  is affiliated only with  $a_{d,0}$ ; since  $Q_d^{(m)} \equiv 0$ , it follows that  $a_{d,0} = 0$ .

Now assume by induction that for the first  $\tau - 1$  powers in (3.5), each coefficient  $a_{ij}$  with which any of these powers is affiliated satisfies  $a_{ij} = 0$ . Let  $x^{rm+s}$  denote the  $\tau$ -th power (for some  $r, s$  with  $0 \leq r \leq d$ ,  $0 \leq s \leq r$ ). Now  $rm + s = \phi(i, j)$  for a unique  $(i, j) \in \mathbb{Z}_d^2$ , and Corollary 3.4 shows that  $x^{rm+s}$  is affiliated with  $a_{ij}$ . Next, suppose  $x^{rm+s}$  is also affiliated with some other coefficient  $a_{i',j'}$ ,  $(i, j) \neq (i', j')$ . Now  $\phi(i', j') = r'm + s'$  for some  $r', s'$  with  $0 \leq r' \leq d$ ,  $0 \leq s' \leq r'$ . Lemma 3.3 shows that  $x^{\phi(i',j')}$  comes first in the sequence of powers affiliated with  $a_{i',j'}$ , and since this sequence is ordered by strictly increasing degree of the powers, we have  $r'm + s' < rm + s$ . By induction, since  $x^{r'm+s'}$  is a lower power than  $x^{rm+s}$ , it follows that  $a_{i',j'} = 0$ . Thus, whether or not  $x^{rm+s}$  is affiliated only with  $a_{ij}$ , we can now represent the total coefficient of  $x^{rm+s}$  in  $Q_d^{(m)}$  as  $\gamma a_{ij}$  for some nonzero absolute constant  $\gamma$ . Since  $Q_d^{(m)} \equiv 0$ , it now follows that  $a_{ij} = 0$ , so every coefficient with which  $x^{rm+s}$  is affiliated equals 0. By induction, we conclude that for every power  $x^{rm+s}$  ( $0 \leq r \leq d$ ,  $0 \leq s \leq r$ ), each coefficient with which  $x^{rm+s}$  is affiliated equals 0. Since, given  $a_{ij}$ ,  $x^{\phi(i,j)}$  is affiliated with  $a_{ij}$  (Corollary 3.4), it follows that each  $a_{ij} = 0$ . Since  $p(x, y) \not\equiv 0$  by hypothesis, this contradiction implies that  $Q_d^{(m)}$  has at most  $md + d$  real roots, which completes the proof.  $\square$

The following example shows that if  $m < d$ , then we may have  $p(x, y) \not\equiv 0$ , but  $Q_d^{(m)} \equiv 0$ . Let  $d = 2$ ,  $m = 1$ , and for  $\alpha \neq 0$  and  $a_{11} \neq 0$ , set  $a_{00} = -\alpha a_{11}$ ,  $a_{02} = -\frac{a_{11}}{\alpha}$ ,  $a_{01} = a_{10} = a_{20} = 0$ . Then  $p(x, y) = -a_{11}(\alpha - xy + \frac{1}{\alpha}y^2) \not\equiv 0$ , but

$q_2^{(1)}(x) \equiv p(x + \frac{1}{x}, \alpha x) = -a_{11}(\alpha - (\alpha x^2 + \alpha) + \frac{1}{\alpha}(\alpha^2 x^2)) = 0$ . Thus  $Q_2^{(1)}(x) \equiv xq_2^{(1)}(x) = 0$ .

We continue this section by identifying another determining sequence that we will use in proving Theorem 1.9.

**Proposition 3.5.** *Let  $m \in \mathbb{N}$ ,  $m \geq d$ . If  $p \in \mathbb{R}[x, y]_d$  and  $p \not\equiv 0$ , then  $p$  has at most  $md + d$  zeros of the form  $(\frac{1}{k^m}, k)$ , where  $k \in \mathbb{N}$ .*

*Proof.* For  $p(x, y) \equiv \sum_{i,j \geq 0, i+j \leq d} a_{ij} x^i y^j$ , let  $r_d^{(m)}(x) := p(\frac{1}{x^m}, x)$ , and define  $R_d^{(m)} \in \mathbb{R}[x]_{md+d}$  by

$$(3.6) \quad R_d^{(m)}(x) := x^{md} r_d^{(m)}(x) = x^{md} \sum_{i,j \geq 0, i+j \leq d} a_{ij} \left(\frac{1}{x^m}\right)^i x^j.$$

We will prove that

$$(3.7) \quad R \equiv R_d(x) = \sum_{i=0}^d \sum_{j=0}^i a_{d-i,j} x^{im+j}.$$

If  $p$  has more than  $md + m$  zeros of the form  $(\frac{1}{k^m}, k)$ , then  $R$  has more than  $md + m$  roots, so  $R \equiv 0$  in  $\mathbb{R}[x]$ . Since  $m \geq d$ , the powers  $x^{im+j}$  ( $0 \leq i \leq d$ ,  $0 \leq j \leq i$ ) are distinct; moreover, each coefficient of  $p$  is of the form  $a_{d-i,j}$  for some  $i, j$  with  $0 \leq i \leq d$ ,  $0 \leq j \leq i$ . It thus follows from (3.7) that each coefficient of  $p$  equals 0, so  $p \equiv 0$ , a contradiction. It remains to establish (3.7). Write  $p(x, y) = u(x, y) + v(x, y)$ , where  $u(x, y) = \sum_{i,j \geq 0, i+j \leq d-1} a_{ij} x^i y^j$  and  $v(x, y) = \sum_{i,j \geq 0, i+j=d} a_{ij} x^i y^j$ . Then  $R_d(x) = x^m x^{(d-1)m} u(\frac{1}{x^m}, x) + x^{md} v(\frac{1}{x^m}, x)$ , and thus

$$(3.8) \quad R_d(x) = x^m R_{d-1}(x) + x^{md} v\left(\frac{1}{x^m}, x\right).$$

Now (3.7) follows from (3.8) by a straightforward induction on  $d \geq 1$ ; we omit the details.  $\square$

We conclude this section by describing a determining sequence, corresponding to a prescribed point  $(x_0, y_0) \in \mathbb{R}^2$ , that we will use in Section 5. For  $r, s \in \mathbb{N}$ , with  $s > 0$  and  $r > (d-1)s$ , consider the sequence  $\Lambda := \{(x_0 + \frac{1}{k^r}, y_0 + \frac{1}{k^{r+s}})\}_{k=1}^\infty$ .

**Proposition 3.6.**  *$\Lambda$  is a determining sequence for  $\mathbb{R}[x, y]_d$ . If  $p \in \mathbb{R}[x, y]_d$  and  $p \not\equiv 0$ , then  $p$  has at most  $(r+s)d$  zeros of the form  $(x_0 + \frac{1}{k^r}, y_0 + \frac{1}{k^{r+s}})$ , where  $k \in \mathbb{N}$ .*

The hypothesis  $s > 0$  is necessary: for  $d = 1$ , if  $a, b, c \in \mathbb{R}$  are nonzero, with  $b = -c$ , if  $a + bx_0 + cy_0 = 0$ , and if we take  $r = 1$  and  $s = 0$ , then  $p(x, y) := a + bx + cy$  vanishes on  $\Lambda$ . We will use notation similar to that in the proof of Proposition 3.2. For  $p \in \mathbb{R}[x, y]_d$ ,  $p(x, y) = \sum_{i,j \geq 0, i+j \leq d} a_{ij} x^i y^j$ , let  $q(x) = p(x_0 + \frac{1}{x^r}, y_0 + \frac{1}{x^{r+s}})$ , and define  $Q_d \in \mathbb{R}[x]_{(r+s)d}$  by

$$(3.9) \quad Q_d(x) := x^{(r+s)d} q(x) = x^{(r+s)d} \sum_{i,j \geq 0, i+j \leq d} a_{ij} \left(x_0 + \frac{1}{x^r}\right)^i \left(y_0 + \frac{1}{x^{r+s}}\right)^j.$$

It follows readily that

$$(3.10) \quad Q_d = x^{r+s} Q_{d-1} + x^{(r+s)d} \sum_{i=0}^d a_{i,d-i} \left(x_0 + \frac{1}{x^r}\right)^i \left(y_0 + \frac{1}{x^{r+s}}\right)^{d-i}.$$

**Lemma 3.7.** *The powers that appear in  $Q_d$  are  $x^{ir+js}$  with  $0 \leq i \leq j \leq d$ , and*

$$(3.11) \quad x^{ir+js} \rightsquigarrow a_{j-i+a,d-j+b} \quad (a, b \geq 0, \quad a+b \leq i).$$

Moreover, the complete expansion of (3.10) includes the term  $a_{j-i,d-j} x^{ir+js}$  (corresponding to  $a=b=0$  in (3.11)).

We have already seen that  $s > 0$  is necessary for Proposition 3.6. In order for the exponents  $ir+js$  ( $0 \leq i \leq j \leq d$ ) to be distinct, it is necessary and sufficient that  $r > (d-1)s$ , which is the requirement in the definition of  $\Lambda$ .

*Proof of Lemma 3.7.* The proof is by induction on  $d \geq 1$ . For  $d=1$ ,  $p(x, y) = a_{00} + a_{10}x + a_{01}y$ , so  $Q_1(x) = a_{00}x^{r+s} + a_{10}x_0x^{r+s} + a_{10}x^s + a_{01}y_0x^{r+s} + a_{01}$ . Thus  $x^0$  is affiliated only with  $a_{01}$  (in agreement with  $i=j=0$ ,  $a=b=0$  in (3.11)), and  $x^s$  is affiliated only with  $a_{10}$  (in agreement with  $i=0$ ,  $j=1$ ,  $a=b=0$ ). For  $x^{r+s}$ , we have  $i=j=1$ , and this power is affiliated with  $a_{00}$  ( $a=b=0$ ),  $a_{01}$  ( $a=0$ ,  $b=1$ ), and  $a_{10}$  ( $a=1$ ,  $b=0$ ). By inspection, we also see that in  $Q_1(x)$  we have the required terms  $a_{01}x^0$ ,  $a_{10}x^s$ , and  $a_{00}x^{r+s}$ .

Now assume the result holds for  $d-1$ . Thus, in  $Q_{d-1}$ , for  $0 \leq i \leq j \leq d-1$ , the power  $x^{ir+js}$  is affiliated precisely with the coefficients  $a_{j-i+a,(d-1)-j+b}$ , where  $a, b \geq 0$ ,  $a+b \leq i$ ; moreover, the expansion of (3.10) (with  $d-1$  taking the place of  $d$ ) contains the term  $a_{j-i,(d-1)-j} x^{ir+js}$ . It follows from (3.10) that in  $Q_d$  the power  $x^{(i+1)r+(j+1)s}$  is also affiliated with these coefficients. Setting  $i' := i+1$ ,  $j' := j+1$ , in  $Q_d$ , for  $1 \leq i' \leq j' \leq d$ , the power  $x^{i'r+j's}$  is affiliated with

$$a_{j'-i'+a,d-j'+b} \quad (a, b \geq 0, \quad a+b \leq i'-1);$$

moreover, the expansion of (3.10) for  $Q_d$  contains the terms

$$a_{j-i,(d-1)-j} x^{(i+1)r+(j+1)s} = a_{j'-i',d-j'} x^{i'r+j's}.$$

To complete the proof of (3.11) (but with  $i$  and  $j$  in (3.11) replaced by  $i'$  and  $j'$ ), it is necessary to show that i) for the case  $i'=0$ ,  $0 \leq j' \leq d$ ,  $x^{j's} \rightsquigarrow a_{j',d-j'}$  and  $Q_d$  contains the term  $a_{j',d-j'} x^{j's}$ ; and ii)  $x^{i'r+j's}$  is affiliated with  $a_{j'-i'+a,d-j'+b}$  when  $0 \leq i' \leq j' \leq d$  and  $a+b=i'$ .

Note that in (3.10), the expansion of  $x^{(r+s)d} \sum_{i=0}^d a_{i,d-i} \left(x_0 + \frac{1}{x^r}\right)^i \left(y_0 + \frac{1}{x^{r+s}}\right)^{d-i}$  is of the form

$$(3.12) \quad x^{(r+s)d} \sum_{i=0}^d a_{i,d-i} \left( \sum_{u=0}^i \binom{i}{u} x_0^u x^{-r(i-u)} \right) \left( \sum_{v=0}^{d-i} \binom{d-i}{v} y_0^v x^{-(r+s)(d-i-v)} \right).$$

Upon further expansion, this shows that

$$(3.13) \quad x^{(u+v)r+(v+i)s} \rightsquigarrow a_{i,d-i} \quad (0 \leq i \leq d, \quad 0 \leq u \leq i, \quad 0 \leq v \leq d-i).$$

To show i), given  $0 \leq j' \leq d$ , in (3.13) we let  $i=j'$ ,  $u=v=0$  and see that  $x^{is}$  is affiliated with  $a_{i,d-i}$ . Further, with  $u=v=0$ , it is clear from (3.12) that  $Q_d$  contains the term  $a_{i,d-i} x^{is}$ . For ii), we have  $1 \leq i' \leq j' \leq d$ ,  $a+b=i'$ , and we seek to show that  $x^{i'r+j's} \rightsquigarrow a_{j'-i'+a,d-j'+b}$ . Set  $\tilde{i} \equiv j'-b$  so that  $0 \leq \tilde{i} \leq d$  and

$a_{j'-i'+a, d-j'+b} = a_{\tilde{i}, d-\tilde{i}}$ . Let  $u := i' - b$  ( $0 \leq u \leq d$ ) and  $v := b$  ( $0 \leq v \leq d - \tilde{i}$ ). It now follows from (3.13) that  $x^{i'r+j's} = x^{(u+v)r+(v+\tilde{i})s} \rightsquigarrow a_{\tilde{i}, d-\tilde{i}}$ , which proves ii) and completes the induction.  $\square$

*Proof of Proposition 3.6.* If  $p(x, y)$  has more than  $(r+s)d$  zeros in  $\Lambda$ , then  $Q_d$  has more than  $(r+s)d$  distinct real roots, so  $Q_d \equiv 0$ . Under this assumption, we will prove that each coefficient of  $p(x, y)$  equals 0, in contradiction to  $p \not\equiv 0$ . From Lemma 3.7, with  $i = 0$ ,  $0 \leq j \leq d$ ,  $a = b = 0$ , we see that in  $Q_d$ ,  $x^{js}$  is affiliated solely with  $a_{j, d-j}$ , so  $a_{j, d-j} = 0$  ( $0 \leq j \leq d$ ). Let us assume by induction that for some  $i$ ,  $1 \leq i \leq d$ , we have  $a_{uv} = 0$  for  $u, v \geq 0$ ,  $d \geq u+v \geq d-i+1$ . We will show that each coefficient of degree  $d-i$  equals 0. Lemma 3.7 implies that for  $i \leq j \leq d$ , the coefficient of  $x^{ir+j's}$  in  $Q_d$  is of the form

$\alpha_{00}a_{j-i, d-j} + \alpha_{10}a_{j-i+1, d-j} + \alpha_{01}a_{j-i, d-j+1} + \cdots + \alpha_{i0}a_{j, d-j} + \cdots + \alpha_{0i}a_{j-i, d-j+i}$  for certain absolute constants  $\alpha_{uv}$  with  $\alpha_{00} = 1$ . Except for  $a_{j-i, d-j}$ , each coefficient of  $p(x, y)$  that appears in the preceding expression has degree at least  $d-i+1$ , so by induction each such coefficient equals 0. It follows that the coefficient of  $x^{ir+j's}$  in  $Q_d$  reduces to  $\alpha_{00}a_{j-i, d-j}$ , whence  $a_{j-i, d-j} = 0$ . Thus, each coefficient of  $p$  of degree  $d-i$  equals 0; the result now follows by induction.  $\square$

#### 4. LIMITS OF POSITIVE FLAT MOMENT MATRICES

In this section we prove Theorem 1.9. Throughout this section  $M_d \equiv M_d(\beta)$  (as in (2.3)) denotes a bivariate moment matrix of degree  $2d$ . Now suppose  $\beta^{(2d-1)}$  has a  $\kappa$ -atomic representing measure  $\mu$ , so that  $M_d[\mu]$  is of the form

$$M_d[\mu] := \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d[\mu] \end{pmatrix}.$$

Let  $\Delta \equiv \Delta[\mu] := C_d - C_d[\mu]$ ,  $s \equiv \text{rank } \Delta[\mu]$ ,  $\rho[\mu] \equiv s + \kappa$ , and

$$M_\Delta := \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}.$$

We now re-state Theorem 1.9 for ease of reference.

**Theorem 4.1.** *Let  $n = 2$ ,  $d \geq 1$ . Suppose that  $M_d(\beta) \succeq 0$  and that  $\beta^{(2d-1)}$  has a  $\kappa$ -atomic representing measure  $\mu$ . If  $\Delta \equiv \Delta[\mu] \succeq 0$  and  $\rho[\mu] \leq \rho_{d-1}$ , then  $M_d \in \overline{\mathcal{F}}_d$ . Moreover, if  $\rho[\mu] \leq \tau \leq \rho_{d-1}$ , then there exists a sequence of positive flat moment matrices,  $\{M_d^{(k)}\}$ , such that  $M_d(\beta) = \lim_{k \rightarrow \infty} M_d^{(k)}$  and for each  $k$ ,  $\text{rank } M_d^{(k)} = \text{rank } M_{d-1}^{(k)} = \tau$ .*

*Proof.* Denote  $\mu$  as  $\mu \equiv \sum_{i=1}^{\kappa} \alpha_i \delta_{w_i}$ , where the points  $w_i$  are distinct and each  $\alpha_i > 0$ .

If  $\rho[\mu] < \rho_{d-1}$  and  $\rho[\mu] < \tau \leq \rho_{d-1}$ , choose additional points  $w_{\rho[\mu]+1}, \dots, w_\tau$  so that all of the  $\tau - s$  points are distinct. (If  $\tau = \rho[\mu]$ , omit all reference to these additional points in the sequel.) Let  $R, S \in \mathbb{N}$ , with  $S > 0$  and  $R > (d-1)S$ . With  $w_j \equiv (x_j, y_j)$  ( $1 \leq j \leq \kappa$  and  $\kappa + s + 1 \leq j \leq \tau$ ), we apply Proposition 3.6 to define the determining sequences  $w^{(j)} \equiv \{w_k^{(j)}\}_{k=1}^\infty$  ( $1 \leq j \leq \kappa$  and  $\kappa + s + 1 \leq j \leq \tau$ ), where  $w_k^{(j)} := (x_j + \frac{1}{k^R}, y_j + \frac{1}{k^{R+S}})$ .

Next, note that  $\Delta$  is a Hankel matrix, say  $\Delta = (u_{i+j})_{0 \leq i, j \leq d}$ . We first consider the case when  $\Delta$  is a moment matrix, i.e.,  $u_0 > 0$ . If  $\Delta \succ 0$  (positive definite),

then  $s = d + 1$  and  $\Delta$  has an  $s$ -atomic representing measure  $\nu$  by Theorem 2.3 (cf. [CF1]). Otherwise,  $\Delta$  is positive and singular, i.e.,  $s \leq d$ , so [FN2, Theorem 3.1] implies that there is a sequence,  $\{H_d(y^{[k]})\}$ , of positive Hankel matrices such that  $\Delta = \lim_{k \rightarrow \infty} H_d(y^{[k]})$  and  $\text{rank } H_d(y^{[k]}) = \text{rank } H_{d-1}(y^{[k]}) = s$ . It follows from

Theorem 1.4 that  $H_d(y^{[k]})$  admits an  $s$ -atomic representing measure  $\nu_k$ . Thus, for the remainder of the proof of this case, replacing  $\Delta$  by  $H_d(y^{[k]})$  and  $C_d$  by  $C_d[\mu] + H_d(y^{[k]})$  if necessary, we may assume that  $\text{rank } \Delta = s$  and that  $\Delta$  admits

an  $s$ -atomic representing measure  $\nu \equiv \sum_{i=\kappa+1}^{\kappa+s} a_i \delta_{\{\alpha_i\}}$  (where each  $a_i > 0$  and the

$\alpha_i$  are distinct). Further, for the purposes of approximation, in the sequel we may assume that  $\alpha_i \neq 0$  ( $\kappa + 1 \leq i \leq \kappa + s$ ). For if a (unique)  $\alpha_{i'}$  satisfies  $\alpha_{i'} = 0$ , then consider the measures  $\nu^{(l)}$  ( $l \geq 1$ ) obtained from  $\nu$  by replacing  $\alpha_{i'} = 0$  by  $\alpha_{i'} = \frac{1}{l}$  (with  $l$  large enough so that the  $\alpha_i$  remain distinct). Clearly,  $\Delta = \lim_{l \rightarrow \infty} H_d[\nu^{(l)}]$ ,

and we claim that  $\text{rank } H_d[\nu^{(l)}] = s$  for all sufficiently large  $l$ . Since  $\nu^{(l)}$  is  $s$ -atomic, then (2.5) implies that  $\text{rank } H_d[\nu^{(l)}] \leq s$ . The reverse inequality follows from lower semicontinuity of rank (cf. Section 1), since  $\Delta = \lim_{l \rightarrow \infty} H_d[\nu^{(l)}]$  and

$\text{rank } \Delta = s$ . In the sequel, by replacing  $\Delta$  by  $H_d[\nu^{(l)}]$  (for sufficiently large  $l$ ), we may thus assume that  $\text{rank } \Delta = s$  and that  $\Delta$  has an  $s$ -atomic measure  $\nu$  (as above) in which the  $\alpha_i$  are distinct and *nonzero*. Now, for  $\kappa + 1 \leq j \leq \kappa + s$ , choose an integer  $m_j \geq d - 1$ . Applying Proposition 3.2 with  $d$  replaced by  $d - 1$ , we define the determining sequences  $w^{(j)} \equiv \{w_k^{(j)}\}_{k=1}^{\infty}$  ( $\kappa + 1 \leq j \leq \kappa + s$ ), where  $w_k^{(j)} := (k + \frac{1}{k^{m_j}}, \alpha_j k)$ .

Let  $v \equiv \rho_{d-1}$  and  $t \equiv \tau$ . Let  $p_1(x, y), \dots, p_v(x, y)$  denote the basis for  $\mathbb{R}[x, y]_{d-1}$  consisting of all of the monomials, and let  $\mathcal{P} := \{p_1, \dots, p_t\}$ . We next apply Proposition 3.1 to  $\mathcal{P}$  and the determining sequences  $w^{(j)}$  ( $1 \leq j \leq t$ ). It follows that given  $k > 0$ , there exist  $k_1, \dots, k_t \geq k$  such that  $V \equiv V[\mathcal{P}, w_{k_1}^{(1)}, \dots, w_{k_t}^{(t)}]$  (as defined in Section 2) is invertible; in particular, the points  $w_j \equiv w_{k_j}^{(j)}$  ( $1 \leq j \leq t$ ) are distinct.

We define three atomic measures that will be used in constructing the flat approximants to  $M_d$ . Let

$$\begin{aligned} \mu^{(k)} &:= \sum_{j=1}^{\kappa} \alpha_j \delta_{w_j} = \sum_{j=1}^{\kappa} \alpha_j \delta_{(x_j + \frac{1}{k^R}, y_j + \frac{1}{k^{R+S}})}, \\ \nu^{(k)} &:= \sum_{j=\kappa+1}^{\kappa+s} \frac{a_j}{k_j^{2d}} \delta_{w_j} \equiv \sum_{j=\kappa+1}^{\kappa+s} \frac{a_j}{k_j^{2d}} \delta_{\{(k_j + \frac{1}{k_j^{m_j}}, \alpha_j k_j)\}}, \\ \sigma^{(k)} &:= \sum_{j=\kappa+s+1}^t \frac{1}{k} \delta_{w_j} \equiv \sum_{j=\kappa+s+1}^t \frac{1}{k} \delta_{(x_j + \frac{1}{k^R}, y_j + \frac{1}{k^{R+S}})}. \end{aligned}$$

Now let  $\omega^{(k)} := \mu^{(k)} + \nu^{(k)} + \sigma^{(k)}$  and set

$$M_d^{(k)} \equiv M_d[\omega^{(k)}] = M_d[\mu^{(k)}] + M_d[\nu^{(k)}] + M_d[\sigma^{(k)}].$$

Straightforward calculations show that  $\lim_{k \rightarrow \infty} M_d[\mu^{(k)}] = M_d[\mu]$ ,  $\lim_{k \rightarrow \infty} M_d[\nu^{(k)}] = M_\Delta$ , and  $\lim_{k \rightarrow \infty} M_d[\sigma^{(k)}] = 0$ . Thus,  $\lim_{k \rightarrow \infty} M_d^{(k)} = M_d[\mu] + M_\Delta = M_d(\beta)$ .

To complete the proof of this case, it remains to show that  $M_d^{(k)}$  is flat. Since  $\omega^{(k)}$  is a representing measure for  $M_d^{(k)}$ ,  $\text{supp } \omega^{(k)} = \{w_1, \dots, w_t\}$ ,  $\mathcal{P} \subset \mathbb{R}[x, y]_{d-1}$ , and  $V[\mathcal{P}; w_1, \dots, w_t]$  is invertible, the conclusion that  $M_d^{(k)}$  is flat follows directly from Theorem 2.7. This completes the proof when  $u_0 > 0$ .

We next consider the case when  $u_0 = 0$ . In this case, it follows from Theorem 2.3 that  $u_j = 0$  ( $0 \leq j \leq 2d-1$ ) and  $\gamma := u_{2d} \geq 0$ . If  $\gamma = 0$ , then  $\Delta = 0$  and  $s = 0$ ; in this case, we may proceed as above except that we skip the step involving  $\Delta$  and  $\nu^{(k)}$  and, in particular, we do not define the sequences  $w^{(j)}$  ( $\kappa+1 \leq j \leq \kappa+s$ ). Assuming that  $\gamma > 0$ , we have  $s = 1$ , and for fixed  $m \geq d-1$ , we use Proposition 3.5 to define the determining sequence  $w^{(\kappa+1)} \equiv \{w_k^{(\kappa+1)}\}_{k=1}^\infty$ , where  $w_k^{(\kappa+1)} = (\frac{1}{k^m}, k)$ . We next apply Proposition 3.1 exactly as before to produce the points  $w_1, \dots, w_t$  with an invertible Vandermonde  $V$ . We define  $\mu^{(k)}$  and  $\sigma^{(k)}$  as before, but we now define

$$\nu^{(k)} := \frac{\gamma}{k^{2d}} \delta_{w_{k_j}^{(j)}},$$

where  $j = \kappa + 1$ . The proof now continues exactly as in the case  $u_0 > 0$ , beginning with the definition of  $\omega^{(k)} := \mu^{(k)} + \nu^{(k)} + \sigma^{(k)}$ .  $\square$

We illustrate Theorem 4.1, and particularly Corollary 1.11, with a continuation of Example 2.1.

**Example 4.2.** In Example 2.1 we have  $M_3 \succeq 0$  and  $\text{rank } M_3 = 6$ , with column relations  $X^2 = 1$ ,  $X^3 = X$ ,  $X^2Y = Y$ , and  $Y^3 = 2Y$ . It is straightforward to check that by propagating these column relations forward, i.e., by defining  $X^4 := X^2$ ,  $X^3Y := XY$ ,  $X^2Y^2 := Y^2$ ,  $XY^3 := 2XY$ ,  $Y^4 := 2Y^2$ , we construct a positive flat moment matrix extension  $M_4$ . Thus  $\beta$  has a 6 atomic representing measure  $\mu$  which may be explicitly computed as described in Section 2 (cf. [CF7]). The support of  $\mu$  is the variety of  $M_4$ , which consists of the common solutions to  $x^2 = 1$  and  $y^3 = 2y$ , namely  $(x_1, y_1) \equiv (-1, 0)$ ,  $(x_2, y_2) \equiv (-1, -\sqrt{2})$ ,  $(x_3, y_3) \equiv (-1, \sqrt{2})$ ,  $(x_4, y_4) \equiv (1, 0)$ ,  $(x_5, y_5) \equiv (1, -\sqrt{2})$ ,  $(x_6, y_6) \equiv (1, \sqrt{2})$ . Note that  $\mathcal{B} \equiv \{1, X, Y, XY, Y^2, XY^2\}$  is a basis for the column space of  $M_3$  and define corresponding polynomials  $p_1(x, y) := 1$ ,  $p_2(x, y) = x$ ,  $p_3(x, y) := y$ ,  $p_4(x, y) = xy$ ,  $p_5(x, y) := y^2$ ,  $p_6(x, y) = xy^2$ . The Vandermonde  $V \equiv (p_i(x_j, y_j))_{1 \leq i, j \leq 6}$  is invertible. Let  $v$  be the vector of moments corresponding to the  $p_i$ , i.e.,  $v := (\beta_{00}, \beta_{10}, \beta_{01}, \beta_{11}, \beta_{02}, \beta_{12})^T = (1, 0, 0, 0, 1, 0)^T$ . We compute  $\alpha \equiv (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)^T := V^{-1}v = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$ . Then  $\mu := \sum_{i=1}^6 \alpha_i \delta_{(x_i, y_i)}$  is a representing measure for  $M_3$ .

We now use the method of Theorem 4.1 to approximate  $M_3$  with positive flat moment matrices. Using Proposition 3.6 with  $d = 3$ ,  $r = 3$ ,  $s = 1$  ( $r$  and  $s$  here have the same meaning as in Proposition 3.6), we perturb the support of  $\mu$  as follows. For  $k > 0$ ,  $1 \leq j \leq 6$ , let  $x_k^{(j)} = x_j + \frac{1}{(jk)^3}$  and  $y_k^{(j)} = y_j + \frac{1}{(jk)^2}$ . We now define

$$\omega^{(k)} \equiv \mu^{(k)} := \sum_{j=1}^6 \alpha_j \delta_{(x_k^{(j)}, y_k^{(j)})}.$$

Since  $\Delta[\mu] = 0$ , we set  $\nu^{(k)} = 0$ , and since  $\rho[\mu] = 6 = \rho_2$ , we set  $\sigma^{(k)} = 0$ . With  $M_3^{(k)} := M_3[\mu^{(k)}]$ , it is clear that  $\lim_{k \rightarrow \infty} M_3^{(k)} = M_3$ . We performed a numerical test

using *Mathematica*, where all calculations are performed exactly in  $\mathbb{Q}[\sqrt{2}]$ . In particular, calculations of the ranks of the approximants are exact because there are no roundoff errors in the matrices. Let  $\beta^{[k]}$  denote the moment sequence corresponding to  $M_3^{(k)}$  and let  $\Delta_k = \max_{a,b \geq 0, a+b \leq 6} |\beta_{ab} - \beta_{ab}^{[k]}|$ . We found  $\Delta_1 \approx 0.4$ ,  $\Delta_2 \approx 0.09$ ,  $\Delta_3 \approx 0.05$ ,  $\Delta_4 \approx 0.02$ ,  $\Delta_5 \approx 0.01$ ,  $\Delta_{10} \approx 0.002$ ,  $\Delta_{20} \approx 0.0002$ ,  $\Delta_{50} \approx 10^{-5}$ . Perhaps surprisingly, every  $M_3^{(k)}$  that we tested was flat, even for  $k = 1$ , where the error is large. Note also that in implementing Proposition 3.1, for each  $k > 0$  we used  $k_j = jk$  ( $1 \leq j \leq 6$ ). Experiments with each  $k_j = k$  did not yield flat approximants  $M_3^{(k)}$ .

We next illustrate Theorem 4.1, particularly Corollary 1.10, in a case where  $C^\flat$  is Hankel. In such a case, let  $\mu$  denote the unique,  $(\text{rank } M_{d-1})$ -atomic measure associated with  $M^\flat \equiv M_d[\mu]$ , let  $\Delta \equiv \Delta[\mu] = \Delta^\flat$ , and let  $\kappa = r \equiv \text{rank } M_{d-1}$ . In any example of this type, we may apply the method of Theorem 4.1 directly or we may apply a slightly different and simpler approach that we next describe. In this approach, we do not define the determining sequences  $w^{(1)}, \dots, w^{(r)}$ , and we replace  $\mu^{(k)}$  with  $\mu$ . Further, if  $r + s < \tau$ , we choose  $\alpha_j$  ( $r + s + 1 \leq j \leq t$ ) so that  $\alpha_1, \dots, \alpha_t$  are distinct and nonzero. For  $m \geq d - 1$ , we use Proposition 3.2 to define determining sequences  $w^{(j)} \equiv \{(k + \frac{1}{k^m}, \alpha_j k)\}_{k=1}^\infty$  ( $r + s + 1 \leq j \leq t$ ). Given  $k \geq 1$ , we then apply Proposition 3.1 to  $p_{r+1}, \dots, p_t$  and the sequences  $w^{(r+1)}, \dots, w^{(t)}$  to produce points  $w_{r+1}, \dots, w_t$  (dependent on  $k$ ) leading to an invertible  $(\tau - r) \times (\tau - r)$  Vandermonde  $V$ . As before, we use  $w_{r+1}, \dots, w_{r+s}$  to define  $\nu^{(k)}$  and  $w_{r+s+1}, \dots, w_t$  to define  $\sigma^{(k)}$ . Setting  $\omega^{(k)} := \mu + \nu^{(k)} + \sigma^{(k)}$ , the proof then proceeds along the lines of the proof of Theorem 4.1; we omit the details.

**Example 4.3.** To define

$$M_4 \equiv M_4(\beta^{(8)}) = \begin{pmatrix} M_3 & B_4 \\ B_4^T & C_4 \end{pmatrix},$$

we begin with

$$M_3 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 5 & 0 \\ 1 & 0 & 2 & 2 & 0 & 5 & 0 & 5 & 0 & 14 \\ 1 & 0 & 2 & 2 & 0 & 5 & 0 & 5 & 0 & 14 \\ 0 & 2 & 0 & 0 & 5 & 0 & 5 & 0 & 14 & 0 \\ 2 & 0 & 5 & 5 & 0 & 14 & 0 & 14 & 0 & 42 \\ 0 & 2 & 0 & 0 & 5 & 0 & 5 & 0 & 14 & 0 \\ 2 & 0 & 5 & 5 & 0 & 14 & 0 & 14 & 0 & 42 \\ 0 & 5 & 0 & 0 & 14 & 0 & 14 & 0 & 42 & 0 \\ 5 & 0 & 14 & 14 & 0 & 42 & 0 & 42 & 0 & 132 \end{pmatrix}.$$

$M_3$  is positive semidefinite, with  $\text{rank } M_3 = 7$ , and column dependence relations  $X^2 = Y$ ,  $X^3 = XY$ , and  $X^2Y = Y^2$ . From Theorem 2.2, a positive  $M_4$  requires that in  $\begin{pmatrix} M_3 & B_4 \end{pmatrix}$  we have  $X^4 = X^2Y$ ,  $X^3Y = XY^2$ ,  $X^2Y^2 = Y^3$ , so  $B_4$  must



be of the form

$$B_4 = \begin{pmatrix} 2 & 0 & 5 & 0 & 14 \\ 0 & 5 & 0 & 14 & 0 \\ 5 & 0 & 14 & 0 & 42 \\ 5 & 0 & 14 & 0 & 42 \\ 0 & 14 & 0 & 42 & 0 \\ 14 & 0 & 42 & 0 & 132 \\ 0 & 14 & 0 & 42 & 0 \\ 14 & 0 & 42 & 0 & 132 \\ 0 & 42 & 0 & 132 & x \\ 42 & 0 & 132 & x & y \end{pmatrix}.$$

A calculation shows that there exists  $W$  satisfying  $B_4 = M_3W$  and

$$C^b = \begin{pmatrix} 14 & 0 & 42 & 0 & 132 \\ 0 & 14 & 0 & 132 & x \\ 42 & 0 & 132 & x & y \\ 0 & 132 & x & x^2 + 428 & x(y - 416) \\ 132 & x & y & x(y - 416) & 179464 + x^2 - 844y + y^2 \end{pmatrix}.$$

In the sequel, to satisfy the requirement that  $C^b$  be Hankel, we set  $y = x^2 + 428$ . We next define  $C_4$  by

$$C_4 = \begin{pmatrix} 15 & 0 & 43 & 0 & 133 \\ 0 & 43 & 0 & 133 & x \\ 43 & 0 & 133 & x & x^2 + 429 \\ 0 & 133 & x & x^2 + 429 & x(x^2 + 12) \\ 133 & x & x^2 + 429 & x(x^2 + 12) & x^4 + 13x^2 + 1417 + \epsilon \end{pmatrix} \quad (\epsilon \geq 0),$$

so that

$$\Delta \equiv C_4 - C^b = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 + \epsilon \end{pmatrix}.$$

Since  $\Delta \succeq 0$ , then  $M_4 \succeq 0$ . However, since  $X^4 \neq X^2Y$  in the column space,  $M_4$  is not recursively generated and thus has no representing measure. Since  $M_4 \succeq 0$ , we have  $\text{rank } M_4 = \text{rank } M_3 + \text{rank } \Delta$ . Thus,  $\text{rank } M_4 = 9$  if  $\epsilon = 0$  and  $\text{rank } M_4 = 10$  if  $\epsilon > 0$ , so in either case  $M_4$  satisfies the hypotheses of Theorem 4.1, and we will illustrate how to approximate  $M_4$  with positive flat moment matrices.

We begin with the case when  $\epsilon = 0$  and  $\text{rank } M_4 = 9$ , and we let  $\mu$  denote the unique (7-atomic) representing measure for  $M^b$  (which can be explicitly computed from [CF7]; cf. Section 2). We first choose  $\tau = 9$ , i.e., we will approximate  $M_4$  by flat moment matrices  $M_4^{(k)}$  with  $\text{rank } M_4^{(k)} = 9$ . Since  $\epsilon = 0$ ,  $\Delta$  has the unique representing measure  $\nu := \frac{1}{2}\delta_{\{1\}} + \frac{1}{2}\delta_{\{-1\}}$ . Following the proof of Theorem 4.1 and the modified approach sketched above (in particular, applying Propositions 3.1 and 3.2), we choose  $m = 3$ , and for  $k \geq 1$ , we set  $k_1 = k_2 = k$ ,  $a_1 = a_2 = \frac{1}{2}$ , and  $\alpha_1 = 1$ ,  $\alpha_2 = -1$ . Now we define

$$\nu^{(k)} := \frac{1}{2k^8}\delta_{(k+\frac{1}{k^3}, k)} + \frac{1}{2k^8}\delta_{(k+\frac{1}{k^3}, -k)}.$$

Let  $\omega^{(k)} := \mu + \nu^{(k)}$  and set  $M_4^{(k)} = M_4[\omega^{(k)}]$ . Clearly,  $\lim_{k \rightarrow \infty} M_4^{(k)} = M^b + M_\Delta = M_4$ , and a symbolic calculation shows that  $\text{rank } M_4^{(k)} = \text{rank } M_3^{(k)} = 9$ .

We next choose  $\tau = 10$ . Note that for  $k \geq 1$ , Propositions 3.1 and 3.2 guarantee the existence of  $k_1, k_2, k_3 \geq k$  corresponding to an invertible Vandermonde, but there is no explicit formula for the  $k_i$  values. In this case, if we choose  $k_1 = k_2 = k_3 = k$ , then we find  $\text{rank } M_3^{(k)} = 9$ , so  $M_4^{(k)}$  (which has rank 10) is not flat. Instead, we define  $\nu^{(k)}$  as above, except that we now try  $k_1 = k$  and  $k_2 = 2k$ . Further, following the method discussed above, we try  $k_3 = 3k$ ,  $\alpha_3 = 2$ , and we define  $\sigma^{(k)} := \frac{1}{k^{10}} \delta_{\{(3k + \frac{1}{(3k)^3}, 6k)\}}$ . Setting  $\omega^{(k)} := \mu + \nu^{(k)} + \sigma^{(k)}$  and  $M_4^{(k)} := M_4[\omega^{(k)}]$ , we see that  $\lim_{k \rightarrow \infty} M_4^{(k)} = M^b + M_\Delta + 0 = M_4$ , and a symbolic calculation shows that  $\text{rank } M_4^{(k)} = \text{rank } M_3^{(k)} = 10$ .

We next consider the case when  $\epsilon > 0$ , i.e.,  $\text{rank } \Delta = 3$ . In this case,  $\Delta$  is not recursively generated and so has no representing measure. Nevertheless, [FN2] implies that  $\Delta$  may be approximated by rank 3 positive flat Hankel matrices. Indeed, for  $k > 0$ , if we define

$$\nu_k := \frac{1}{2} \delta_{\{1\}} + \frac{1}{2} \delta_{\{-1\}} + \frac{\epsilon}{k^8} \delta_{\{k\}},$$

then  $\lim_{k \rightarrow \infty} H_4[\nu_k] = \Delta$ , and  $\text{rank } H_4[\nu_k] = \text{rank } H_3[\nu_k] = 3$ . We may therefore assume that  $\Delta = \Delta^{(k)} := H_4[\nu_k]$ , and we proceed as outlined above, with  $r = 7$  and  $\rho = \tau = \rho_3 = 10$ . Let  $\alpha_1 = 1$ ,  $\alpha_2 = -1$ ,  $\alpha_3 = k$ , and  $a_1 = a_2 = \frac{1}{2}$ ,  $a_3 = \frac{\epsilon}{k^8}$ . For fixed  $j \geq k$ , we choose  $k_1 = j$ ,  $k_2 = 2j$ ,  $k_3 = 3j$  (based on Proposition 3.1 and experimentation), and define

$$\nu^{(k)} = \frac{1}{2k_1^8} \delta_{\{(k_1 + \frac{1}{k_1^3}, k_1)\}} + \frac{1}{2k_2^8} \delta_{\{(k_2 + \frac{1}{k_2^3}, -k_2)\}} + \frac{\epsilon}{k^8} \frac{1}{k_3^8} \delta_{\{(k_3 + \frac{1}{k_3^3}, k \cdot k_3)\}}.$$

We now set  $\omega^{(k)} := \mu + \nu^{(k)}$  and  $M_4^{(k)} := M_4[\omega^{(k)}]$ . Clearly,  $\lim_{k \rightarrow \infty} M_4^{(k)} = M^b + \lim_{k \rightarrow \infty} M_{\Delta^{(k)}} = M^b + M_\Delta = M_4$ , and a symbolic calculation shows that  $\text{rank } M_4^{(k)} = \text{rank } M_3^{(k)} = 10$ .

To conclude the example we re-define  $C_4$  as

$$C_4 = \begin{pmatrix} 14 & 0 & 42 & 0 & 132 \\ 0 & 42 & 0 & 132 & x \\ 42 & 0 & 132 & x & x^2 + 428 \\ 0 & 132 & x & x^2 + 428 & x(x^2 + 12) \\ 132 & x & x^2 + 428 & x(x^2 + 12) & x^4 + 13x^2 + 1416 + \epsilon \end{pmatrix} \quad (\epsilon > 0),$$

so that the Hankel matrix  $\Delta$  is no longer a moment matrix, i.e.,  $\Delta_j = 0$  ( $0 \leq j \leq 7$ ) and  $\Delta_8 = \epsilon > 0$ . We have  $r = 7$ ,  $\rho = 8$ , and we choose  $\tau = 9$ . Using Propositions 3.1 and 3.5 (with  $k_1 = k$  and  $k_2 = 2k$  in Proposition 3.1), let  $\nu^{(k)} := \frac{\epsilon}{k^8} \delta_{\{(\frac{1}{k^3}, k)\}}$  and  $\sigma^{(k)} := \frac{1}{k^{10}} \delta_{\{(\frac{1}{(2k)^3}, 2k)\}}$ . Setting  $\omega^{(k)} := \mu + \nu^{(k)} + \sigma^{(k)}$  and  $M_4^{(k)} := M_4[\omega^{(k)}]$ , we see that  $\lim_{k \rightarrow \infty} M_4^{(k)} = M^b + M_\Delta + 0 = M_4$ , and a symbolic calculation shows that  $\text{rank } M_4^{(k)} = \text{rank } M_3^{(k)} = 9$ .

We conclude this section with a proof of Proposition 1.12.

*Proof of Proposition 1.12.* The proof consists of repeating the proof of Theorem 4.1 up to the point where we define  $\omega^{(k)} := \mu^{(k)} + \nu^{(k)} + \sigma^{(k)}$ , set

$$M_d^{(k)} \equiv M_d[\omega^{(k)}] = M_d[\mu^{(k)}] + M_d[\nu^{(k)}] + M_d[\sigma^{(k)}],$$

and show that  $\lim_{k \rightarrow \infty} M_d^{(k)} = M_d[\mu] + M_\Delta = M_d(\beta)$ . This shows that  $M_d$  is in the closure of moment matrices with measures, so  $L_\beta$  is positive. Since we are not interested in the ranks of the approximating moment matrices, it is also possible to re-work the proof in a simpler way, without using determining sequences.  $\square$

## 5. FLAT APPROXIMATION IN THE DEGREE 6 MOMENT PROBLEM

Our main application is Theorem 1.1, which is included in the following result.

**Theorem 5.1.**  $M_3 \equiv M_3(\beta^{(6)})$  belongs to  $\overline{\mathcal{F}_3}$  if and only if  $M_3 \succeq 0$  and  $\rho \equiv \text{rank } M_3 \leq 6$ . In this case, given  $\tau$ ,  $\rho \leq \tau \leq 6$ ,  $M_3$  is in the closure of the rank- $\tau$  positive flat moment matrices, and there exists a sequence of (computable)  $\tau$ -atomic positive measures  $\{\mu_k\}$  such that  $\beta_{ij} = \lim_{k \rightarrow \infty} \int x^i y^j d\mu_k$  ( $i, j \geq 0$ ,  $i + j \leq 6$ ). Further, either  $M_3^b$  is a moment matrix or  $M_3$  admits a flat extension  $M_4$ , whence  $\beta^{(5)}$  has a representing measure.

Let  $M_d = \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d \end{pmatrix}$  denote a positive semidefinite moment matrix. We recall from Section 2 two basic properties that we will use repeatedly. It follows from [CF2, Proposition 3.9] that if  $M_d \succeq 0$ , then

$$(5.1) \quad p \in \mathbb{R}[x]_{d-1}, p(X) = 0 \text{ in } \text{Col } M_{d-1} \implies p(X) = 0 \text{ in } \text{Col } M_d.$$

Further, Theorem 2.2 iii) implies that  $\begin{pmatrix} M_{d-1} & B_d \end{pmatrix}$  is recursively generated, i.e.,

$$(5.2) \quad p, q, pq \in \mathbb{R}[x]_d, p(X) = 0 \text{ in } \text{Col } \begin{pmatrix} M_{d-1} & B_d \end{pmatrix} \implies (pq)(X) = 0 \text{ in } \text{Col } \begin{pmatrix} M_{d-1} & B_d \end{pmatrix}.$$

If  $M_d \succeq 0$ , then  $\text{Ran } B_d \subseteq \text{Ran } M_{d-1}$ , so there exists  $W$  such that  $B_d = M_{d-1}W$ . Let  $C_d^b := B_d^T W = W^T M_{d-1} W$ ; note that  $C_d^b = C_d^b W^T$  and that  $C_d^b$  is independent of  $W$  satisfying  $B_d = M_{d-1}W$ . Let  $M_d^b := \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d^b \end{pmatrix}$ . The following property is useful in computing  $M_d^b$ :

$$(5.3) \quad p \in \mathbb{R}[x]_d, p(X) = 0 \text{ in } \text{Col } \begin{pmatrix} M_{d-1} & B_d \end{pmatrix} \implies p(X) = 0 \text{ in } \text{Col } M_d^b.$$

As discussed in Section 2,  $\text{rank } M_d^b = \text{rank } M_{d-1}$ . Thus, if  $C_d^b$  is Hankel, then  $M_d^b$  is a flat moment matrix extension of  $M_{d-1}$  (using the data in  $B_d$ ) and thus has a representing measure (cf. Theorem 1.4). Note that in the case when  $d = 3$ , since  $C_3^b \equiv (c_{ij})_{1 \leq i, j \leq 4}$  is real symmetric,  $C_3^b$  is Hankel if and only if  $c_{31} = c_{22}$ ,  $c_{41} = c_{32}$ , and  $c_{42} = c_{33}$ .

We begin the proof of Theorem 5.1 with a series of preliminary results based on the value of  $r \equiv \text{rank } M_2$ .

**Proposition 5.2.** If  $M_3 \succeq 0$ ,  $\text{rank } M_3 \leq 6$  and  $r \equiv \text{rank } M_2 \leq 3$ , then  $C_3^b$  is Hankel.

*Proof.* Suppose  $r=1$ . Then in  $\text{Col } M_3$ ,  $X=\alpha 1$ ,  $Y=\beta 1$  (for certain scalars  $\alpha, \beta$ ), so (5.2) and (5.3) imply that in  $\text{Col } M_3^b$ ,  $X^i Y^j = \alpha^i \beta^j 1$ . Then for  $i+j=k+l=3$  with  $1 \leq i, l \leq 3$ ,  $0 \leq j, k \leq 2$ ,  $\langle X^i Y^j, X^k Y^l \rangle = \alpha^{i+k} \beta^{j+l} = \langle X^{i-1} Y^{j+1}, X^{k+1} Y^{l-1} \rangle$ , so  $C_3^b$  is Hankel.

Next, let  $r=2$ . If  $\text{rank } M_1 = 1$ , then by recursiveness in  $M_2$ , it follows that  $\text{rank } M_2 = 1$ , a contradiction. Thus  $\text{rank } M_1 = \text{rank } M_2 = 2$ , so  $M_2$  is flat. Thus  $M_2$  has a unique flat extension  $\widetilde{M}_3$ . Since  $\widetilde{M}_3$  is recursively generated, it follows readily from (5.2) that  $B_3 = \widetilde{B}_3$ . Now (5.3) implies that  $M_3^b = \widetilde{M}_3$ , so  $C_3^b = \widetilde{C}_3$ , whence  $C_3^b$  is Hankel.

Now suppose  $r=3$ . As above, if  $\text{rank } M_1 = 1$ , then by recursiveness of  $M_2$ ,  $\text{rank } M_2 = 1$ , a contradiction. Further, if  $\text{rank } M_1 = 3$ , then  $M_2$  is flat, so we may proceed exactly as in the  $r=2$  case (above) to conclude that  $C_3^b$  is Hankel. We may thus assume that  $\text{rank } M_1 = 2$ , and we first consider the case when  $\{1, X\}$  is a column basis, with  $Y = \alpha 1 + \beta X$  for some  $\alpha, \beta \in \mathbb{R}$ . By recursiveness in  $M_2$ ,  $XY = \alpha X + \beta X^2$  and  $Y^2 = \alpha Y + \beta XY$ , so  $\{1, X, X^2\}$  is a basis for  $\text{Col } M_2$ . From (5.2), in  $\text{Col } B_3$  we have

$$(5.4) \quad X^2 Y = \alpha X^2 + \beta X^3,$$

$$(5.5) \quad XY^2 = \alpha XY + \beta X^2 Y,$$

$$(5.6) \quad Y^3 = \alpha Y^2 + \beta XY^2.$$

These relations and the value of  $\beta_{50}$  completely determine the other moments of degree 5; thus from (5.4),  $\beta_{41} = \alpha\beta_{40} + \beta\beta_{50}$ ,  $\beta_{32} = \alpha\beta_{31} + \beta\beta_{41}$ ,  $\beta_{23} = \alpha\beta_{22} + \beta\beta_{32}$ ; from (5.5),  $\beta_{14} = \alpha\beta_{13} + \beta\beta_{23}$ ; and from (5.6),  $\beta_{05} = \alpha\beta_{04} + \beta\beta_{14}$ . With these values, a calculation shows that there exists  $W$  such that  $B_3 = M_2 W$ , and a further calculation shows that  $C_3^b \equiv W^T M_2 W$  is Hankel. In the remaining case, a basis is  $\{1, Y\}$ , and the proof is entirely analogous.  $\square$

We next analyze cases of Theorem 5.1 with  $r=4$  or  $r=5$ . Since  $M_3 \succeq 0$ , then  $M_2$  is recursively generated, so in these cases we must have  $\{1, X, Y\}$  independent in  $\text{Col } M_2$ . Thus, in  $\text{Col } M_2$  there is a dependence relation of the form  $p(X, Y) = 0$ , with  $\deg p(x, y) = 2$ . Given a degree-one map  $\mathcal{T}$ , let  $\widetilde{M}_2$  denote the moment matrix corresponding to  $M_2$  under  $\mathcal{T}$  (cf. Proposition 2.8), and let  $q(\widetilde{x}, \widetilde{y}) = p \circ \mathcal{T}^{-1}(\widetilde{x}, \widetilde{y}) \in \mathbb{R}[\widetilde{x}, \widetilde{y}]_d$ ; clearly  $q(\widetilde{x}, \widetilde{y}) = 0$  if and only if  $p(x, y) = 0$ . It is well known that corresponding to  $p(x, y)$  there is a degree-one map  $\mathcal{T}$  such that the variety  $q(\widetilde{x}, \widetilde{y}) = 0$  is one of the following:  $\widetilde{x}\widetilde{y} = 0$ ,  $\widetilde{x}\widetilde{y} = 1$ ,  $\widetilde{y} = \widetilde{x}^2$ ,  $\widetilde{x}^2 + \widetilde{y}^2 = 1$ ,  $\widetilde{x}^2 = 1$ ,  $\widetilde{x}^2 = 0$ ,  $\widetilde{x}^2 = -1$ ,  $\widetilde{x}^2 + \widetilde{y}^2 = 0$ ,  $\widetilde{x}^2 + \widetilde{y}^2 = -1$  (cf. [SH, p. 405]). We note that column dependence relations corresponding to any of the last four cases cannot occur in  $\widetilde{M}_2$  if  $\widetilde{M}_2 \succeq 0$  and  $\text{rank } \widetilde{M}_2 \geq 4$ . To see this, we may scale  $\widetilde{\beta}$  so that  $\widetilde{\beta}_{00} = 1$ , and we denote  $\widetilde{M}_2$  as

$$(5.7) \quad \widetilde{M}_2 \equiv \begin{pmatrix} 1 & a & b & c & d & e \\ a & c & d & f & g & h \\ b & d & e & g & h & k \\ c & f & g & p & q & w \\ d & g & h & q & w & s \\ e & h & k & w & s & t \end{pmatrix}.$$

If  $\tilde{X}^2 = 0$ , then  $c = f = g = p = q = w = 0$ ; since  $c = 0$ , positivity of  $\tilde{M}_2$  implies  $a = d = h = 0$ , and, similarly,  $w = 0$  implies  $s = 0$ . Thus,  $\text{rank } \tilde{M}_2 \leq 3$ , a contradiction. Similarly, if  $\tilde{X}^2 + \tilde{Y}^2 = 0$ , then  $c + e = 0$  and  $p + w = 0$  imply  $c = e = p = w = 0$ , and it follows as above (via positivity) that  $\text{rank } \tilde{M}_2 \leq 2$ . If  $\tilde{X}^2 = -1$ , then  $c = -1$ , which violates positivity in  $\tilde{M}_2$ ; similarly, if  $\tilde{X}^2 + \tilde{Y}^2 = -1$ , then  $c + e = -1$ , which is impossible. Motivated by the preceding discussion, in the following sequence of results we assume that  $M_2$  admits a column dependence relation corresponding to one of the following varieties:  $xy = 0$ ,  $xy = 1$ ,  $x^2 = y$ ,  $x^2 = 1$ , or  $x^2 + y^2 = 1$ . In each case we will show that under the hypotheses of Theorem 5.1,  $M_3$  satisfies the following property:

$$(5.8) \quad C_3^b \text{ is Hankel or } M_3 \text{ admits a flat extension } M_4.$$

In view of Theorem 1.4 and Corollaries 1.10 and 1.11, it is clear that (5.8) implies the conclusions of Theorem 5.1, including the existence of a representing measure for  $\beta^{(5)}$ .

In the sequel, to simplify certain symbolic calculations, we always scale  $\beta$  so that  $\beta_{00} = 1$  (without loss of generality). If  $\mathcal{B}$  is a collection of columns of  $M_d$  which forms a basis of the column space of  $M_d$ , we will denote by  $[M_d]_{\mathcal{B}}$  the compression of  $M_d$  to the rows and columns indexed by the elements of  $\mathcal{B}$ . Thus, if  $M_d \succeq 0$ , then  $[M_d]_{\mathcal{B}} \succ 0$ .

**Proposition 5.3.** *Suppose  $M_3 \succeq 0$ , with  $\rho \leq 6$  and  $r = 4$ . If  $M_2$  has a column relation  $XY = 0$ , then  $C_3^b$  is Hankel.*

*Proof.* As noted above,  $\{1, X, Y\}$  is independent. Suppose first that  $\text{Col } M_2$  has the basis  $\mathcal{B} \equiv \{1, X, Y, X^2\}$ . Writing  $Y^2 = c_1 1 + c_2 X + c_3 Y + c_4 X^2$ , in  $\begin{pmatrix} M_2 & B_3 \end{pmatrix}$  we have  $X^2 Y = 0$ ,  $0 = XY^2 = c_1 X + c_2 X^2 + c_4 X^3$  and  $Y^3 = c_1 Y + c_3 Y^2$ . Further,  $M_2$  is of the form

$$(5.9) \quad M_2 = \begin{pmatrix} 1 & a & b & c & 0 & d \\ a & c & 0 & e & 0 & 0 \\ b & 0 & d & 0 & 0 & f \\ c & e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & f & 0 & 0 & h \end{pmatrix},$$

with

$$J \equiv [M_2]_{\mathcal{B}} = \begin{pmatrix} 1 & a & b & c \\ a & c & 0 & e \\ b & 0 & d & 0 \\ c & e & 0 & g \end{pmatrix} \succ 0,$$

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}^T = J^{-1} \begin{pmatrix} d & 0 & f & 0 \end{pmatrix}^T, \quad h = \begin{pmatrix} d & 0 & f & 0 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}^T.$$

From the above column relations,  $B_3$  must be of the form

$$(5.10) \quad B_3 = \begin{pmatrix} e & 0 & 0 & f \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & h \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \end{pmatrix},$$

for certain  $t$  and  $u$ . Since  $X^3$  in  $B_3$  belongs to  $\text{Ran } M_2$ , it follows that  $X^3 = d_1 1 + d_2 X + d_3 Y + d_4 X^2$ , with  $(d_1 d_2 d_3 d_4) = J^{-1}(e, g, 0, t)^T$ . In particular,  $\gamma := \langle X^3, Y^2 \rangle = 0$ , and a symbolic calculation shows that

$$\gamma = (d, 0, f, 0)(d_1 d_2 d_3 d_4)^T = \frac{(bf - d^2)(e^3 - 2ceg + ag^2 + (c^2 - ac)t)}{\det(J)}.$$

Further, since  $Y^3 = c_1 Y + c_3 Y^2$ , it follows that  $u = c_1 f + c_3 h$ . Now let

$$W = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ d_2 & 0 & 0 & 0 \\ d_3 & 0 & 0 & c_1 \\ d_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \end{pmatrix}.$$

Then  $M_2 W = B_3$ , and in  $C_3^\flat \equiv W^T M_2 \equiv (c_{ij})_{1 \leq i, j \leq 4}$ , we have  $c_{31} = 0 = c_{22}$  and  $c_{42} = 0 = c_{32}$ . Thus  $C_3^\flat$  is Hankel if and only if  $c_{41} = 0$  ( $= c_{32}$ ). A symbolic calculation shows that

$$c_{41} = \frac{-\gamma\omega}{\det(J)} (\omega \equiv -bde^2 + (c^3 - 2ace + e^2)f + (bcd + a^2f - cf)g),$$

and since  $\gamma = 0$ , then  $c_{41} = 0$ . Thus  $C_3^\flat$  is Hankel, which completes the proof in this case.

Suppose next that  $XY = 0$  and that  $B \equiv \{1, X, Y, Y^2\}$  is a basis for  $\text{Col } M_2$ . In view of the previous case, we may assume that  $X^2 \in \langle 1, X, Y \rangle$ . Thus,  $X^2 = c_1 1 + c_2 X + c_3 Y$ , with  $(c_1 c_2 c_3) = M_1^{-1}(c e 0)^T$ . In particular,  $g = (c e 0)(c_1 c_2 c_3)^T$  and  $0 = \langle X^2, Y^2 \rangle = (d 0 f)(c_1 c_2 c_3)^T$  (which entails  $c^2 - ae)(d^2 - bf) = 0$ ). Denoting  $M_2$  and  $B_3$  as in (5.9) and (5.10),  $t$  in  $X^3$  is uniquely determined via recursiveness in  $(M_2 B_3)$  by

$$(5.11) \quad X^3 = c_1 X + c_2 X^2,$$

i.e.,  $t = c_1 e + c_2 g$ .

Let  $K = [M_2]_B (> 0)$ . Since  $M_3 \succeq 0$ ,  $Y^3$  in  $B_3$  is in  $\text{Ran } M_2$ , so  $Y^3 = k_1 1 + k_2 X + k_3 Y + k_4 Y^2$ , where  $(k_1 k_2 k_3 k_4) = K^{-1}(f, 0, h, u)^T$ . This relation and (5.10) imply that  $0 = \langle Y^3, X^2 \rangle = k_1 c + k_2 e \equiv \lambda$ . A symbolic calculation now shows that

$$\lambda = \frac{(ae - c^2)(f^3 - 2dfh + bh^2 + (d^2 - bf)u)}{\det(K)}.$$

Setting

$$W = \begin{pmatrix} 0 & 0 & 0 & k_1 \\ c_1 & 0 & 0 & k_2 \\ 0 & 0 & 0 & k_3 \\ c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix},$$

we have  $M_2 W = B_3$ , and a further symbolic calculation shows that

$$C_3^\flat \equiv B_3^T W = \begin{pmatrix} * & 0 & 0 & c_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{pmatrix},$$

with

$$c_{14} \equiv \langle Y^3, X^3 \rangle = k_1 e + k_2 g = \frac{-\lambda(acd + b^2e - de)}{\det(K)}.$$

Since  $\lambda = 0$ , it follows that  $C_3^\flat$  is Hankel, whence the result follows.  $\square$

**Proposition 5.4.** *Suppose  $M_3 \succeq 0$ , with  $\rho \leq 6$  and  $r = 4$ . If  $M_2$  has a column relation  $XY = 1$ , then  $C_3^\flat$  is Hankel.*

*Proof.* Since  $XY = 1$ , we may denote  $M_2$  as

$$(5.12) \quad M_2 = \begin{pmatrix} 1 & a & b & c & 1 & d \\ a & c & 1 & e & a & b \\ b & 1 & d & a & b & f \\ c & e & a & g & c & 1 \\ 1 & a & b & c & 1 & d \\ d & b & f & 1 & d & h \end{pmatrix}.$$

Since  $r = 4$ ,  $\{1, X, Y\}$  is independent, and we first consider the case when  $\mathcal{B} \equiv \{1, X, Y, X^2\}$  is a column basis. Let

$$J \equiv [M_2]_{\mathcal{B}} = \begin{pmatrix} 1 & a & b & c \\ a & c & 1 & e \\ b & 1 & d & a \\ c & e & a & g \end{pmatrix} \succ 0,$$

and set  $(c_1 \ c_2 \ c_3 \ c_4)^T := J^{-1}(d \ b \ f \ 1)^T$ . Then

$$(5.13) \quad h = (d \ b \ f \ 1)(c_1 \ c_2 \ c_3 \ c_4)^T$$

and  $Y^2 = c_1 1 + c_2 X + c_3 Y + c_4 X^2$ . By recursiveness, in  $\text{Col}(M_2 \ B_3)$  we have  $X^2 Y = X$ ,  $XY^2 = Y$ ,  $Y^3 = c_1 Y + c_2 XY + c_3 Y^2 + c_4 X^2 Y$ . Now  $B_3$  is of the form

$$B_3 = \begin{pmatrix} e & a & b & f \\ g & c & 1 & d \\ c & 1 & d & h \\ t & e & a & b \\ e & a & b & f \\ a & b & f & u \end{pmatrix}$$

for some  $t \equiv \beta_{50}$  and  $u \equiv \beta_{05}$ . Setting  $(k_1 \ k_2 \ k_3 \ k_4)^T := J^{-1}(e \ g \ c \ t)^T$ , the condition  $X^3 \in \text{Ran } M_2$ , i.e.,  $X^3 = k_1 1 + k_2 X + k_3 Y + k_4 X^2$ , is equivalent to the condition that  $\lambda := k_1 d + k_2 b + k_3 f + k_4 1 - a$  satisfies  $\lambda = 0$ . Setting

$$W \equiv \begin{pmatrix} k_1 & 0 & 0 & 0 \\ k_2 & 1 & 0 & c_4 \\ k_3 & 0 & 1 & c_1 \\ k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_3 \end{pmatrix},$$

we have  $M_2 W = B_3$ , so that  $C_3^\flat = B_3^T W$ . Denoting  $C_3^\flat \equiv (c_{ij})_{1 \leq i, j \leq 4}$ , a symbolic calculation shows that  $c_{31} = c_{22}$ ,  $c_{42} = c_{33}$ , and  $c_{32} = 1$ . Thus,  $C_3^\flat$  is Hankel if and



only if  $c_{41} \equiv k_1 f + k_2 d + k_3 h + k_4 b$  satisfies  $c_{41} = 1$  ( $= c_{32}$ ). Now, using (5.13), a symbolic calculation shows that

$$c_{41} - 1 = \frac{\lambda\gamma}{\det(J)},$$

where  $\gamma$  is a polynomial in the moments of  $M_2$ . Since, from above,  $\lambda = 0$ , we have  $c_{41} = 1$ , whence  $C_3^b$  is Hankel.

In the remaining case with  $XY = 1$ , we may assume that  $M_2$  has the column basis  $\mathcal{B} \equiv \{1, X, Y, Y^2\}$ . In view of the previous case, we may also assume that  $X^2 = c_1 1 + c_2 X + c_3 Y$ , where  $(c_1 \ c_2 \ c_3)^T = M_1^{-1} (c \ e \ a)^T$ , so, in particular, we have

$$(5.14) \quad c_1 c + c_2 e + c_3 a = g, \quad c_1 d + c_2 b + c_3 f = 1.$$

In  $(M_2 \ B_3)$ , by recursiveness, we have  $X^3 = c_1 X + c_2 X^2 + c_3 1$ ,  $X^2 Y = X$ ,  $XY^2 = Y$ . Further, since  $Y^3 \in \text{Ran } M_2$ , in  $\text{Col } (M_2 \ B_3)$  we have

$$(5.15) \quad Y^3 = k_1 1 + k_2 X + k_3 Y + k_4 Y^2,$$

where  $(k_1 \ k_2 \ k_3 \ k_4)^T = L^{-1} (f \ d \ h \ u)^T$  for

$$L \equiv [M_2]_{\mathcal{B}} = \begin{pmatrix} 1 & a & b & d \\ a & c & 1 & b \\ b & 1 & d & f \\ d & b & f & h \end{pmatrix} (\succ 0).$$

In particular, (5.15) entails

$$(5.16) \quad b \equiv \langle Y^3, X^2 \rangle = k_1 c + k_2 e + k_3 a + k_4.$$

Define  $W$  by

$$W = \begin{pmatrix} c_3 & 0 & 0 & k_1 \\ c_1 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \\ c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix},$$

so that  $M_2 W = B_3$ . A symbolic calculation shows that  $C_3^b := W^T M_2 W \equiv (c_{ij})_{1 \leq i, j \leq 4}$  is Hankel if and only if  $1 = c_{41} \equiv \langle X^3, Y^3 \rangle = c_1 d + c_2 b + c_3 f$  and  $c_{41} = c_{14}$ ; the former condition follows from (5.14) and the latter from the fact that  $C_3^b$  is real symmetric, so the proof is complete.  $\square$

**Proposition 5.5.** *Suppose  $M_3 \succeq 0$ , with  $\rho \leq 6$  and  $r = 4$ . If  $M_2$  has a column relation  $Y = X^2$ , then  $C_3^b$  is Hankel.*

*Proof.* Since  $r = 4$  and  $M_2$  is recursively generated, then  $\{1, X, Y\}$  is independent, and we first consider the case when  $\{1, X, Y, Y^2\}$  is a column basis for  $M_2$ . Thus, there is a column relation of the form  $XY = c_1 1 + c_2 X + c_3 Y + c_4 Y^2$ . The corresponding curve  $xy = c_1 + c_2 x + c_3 y + c_4 y^2$  has no  $x^2$  term, so its discriminant is positive, and it thus represents a (possibly degenerate) hyperbola. Thus, by applying an appropriate degree-one map and Proposition 2.8 iii), iv), vii),  $M_2$  may be transformed into a rank-4 positive  $\widetilde{M}_2$  with a column relation of the form  $\widetilde{X}\widetilde{Y} = 0$  or  $\widetilde{X}\widetilde{Y} = \widetilde{1}$ . It thus follows from Propositions 5.3 and 5.4 that the  $C$  block of  $(\widetilde{M}_d)^b$  is Hankel, whence Proposition 2.8 x) implies that  $C_3^b$  is Hankel.

We may now assume that  $\{1, X, Y, XY\}$  is a column basis, with a column relation of the form  $Y^2 = k_1 1 + k_2 X + k_3 Y + k_4 XY$ . Denoting  $M_2$  as  $M_d$  with  $d = 2$ , we see that  $M_d$  is *recursively determinate* in the sense of [CF10]; i.e., there is a degree-reducing relation  $X^n = p(X, Y)$  ( $n = 2$ ,  $\deg p = 1 < n$ ) and a degree-preserving relation  $Y^m = q(X, Y)$  ( $m = 2$ ,  $\deg q = 2 = m$ ). Since  $M_d$  is positive, recursively generated, and recursively determinate with  $n + m - 2 = d$ , it follows from [CF10, Corollary 2.4] that  $M_2$  admits a unique flat extension  $\widetilde{M}_3$ . In this extension,  $\widetilde{B}_3$  is completely determined from  $M_2$  by recursiveness, i.e.,  $\widetilde{X}^3 = \widetilde{X}\widetilde{Y}$ ,  $\widetilde{X}^2\widetilde{Y} = \widetilde{Y}^2$ ,  $\widetilde{X}\widetilde{Y}^2 = (xq)(\widetilde{X}, \widetilde{Y})$ ,  $\widetilde{Y}^3 = (yq)(\widetilde{X}, \widetilde{Y})$ . Since  $M_2 = \widetilde{M}_2$  and  $(M_2 \ B_3)$  is recursively generated, the same column relations as just above determine  $B_3$  so that  $B_3 = \widetilde{B}_3$ . It now follows immediately from (5.3) that  $C_3^b = \widetilde{C}_3$ , and since  $\widetilde{C}_3$  is Hankel, the result follows.  $\square$

**Proposition 5.6.** *Suppose  $M_3 \succeq 0$ , with  $\rho \leq 6$  and  $r = 4$ . If  $M_2$  has a column relation  $X^2 = 1$ , then  $C_3^b$  is Hankel.*

*Proof.* The proof is essentially the same as that of the preceding result. If  $\{1, X, Y, Y^2\}$  is a column basis for  $M_2$ , then there is a hyperbola relation  $XY = c_1 1 + c_2 X + c_3 Y + c_4 Y^2$ , so it follows as in the proof of Proposition 5.5 that  $C_3^b$  is Hankel. In the remaining case,  $\{1, X, Y, XY\}$  is a column basis, and it follows exactly as in the proof of Proposition 5.5 that  $M_3^b$  is a moment matrix, whence  $C_3^b$  is Hankel.  $\square$

**Proposition 5.7.** *Suppose  $M_3 \succeq 0$ , with  $\rho \leq 6$  and  $r = 4$ . If  $M_2$  has a column relation  $X^2 + Y^2 = 1$ , then  $C_3^b$  is Hankel.*

*Proof.* As in the previous proof,  $\{1, X, Y\}$  is independent. We consider first the case when  $\{1, X, Y, X^2\}$  is a column basis for  $M_2$ . It follows that  $XY$  is a linear combination of the basis columns, but since this relation has no  $Y^2$  term, the relation represents a hyperbola, so the result follows from the hyperbola cases above (exactly as in the proof of Proposition 5.5, via Proposition 2.8 x)). In the remaining case,  $\{1, X, Y\}$  is independent and  $\{1, X, Y, X^2\}$  is dependent, so there is a degree-reducing column relation of the form  $X^2 = c_1 1 + c_2 X + c_3 Y$ . Since we also have the degree-preserving relation  $Y^2 = 1 - X^2$ ,  $M_2$  is recursively determinate, and the proof proceeds exactly as in the conclusion of the proof of Proposition 5.5.  $\square$

We next present cases where  $r = 5$ . Since there is nothing to prove in Theorem 5.1 if  $M_3$  is flat, and (5.8) clearly holds as well in this case, we may assume  $\text{rank } M_3 = 6$  in these cases.

**Proposition 5.8.** *Suppose  $M_3 \succeq 0$ , with  $\rho = 6$  and  $r = 5$ . If  $M_2$  has a column relation  $XY = 0$ , then  $M_3$  satisfies (5.8).*

*Proof.* We have  $M_2$  as in (5.9), and  $(M_2 \ B_3)$  is recursively generated, so  $B_3$  is as in (5.10). Since  $X^2 Y = XY^2 = 0$  in  $B_3$ , (5.3) implies that the same column relations hold in  $C_3^b$ , so  $C_3^b$  has the form

$$(5.17) \quad C_3^b = \begin{pmatrix} \kappa & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & \delta \end{pmatrix}.$$

We may suppose  $\tau \neq 0$ , for otherwise  $C_3^b$  is Hankel and the result follows. A symbolic calculation shows that  $\tau = \tau_1 \tau_2$  where

$$\tau_1 = (e^3 - 2ceg + ag^2 + (c^2 - ae)t), \quad \tau_2 = (f^3 - 2dfh + bh^2 + (d^2 - bf)u).$$

Let  $J = [M_2]_{\mathcal{B}}$ , where  $\mathcal{B}$  denotes the column basis  $\{1, X, Y, X^2, Y^2\}$ . Now  $J \succ 0$ , so  $\kappa = [X^3]J^{-1}[X^3]^T$ , where  $[X^3] = (e \ g \ 0 \ t \ 0)$ ; note that since  $\tau \neq 0$  and  $C_3^b \succeq 0$ , then  $\kappa > 0$ . Consider  $\Delta := C_3 - C_3^b$  and write  $\Delta \equiv (\Delta_{ij})_{1 \leq i, j \leq 4}$ .

We claim that  $\Delta_{11} \neq 0$ . Let us denote  $C_3^b$  as  $(c_{ij})_{1 \leq i, j \leq 4}$ , so that  $c_{31} = c_{22} = c_{13} = 0$ . Since  $C_3 (= C_3^b + \Delta)$  is Hankel, it follows that  $\Delta_{31} = \Delta_{22} = \Delta_{13}$ . If  $\Delta_{11} = 0$ , then, since  $\Delta \succeq 0$ , it follows that  $\Delta_{21} = \Delta_{31} = \Delta_{41} = 0$ , whence  $\Delta_{22} = 0$ . Positivity of  $\Delta$  now implies  $\Delta_{32} = \Delta_{42} = 0$ . Since  $C_3 \equiv C_3^b + \Delta$  is Hankel, it follows that  $\Delta_{32} = \tau$ , which contradicts  $\tau \neq 0$ .

Now  $M_3 \succeq 0$  and  $\text{rank } M_3 = 6$ , so

$$(5.18) \quad \Delta \succeq 0, \quad \text{rank } \Delta = 1.$$

Since  $C_3^b + \Delta = C_3$  and  $C_3$  is Hankel, (5.17) and (5.18) imply that for  $p \equiv \Delta_{11} > 0$  and certain scalars  $q$  and  $\epsilon$ ,  $\Delta$  has the form

$$\Delta = \begin{pmatrix} p & qp & q^2p & \epsilon \\ qp & q^2p & q^3p & q\epsilon \\ q^2p & q^3p & q\epsilon & q^2\epsilon \\ \epsilon & q\epsilon & q^2\epsilon & k \end{pmatrix}, \quad \epsilon \neq q^3p.$$

Since  $\text{rank } \Delta = 1$ ,  $XY^2 = q^2X^3$  in  $\text{Col } \Delta$ , whence  $q^4p = q\epsilon$ . Now, if  $q \neq 0$ , then  $\epsilon = q^3p$ , a contradiction. Thus  $q = 0$ , so the Hankel matrix  $C_3$  is of the form

$$C_3 = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w \end{pmatrix}.$$

We now have  $X^2Y = XY^2 = 0$  in  $M_3$ , and we next construct a flat extension  $M_4$ . Since  $(M_3 \ B_4)$  must be recursively generated in any flat extension  $M_4$ , in  $B_4$  we must define  $X^3Y = X^2Y^2 = XY^3 = 0$ . From this structure, it follows that in  $B_4$ ,  $X^4$  must be of the form  $X^4 = (g \ t \ 0 \ v \ 0 \ 0 \ x \ 0 \ 0 \ 0)^T$  (for some  $x$ ), and  $Y^4$  must have the form  $Y^4 = (h \ 0 \ u \ 0 \ 0 \ w \ 0 \ 0 \ 0 \ y)^T$  (for some  $y$ ). Since  $p > 0$ ,  $M_3$  has a column basis  $\{1, X, Y, X^2, Y^2, X^3\}$ . Let  $\tilde{J}$  denote the compression of  $M_3$  to these rows and columns, and let  $\tilde{X}^4 \equiv (g \ t \ 0 \ v \ 0 \ x)^T$ . For each  $x$ , let  $w_x = \tilde{J}^{-1}\tilde{X}^4$ . For  $M_4$  to be a positive extension of  $M_3$  it is necessary that  $X^4 \in \text{Ran } M_3$ , and this is equivalent to the condition that  $\psi \equiv \langle \tilde{Y}^3, w_x \rangle = \langle X^4, Y^3 \rangle = 0$ , where  $\tilde{Y}^3 \equiv (f \ 0 \ h \ 0 \ u \ 0)^T$ . A symbolic calculation shows that

$$\psi = \frac{\tau_2(Q - \tau_1 x)}{\det(\tilde{J})}$$

(where  $Q$  is some polynomial in the moments of  $M_3$ ). Since  $\tau_1 \neq 0$ , it follows that there exists a unique  $x (= \frac{Q}{\tau_1})$  such that  $X^4 \in \text{Ran } M_3$ . Further, setting  $\tilde{Y}^4 = (h \ 0 \ u \ 0 \ w \ 0)^T$  and  $w_y = \tilde{J}^{-1}\tilde{Y}^4$ , if we define  $y := \langle \tilde{Y}^3, w_y \rangle$ , then the resulting  $Y^4$  satisfies  $Y^4 \in \text{Ran } M_3$ , as required for positivity of  $M_4$ . With these values for the moments of  $B_4$ , we now compute  $C_4^b \equiv (c_{ij})_{1 \leq i, j \leq 5}$  via (5.3). Since  $X^3Y = X^2Y^2 = XY^2 = 0$  in  $C_4^b$ , it is easy to see that  $C_4^b$  is Hankel if and only if

$c_{51} = 0$ . A symbolic calculation of  $c_{51}$  using (5.3) shows that  $c_{51} = \psi$ , and by our choice of  $x$ ,  $\psi = 0$ , so the existence of a flat extension  $M_4$  is established.  $\square$

**Proposition 5.9.** *Suppose  $M_3 \succeq 0$ , with  $\rho = 6$  and  $r = 5$ . If  $M_2$  has a column relation  $XY = 1$ , then  $M_3$  satisfies (5.8).*

*Proof.* The proof is very similar to that of the previous result, so we omit many of the details. Symbolic calculation shows that either  $C_3^b$  is Hankel, in which case the result follows, or  $C_3^b$  is not Hankel because  $\tau \equiv (C_3^b)_{41} - 1 \neq 0$ . In the latter case we proceed to construct a flat extension  $M_4$  as follows. In  $(M_2 \ B_3)$  we have  $X^2Y = X$  and  $XY^2 = Y$ . As in the previous proof, by examining the structures of  $C_3^b$  and  $\Delta$  in detail, we see that  $X^2Y = XY^2 = 0$  in  $\Delta$ , whence  $X^2Y = X$  and  $XY^2 = Y$  hold in  $M_3$ . In  $M_4$  we must then have  $X^3Y = X^2$ ,  $X^2Y^2 = 1$ ,  $XY^3 = Y^2$  in  $(M_3 \ B_4)$  (due to recursiveness), so in  $B_4$  it remains to define  $x \equiv \beta_{70}$  and  $y \equiv \beta_{07}$ . A symbolic calculation shows that there exists  $x$  such that  $X^4 \in \text{Ran } M_3$  if and only if an equation of the form  $Q - \tau x = 0$  has a solution (for a particular polynomial expression  $Q$  in the moments of  $M_3$ ). Since  $\tau \neq 0$ , there is a unique solution  $x$ . Next,  $\beta_{07}$  can be defined so that  $Y^4 \in \text{Ran } M_3$  exactly as in the proof of the preceding result. Now (5.3) implies that in  $\text{Col } C_4^b$  we have  $X^3Y = X^2$ ,  $X^2Y^2 = 1$ , and  $XY^3 = Y^2$ , so  $C_4^b$  is Hankel if and only if  $(C_4^b)_{51} = 1$ . A symbolic calculation now shows that  $(C_4^b)_{51} - 1 = (Q - \tau x)R$  (for some well-defined rational expression  $R$  in the moment data). Since  $Q - \tau x = 0$ , a flat extension  $M_4$  is established, so the result follows.  $\square$

**Proposition 5.10.** *Suppose  $M_3 \succeq 0$ , with  $\rho = 6$  and  $r = 5$ . If  $M_2$  has a column relation  $Y = X^2$ , then  $M_3$  satisfies (5.8).*

*Proof.* By recursiveness, we have  $X^3 = XY$  and  $X^2Y = Y^2$  in  $(M_2 \ B_3)$ , and so, by (5.3), these relations also hold in  $C_3^b$ . Thus,  $X^3$  and  $X^2Y$  in  $C_3^b$  are Hankel with respect to each other, and it follows that  $C_3^b \equiv (c_{ij})_{1 \leq i, j \leq 4}$  is Hankel if and only if  $c_{42} = c_{33}$ , in which case we are done. Assuming this is not the case, we will establish a flat extension  $M_4$ . Consider  $\Delta := C_3 - C_3^b$ ,  $\text{rank } \Delta = 1$ , and note that the leftmost two columns of  $\Delta$  must be Hankel with respect to each other. If  $p \equiv \Delta_{11} \neq 0$ , then from the form of  $C_3$  and  $C_3^b$ , it follows that  $\Delta$  is of the form

$$\Delta = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & w \\ 0 & 0 & w & s \end{pmatrix}, \quad \delta > 0,$$

or

$$\Delta = \begin{pmatrix} p & qp & q^2p & q^3p \\ qp & q^2p & q^3p & q^4p \\ q^2p & q^3p & \delta & q^5p \\ q^3p & q^4p & q^5p & q^6p \end{pmatrix}, \quad \delta \neq q^4p, \quad q \neq 0.$$

In the former case,  $\text{rank } \Delta \geq 2$ , a contradiction. In the latter case, since  $p \neq 0$  and  $\text{rank } \Delta = 1$ , then  $\text{Col } 3 = q^2 \text{Col } 1$ , which implies  $\delta = q^4p$ , a contradiction. Thus,  $p = 0$ , so the positivity of  $\Delta$  and the Hankel property of its leftmost two columns

imply that

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & w \\ 0 & 0 & w & s \end{pmatrix}, \quad \delta > 0, \quad \delta s = w^2.$$

It follows that in  $C_3 \equiv C_3^b + \Delta$ , we have  $X^3 = XY$  and  $X^2Y = Y^2$ . Thus  $M_3$  is positive and recursively generated. Since  $\delta > 0$ , it follows that a column basis for  $M_3$  is given by  $\{1, X, Y, XY, Y^2, XY^2\}$ . We thus have a column relation of the form  $Y^3 = k_1 1 + k_2 X + k_3 Y + k_4 XY + k_5 Y^2 + k_6 XY^2$ . Denoting  $M_3$  as  $M_d$  with  $d = 3$ , we see that  $M_d$  is recursively determinate, i.e.,  $X^n = p(X, Y)$  ( $n = 2$ ,  $\deg p = 1$ ) and  $Y^m = q(X, Y)$  ( $m = 3$ ,  $\deg q = 3$ ). Since  $M_d$  is positive, recursively generated, and recursively determinate with  $n + m - 2 = d$ , it follows from [CF10, Corollary 2.4] that  $M_3$  admits a unique flat extension  $M_4$ , whence the result follows.  $\square$

**Proposition 5.11.** *Suppose  $M_3 \succeq 0$ , with  $\rho = 6$  and  $r = 5$ . If  $M_2$  has a column relation  $X^2 = 1$ , then  $M_3$  satisfies (5.8).*

*Proof.* The proof is essentially the same as that of the preceding result; replacing  $X^2 = Y$  by  $X^2 = 1$  has no effect on the argument there that either  $C_3^b$  is Hankel or  $M_3$  is recursively determinate with  $n + m - 2 = d$  (in the sense of [CF10, Corollary 2.4]), whence  $M_3$  admits a flat extension  $M_4$ .  $\square$

**Proposition 5.12.** *Suppose  $M_3 \succeq 0$ , with  $\rho = 6$  and  $r = 5$ . If  $M_2$  has a column relation  $X^2 + Y^2 = 1$ , then  $M_3$  admits a flat extension  $M_4$ .*

*Proof.* Since  $\{1, X, Y\}$  is independent in  $\text{Col } M_3$ , it follows from [CF5, Proposition 1.12] that  $M_3(\beta)$  corresponds to a complex moment matrix  $\mathcal{M}_3(\gamma)$  with columns  $1, Z, \bar{Z}$  independent and  $\bar{Z}Z = 1$  in the column space. [CF5, Theorem 1.1] now implies that  $\mathcal{M}_3(\gamma)$  has a flat extension  $\mathcal{M}_4$ . Under the inverse correspondence of [CF5, Proposition 1.12],  $\mathcal{M}_4$  corresponds to a real moment matrix  $\widetilde{M}_4$  that is a flat extension of  $M_3$ .  $\square$

We are now prepared to prove Theorem 5.1.

*Proof of Theorem 5.1.* The “only if” direction is clear from the discussion in Section 1. For the converse, let  $r = \text{rank } M_2$ . If  $r = 6$ , then  $M_3$  is flat, so there is nothing to prove. For  $1 \leq r \leq 3$ , the result follows from Proposition 5.2 and Corollary 1.10. Suppose  $4 \leq r \leq 5$ . As noted earlier, in these cases recursiveness in  $M_2$  implies that  $\{1, X, Y\}$  is independent in  $\text{Col } M_2$ . Since  $M_2$  is singular, there is a column dependence relation of the form  $AX^2 + BXY + CY^2 + DX + EY + F1 = 0$ , with  $A^2 + B^2 + C^2 \neq 0$ . Let  $p(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$  and consider the planar variety  $p(x, y) = 0$ . As discussed earlier in this section (following Proposition 5.2), there is a degree-one map  $\mathcal{T}$  ( $\mathcal{T}(x, y) \equiv (\tilde{x}, \tilde{y})$ ), such that the variety corresponding to  $q(\tilde{x}, \tilde{y}) := (p \circ \mathcal{T}^{-1})(\tilde{x}, \tilde{y})$  is one of the following:  $\tilde{x}\tilde{y} = 0$ ,  $\tilde{x}\tilde{y} = 1$ ,  $\tilde{y} = \tilde{x}^2$ ,  $\tilde{x}^2 = 1$ ,  $\tilde{x}^2 + \tilde{y}^2 = 1$ . Applying  $\mathcal{T}$  to  $M_3$  via Proposition 2.8, it follows from Proposition 2.8 iii), iv), vii) that the resulting moment matrix  $\widetilde{M}_3$  is positive, with  $\text{rank } \widetilde{M}_3 = \text{rank } M_3$ ,  $\{\tilde{1}, \tilde{X}, \tilde{Y}\}$  independent in  $\text{Col } \widetilde{M}_2$ , and with one of the following column dependence relations:  $\widetilde{XY} = \tilde{0}$ ,  $\widetilde{XY} = \tilde{1}$ ,  $\tilde{Y} = \tilde{X}^2$ ,  $\tilde{X}^2 = \tilde{1}$ ,  $\tilde{X}^2 + \tilde{Y}^2 = \tilde{1}$ . It now follows from Propositions 5.3-5.12 that  $\widetilde{M}_3$  satisfies (5.8). Proposition 2.8 vi) and x) thus imply that  $M_3$  satisfies (5.8), so the proof is completed by applications of Corollaries 1.10 and 1.11.  $\square$

The bivariate singular quartic problem was solved in [CF5] (with one exception that we note below). We conclude with a new formulation based on Theorem 5.1.

**Proposition 5.13.** *For  $n = 2$ , let  $\beta \equiv \beta^{(4)}$  and suppose  $M_2(\beta)$  is singular. The following are equivalent:*

- i)  $\beta$  admits a representing measure;
- ii)  $M_2$  is positive and recursively generated, and  $\text{rank } M_2 \leq \text{card } \mathcal{V}(\beta)$ ;
- iii)  $M_2$  admits a positive extension  $M_3 \equiv M_3(\tilde{\beta})$  satisfying  $\text{rank } M_3 \leq 6$ .

The equivalence of i) and ii) is proved in [CF5], except that [CF5] neglects to treat the case when  $M_2$  has a column relation equivalent to  $X^2 = 1$  under a degree-one map; we treat the  $X^2 = 1$  case now.

**Lemma 5.14.** *Suppose  $X^2 = 1$  in  $\text{Col } M_2$ . Then  $\beta \equiv \beta^{(4)}$  has a representing measure if and only if  $M_2$  is positive, recursively generated, and satisfies  $\text{rank } M_2 \leq \text{card } \mathcal{V}(\beta)$ .*

*Proof.* The necessity of the conditions is clear. For the converse, if  $M_1$  is singular, the existence of a flat extension follows from [CF3] and from the equivalence of the real and complex truncated moment problems [CF7]. We may thus assume that  $M_1 \succ 0$ , and we consider next the case when  $\{1, X, Y, XY\}$  is a basis for  $\text{Col } M_2$ . Then  $Y^2 = q(X, Y)$  for some  $q \in \mathbb{R}[x, y]_2$ , and since also  $X^2 = 1$ , it follows that  $M_2$  is recursively determinate with  $n + m - 2 = d$  (in the sense of [CF10, Corollary 2.4]), so [CF10] implies the existence of a flat extension  $M_3$ . If  $\{1, X, Y, Y^2\}$  is a column basis, then there is a hyperbola relation  $XY = a1 + bX + cY + dY^2$ , so it follows from [CF8] that either  $M_2$  has a flat extension  $M_3$  or  $M_2$  admits a positive, recursively generated extension of rank 5, which in turn has a flat extension  $M_4$ .

In the remaining case, we have a column basis  $\mathcal{B} \equiv \{1, X, Y, XY, Y^2\}$ , and we will construct a flat extension  $M_3(\tilde{\beta})$ . Since  $X^2 = 1$ , we may denote  $M_2$  by

$$(5.19) \quad M_2 = \begin{pmatrix} 1 & a & b & 1 & c & d \\ a & 1 & c & a & b & f \\ b & c & d & b & f & g \\ 1 & a & b & 1 & c & d \\ c & b & f & c & d & p \\ d & f & g & d & p & q \end{pmatrix}.$$

In  $\text{Col } (M_2 \ B_3)$  we must have  $X^3 = X$  and  $X^2Y = Y$ , so  $B_3$  has the form

$$(5.20) \quad B_3 = \begin{pmatrix} a & b & f & g \\ 1 & c & d & p \\ c & d & p & q \\ a & b & f & g \\ b & f & g & w \\ f & g & w & s \end{pmatrix}$$

for certain  $w, s$ . Let  $J = [M_2]_{\mathcal{B}}$ . Since  $J \succ 0$ , it is clear that  $B_3 = M_2W$  for some  $W$ . Let  $C_3^b \equiv W^T M_2 W$ . Since  $X^3 = X$  and  $X^2Y = Y$  in  $\text{Col } (B_3^T \ C_3^b)$ , then  $C_3^b \equiv (c_{ij})_{1 \leq i, j \leq 4}$  is Hankel if and only if  $c_{42} = c_{33}$ . A symbolic calculation shows that  $\tau \equiv c_{42} - c_{33}$  is of the form  $\tau = \frac{\tau_1 \tau_2}{\text{Det } J}$ , where  $\tau_1$  and  $\tau_2$  are polynomials in the moments of  $M_2$ . Further,  $\tau_1$  can be expressed as  $\tau_1 = \sigma + \rho w$ , where  $\sigma$  and  $\rho$  are also polynomials in the moments of  $M_2$ . Let  $K$  denote the compression of  $M_2$  to rows and columns indexed by  $1, X, Y, XY$ . Then  $K \succ 0$ , and a symbolic

calculation shows that  $\text{Det } K = \rho\rho'$  for some moment polynomial  $\rho'$ . Thus,  $\rho \neq 0$ , and it follows that there is a unique  $w \equiv \tilde{w}$  such that  $\tau_1 = 0$ . Setting  $\tilde{\beta}_{14} = \tilde{w}$ , then  $\tau = 0$  and  $C_3^b$  is Hankel, so the existence of a flat extension  $M_3(\tilde{\beta})$  is established.  $\square$

*Proof of Proposition 5.13.* We first show that ii) implies iii). If ii) holds and  $M_1$  is singular, then [CF3] and the equivalence of the real and complex truncated moment problems [CF7] together imply that  $M_2$  has a flat extension, so iii) follows in this case. We now assume that  $M_1 \succ 0$ , so that  $M_2$  has a degree 2 column dependence relation. Note from Proposition 2.8 that all of the properties in i)-iii) (i.e., positivity, rank, variety, existence of measureness, recursiveness, existence of positive extensions with prescribed rank) are invariant under degree-one maps. From Proposition 2.8 and the discussion following Proposition 5.2, we may thus assume that  $M_2$  has a column dependence corresponding to one of the five varieties considered in the proof of Theorem 5.1. In each of these cases, except for  $x^2 = 1$ , the results of [CF5] show that when ii) holds, then either  $M_2$  admits a flat extension  $M_3$  or  $M_2$  has a positive extension  $M_3$  with  $\text{rank } M_3 \leq 1 + \text{rank } M_2 \leq 6$ , and  $M_3$  admits a flat extension  $M_4$ . The proof of Lemma 5.14 establishes the same conclusions in the case when  $x^2 = 1$ . Thus, ii) implies iii). If iii) holds, then Theorem 5.1 implies that  $L_{\tilde{\beta}}$  is positive, so i) follows from Theorem 1.2. Since i) always implies ii) (cf. Section 2), the proof is complete.  $\square$

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#### ADDED IN PROOF

A very general and concrete sufficient condition for positivity of  $L_{\beta}$  was recently discovered by Grigoriy Blekherman [Bl]. Blekherman's results imply that if  $n \geq 1$ ,  $d \geq 3$ , and  $\text{rank } M_d \leq 3d - 3$ , then  $L_{\beta}$  is positive. They also imply that Theorem 1.1 cannot be extended to moment problems of higher degree.

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