Flat Extensions of Positive Moment Matrices: Relations in Analytic or Conjugate Terms*

Raúl E. Curto and Lawrence A. Fialkow

September 15, 1996

Dedicated to our teacher and friend, Carl M. Pearcy, on the occasion of his sixtieth birthday

1. Introduction

Given a doubly indexed finite sequence of complex members $\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ij} = \overline{\gamma_{ji}}$, the *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbf{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \le i+j \le 2n); \tag{1.1}$$

γ is called a *truncated moment sequence (of order* $2n$*)* and μ is called a *representing measure* for γ. The truncated complex moment problem is closely related to several other moment problems: the *full moment problem* prescribes moments of *all* orders, i.e., $\gamma = (\gamma_{ij})_{i,j\ge 0}$, $\gamma_{00} > 0$, $\gamma_{ij} = \overline{\gamma_{ji}}$; the K*-moment problem* (truncated or full) prescribes a closed set $K \subseteq \mathbf{C}$ which is to contain the support of the representing measure ([Atz], [BM], [Cas], [CP], [P3], [Sch2], [StSz], [Sza]); and the *multidimensional moment problem* extends each of these problems to measures supported in $\mathbf{C}^k$ ([Ber], [BCJ], [Cas], [Fug], [Hav1], [Hav2], [McG], [P1], [P2], [P4]); moreover, the k-dimensional complex moment problem is equivalent to the $2k$-dimensional real moment problem [CF4, Section 6]. All of these problems generalize classical power moment problems on the real line, whose study was initiated by Stieltjes, Riesz, Hamburger, and Hausdorff (cf. [AK], [Akh], [Hau], [KrN], [Lan], [Sar], [ShT]). Recently, J. Stochel [Sto] proved that a solution to the multidimensional truncated K-moment problem actually implies a solution to the corresponding full moment problem. For $k = 1$, we may informally paraphrase Stochel's result as follows: If $K \subseteq \mathbf{C}$ is closed, if $\gamma = (\gamma_{ij})_{i,j\ge 0}$ is a full moment sequence, and if for each $n \ge 1$ there exists a representing measure μ_n for $\{\gamma_{ij}\}_{0\le i+j\le 2n}$ such that $\operatorname{supp}\mu_n \subseteq K$, then there exists a subsequence of $\{\mu_n\}$ that converges (in an appropriate weak topology) to a representing measure μ for γ with $\operatorname{supp}\mu \subseteq K$.

In [CF4] we initiated an approach to the truncated complex moment problem based on positivity and extension properties of the *moment matrix* $M(n) \equiv M(n)(\gamma)$ associated to a truncated moment sequence γ (see below for notation). If μ is any representing measure for γ, then $\operatorname{card}\operatorname{supp}\mu \ge \operatorname{rank} M(n)$ (see (1.5) below); the main results of [CF4] characterize the existence of representing measures μ for which $\operatorname{card}\operatorname{supp}\mu = \operatorname{rank} M(n)$.

*Both authors were partially supported by NSF research grants

Theorem 1.1. [CF4, Corollary 5.14] *If* $M(n) \geq 0$ *and* $M(n)$ *is* flat, *i.e.,* $\operatorname{rank} M(n) = \operatorname{rank} M(n-1)$, *then* γ *has a unique representing measure, which is* $\operatorname{rank} M(n)$*-atomic.*

Theorem 1.2. [CF4, Theorem 5.13] γ *has a* $\operatorname{rank} M(n)$*-atomic representing measure if and only if* $M(n) \geq 0$ *and* $M(n)$ *admits a* flat extension $M(n+1)$, *i.e.,* $M(n)$ *can be extended to a moment matrix* $M(n+1)$ *satisfying* $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$.

In [CF4] we conjectured that if γ has *any* representing measure, then it has a $\operatorname{rank} M(n)$-atomic representing measure; this conjecture remains open. In the present note we study concrete sufficient conditions for the existence of flat moment matrix extensions of positive moment matrices; in view of Theorem 1.2, each such condition leads to the solution of a corresponding truncated moment problem.

To explain our results we require some additional notation. For $m \geq 1$, let $M_m(\mathbf{C})$ denote the $m \times m$ complex matrices. For $n \geq 1$, let $m \equiv m(n) := (n+1)(n+2)/2$; we introduce the following lexicographic order on the rows and columns of matrices in $M_{m(n)}(\mathbf{C})$: $1, Z, \bar{Z}, Z^2, Z\bar{Z}, \bar{Z}^2, \ldots, Z^n, \ldots, \bar{Z}^n$; rows or columns indexed by $1, Z, Z^2, \ldots, Z^n$ are said to be *analytic*. Let $\gamma : \gamma_{00}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$ be a truncated moment sequence; given $0 \leq i, j \leq n$ we define the $(i+1) \times (j+1)$ matrix B_{ij} whose entries are the moments of order $i+j$:

$$B_{ij} \equiv \begin{pmatrix} \gamma_{ij} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{ij} & \gamma_{i+1,j-1} & \cdots \\ \vdots & \gamma_{i-1,j+1} & & \vdots \\ \gamma_{0,j+i} & \cdots & & \gamma_{ji} \end{pmatrix}; \tag{1.2}$$

B_{ij} has the Toeplitz-like property of being constant on each diagonal. We now define the *moment matrix* $M(n) \equiv M(n)(\gamma)$ via the block decomposition $M(n) = (B_{ij})_{0 \leq i,j \leq n}$. For example, if $n = 1$, the *quadratic moment problem* for $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ corresponds to

$$M(1) = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}.$$

Note that for $0 \leq i+j \leq n$, $0 \leq k+\ell \leq n$, the entry in row $\bar{Z}^k Z^\ell$, column $\bar{Z}^i Z^j$ of $M(n)$ is equal to $\gamma_{i+\ell, j+k}$.

Let $\mathcal{P}_n \subseteq \mathbf{C}[z, \bar{z}]$ denote the complex polynomials in $z, \bar{z}$ of total degree $\leq n$. For $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$, let $\bar{p}(z, \bar{z}) \equiv \sum \overline{a_{ij}} z^i \bar{z}^j$ and let $\hat{p} \equiv (a_{00}, a_{01}, a_{10}, \ldots, a_{0n}, \ldots, a_{n0})^T \in \mathbf{C}^{m(n)}$. The basic connection between $M(n)(\gamma)$ and any representing measure μ is provided by the identity

$$\int f \bar{g} \, d\mu = (M(n)\hat{f}, \hat{g}) \qquad (f, g \in \mathcal{P}_n); \tag{1.3}$$

in particular $(M(n)\hat{f}, \hat{f}) = \int |f|^2 \, d\mu \geq 0$, so $M(n) \geq 0$. For the quadratic moment problem ($n = 1$), positivity of $M(1)$ implies the existence of $\operatorname{rank} M(1)$-representing measures [CF4, Theorem 6.1], but in general positivity of $M(n)$ does not by itself imply the existence of representing measures.

We next recall from [CF4] some additional necessary conditions for the existence of representing measures. Let $\mathcal{C}_{M(n)}$ denote the column space of $M(n)$, i.e., $\mathcal{C}_{M(n)} = \langle 1, Z, \bar{Z}, \ldots, Z^n, \ldots, \bar{Z}^n \rangle \subseteq \mathbf{C}^{m(n)}$. For $p \in \mathcal{P}_n$, $p \equiv \sum a_{ij} \bar{z}^i z^j$, we define $p(Z, \bar{Z}) \in \mathcal{C}_{M(n)}$ by $p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j$; note that if $p(Z, \bar{Z}) = 0$ then $\bar{p}(Z, \bar{Z}) = 0$ [CF4, Lemma 3.10]. If μ is a representing measure for γ, then

(1.4) For $p \in \mathcal{P}_n$, $p(Z, \bar{Z}) = 0 \Leftrightarrow \operatorname{supp} \mu \subseteq \mathcal{Z}(p) := \{z \in \mathbf{C} : p(z, \bar{z}) = 0\}$ [CF4, Prop. 3.1].

It follows from (1.4) that

(1.5) If μ is a representing measure for γ, then $\operatorname{card} \operatorname{supp} \mu \geq \operatorname{rank} M(n)$ [CF4, Cor. 3.5].

The following Structure Theorem for positive moment matrices provides a basic tool for constructing flat extensions.

Theorem 1.3. [CF4, Theorem 3.14] *Let* $M(n)(\gamma) \geq 0$. *If* f, g, $fg \in \mathcal{P}_{n-1}$ *and* $f(Z, \bar{Z}) = 0$, *then* $(fg)(Z, \bar{Z}) = 0$.

In view of Theorem 1.3 the following condition is *necessary* for the existence of a positive extension $M(n+1)$ of $M(n)(\gamma)$:

(RG) f, g, $fg \in \mathcal{P}_n$, $f(Z, \bar{Z}) = 0 \Rightarrow (fg)(Z, \bar{Z}) = 0$.

A moment matrix satisfying (RG) is said to be *recursively generated*.

For the case of the truncated moment problems in one real variable, where the "moment matrix" associated to moments $\gamma : \gamma_0, \ldots, \gamma_{2n}$ is the *Hankel matrix* $H(n) \equiv (\gamma_{i+j})_{0 \leq i,j \leq n}$, we have the following result.

Theorem 1.4. [CF3, Section 3] *The following are equivalent:*

1) *There exists a positive Borel measure* μ, $\operatorname{supp} \mu \subseteq \mathbf{R}$, *such that* $\gamma_i = \int t^i \, d\mu(t)$ $(0 \leq i \leq 2n)$*;*

2) γ *has a* $\operatorname{rank} H(n)$*-atomic representing measure supported in* $\mathbf{R}$*;*

3) $H(n) \geq 0$ *and* $H(n)$ *is recursively generated (in the one-variable sense);*

4) $H(n) \geq 0$ *and* $H(n)$ *admits a flat (i.e., rank preserving) extension* $H(n+1)$.

In [CF4] we presented several cases in which Theorem 1.4 admits the following analogue for the truncated complex moment problem:

(1.6) If $M(n)$ is positive and satisfies (RG), then $M(n)$ admits a flat extension $M(n+1)$.

Of course, if (1.6) holds for a particular $M(n)(\gamma)$, then by Theorem 1.2, γ has a $\operatorname{rank} M(n)$-atomic representing measure. Theorem 1.1 corresponds to the case of (1.6) in which $M(n) \geq 0$ and for all $i + j = n$, $\bar{Z}^i Z^j \in \langle \bar{Z}^\ell Z^m \rangle_{0 \leq \ell + m \leq n-1}$.

In the present note we establish (1.6) in the following two new cases:

(1.7) $$\bar{Z} = \alpha 1 + \beta Z \text{ for some } \alpha, \beta \in \mathbf{C} \text{ (Theorem 2.1);}$$

(1.8) $$Z^k = p(Z,\bar{Z}) \text{ for some } p \in \mathcal{P}_{k-1}, \text{ where } k \le [\tfrac{n}{2}] + 1 \text{ (Theorem 3.1)}.$$

On the other hand, we show that (1.6) does not always hold. In Section 4 we use an example of Schmüdgen [Sch1] to construct a positive invertible (hence recursively generated) moment matrix $M(3)(\gamma)$ for which γ admits *no* representing measure; hence $M(3)(\gamma)$ admits no flat extension $M(4)$.

We conclude this section with some preliminaries concerning flat moment matrix extensions of positive moment matrices. For $n \ge 1$ and $A \in M_{m(n)}(\mathbf{C})$, $A = A^*$, we define an hermitian sesquilinear form $\langle \cdot, \cdot \rangle_A$ on $\mathcal{P}_n$ by $\langle p, q \rangle_A := (A\hat{p}, \hat{q})$; thus the entry in row $\bar{Z}^k Z^\ell$, column $\bar{Z}^i Z^j$ of A is given by

$$\begin{aligned} (\bar{Z}^i Z^j, \widehat{\bar{z}^k z^\ell}) &= (A\widehat{\bar{z}^i z^j}, \widehat{\bar{z}^k z^\ell}) \\ &= \langle \bar{z}^i z^j, \bar{z}^k z^\ell \rangle_A \qquad (0 \le i+j \le n,\ 0 \le k+\ell \le n). \end{aligned}$$

Moreover, if there exist p, $q \in \mathcal{P}_n$ such that $\bar{Z}^i Z^j = p(Z,\bar{Z})$ and $\bar{Z}^k Z^\ell = q(Z,\bar{Z})$, then $\langle \bar{z}^i z^j, \bar{z}^k z^\ell \rangle_A = \langle p, q \rangle_A$. The following intrinsic characterization of moment matrices provides a useful tool for constructing extensions.

Theorem 1.5. [CF4, Theorem 2.1] *Let $n \ge 1$ and let $A \in M_{m(n)}(\mathbf{C})$. There exists a truncated moment sequence γ such that $A = M(n)(\gamma)$ if and only if*

0) $\langle 1, 1 \rangle_A > 0$;

1) $A = A^*$;

2) $\langle p, q \rangle_A = \langle \bar{q}, \bar{p} \rangle_A$ $(p, q \in \mathcal{P}_n)$ *(symmetric property);*

3) $\langle zp, q \rangle_A = \langle p, \bar{z}q \rangle_A$ $(p, q \in \mathcal{P}_{n-1})$;

4) $\langle zp, zq \rangle_A = \langle \bar{z}p, \bar{z}q \rangle_A$ $(p, q \in \mathcal{P}_{n-1})$ *(normality property).*

For $0 \le i, j \le n$, let $B[i,j] \in M_{(i+1)\times(j+1)}(\mathbf{C})$. Let Z^i, $Z^{i-1}\bar{Z}$, $\ldots$, $\bar{Z}^i$ denote the rows of $B[i,j]$ and let Z^j, $Z^{j-1}\bar{Z}$, $\ldots$, $\bar{Z}^j$ denote the columns of $B[i,j]$. For $0 \le r \le i$, $0 \le s \le j$, we denote the entry in row $\bar{Z}^r Z^{i-r}$, column $\bar{Z}^s Z^{j-s}$ of $B[i,j]$ by $\langle \bar{z}^s z^{j-s}, \bar{z}^r z^{i-r} \rangle_{B[i,j]}$. For $i = j$, this notation is consistent with our previous definition of $\langle \cdot, \cdot \rangle_{B[i,j]}$. We say that $B[i,j]$ is *symmetric* if

(1.9) $$\langle \bar{z}^s z^{j-s}, \bar{z}^r z^{i-r} \rangle_{B[i,j]} = \langle \overline{z^s \bar{z}^{j-s}, z^r \bar{z}^{i-r}} \rangle_{B[i,j]} \qquad (0 \le r \le i,\ 0 \le s \le j),$$

and we say that $B[i,j]$ satisfies *normality* if it is constant on diagonals, i.e.,

(1.10) $$\langle \bar{z}^s z^{j-s}, \bar{z}^r z^{i-r} \rangle_{B[i,j]} = \langle \bar{z}^{s+1} z^{j-s-1}, \bar{z}^{r+1} z^{i-r-1} \rangle_{B[i,j]} \qquad (0 \le r < i,\ 0 \le s < j).$$

Note that if $B[i,j]$ is symmetric and constant on upper diagonals (i.e., where $r \le s$ in (1.10)), then $B[i,j]$ satisfies normality. More generally, given $B[i,j]$, there exist scalars $\{\gamma_{\ell,m}\}_{\ell+m=i+j}$, $\gamma_{\ell m} = \overline{\gamma_{m\ell}}$, such that for all $s + t = i$, $u + v = j$, we have $\langle \bar{z}^u z^v, \bar{z}^s z^t \rangle_{B[i,j]} = \gamma_{u+t,v+s}$ if and only if $B[i,j]$ is symmetric and satisfies normality.

Given $\gamma \equiv \gamma^{(2n)}$, in addition to $M(n)$ we may also define blocks $B_{0,n+1}, \ldots, B_{n-1,n+1}$ via (1.2). Given $B \equiv B[n, n+1]$, let

$$\tilde{B} := \begin{pmatrix} B_{0,n+1} \\ \vdots \\ B_{n-1,n+1} \\ B \end{pmatrix}.$$

Given $C := B[n+1, n+1]$, let $M = \begin{pmatrix} M(n) & \tilde{B} \\ \tilde{B}^* & C \end{pmatrix}$; M is an *extension* of $M(n)$; M is a *flat extension* if $\operatorname{rank} M = \operatorname{rank} M(n)$. Note that if C is self-adjoint and constant on upper diagonals, then C is constant on all diagonals and is thus also symmetric.

In the sequel we seek to construct M so that it is a positive flat extension of the form $M(n+1)$. The structure theory of positive operator matrices (cf. [Fia], [Smu]) implies that if $M \geq 0$, then $\operatorname{Ran} \tilde{B} \subseteq \mathcal{C}_{M(n)}$; equivalently, there exists a matrix W such that $\tilde{B} = M(n)W$. Conversely, given $M(n) \geq 0$ and $\tilde{B} = M(n)W$, then $M \geq 0 \Leftrightarrow C \geq W^* M(n) W$; moreover, M is a flat extension of $M(n) \geq 0 \Leftrightarrow C = W^* M(n) W$. (In this case, C is independent of W.) Thus a flat extension of a positive moment matrix is positive. The block structures of $\tilde{B}$ and $M(n)$, Theorem 1.5, and the preceding remarks, imply that to construct a flat moment matrix extension $M(n+1)$ of $M(n) \geq 0$ it is necessary and sufficient to construct a block $B[n, n+1]$ such that

(1.11) $$B[n, n+1] \text{ is symmetric and satisfies normality;}$$

(1.12) $$\operatorname{Ran} \tilde{B} \subseteq \mathcal{C}_{M(n)} \text{ (so that } \tilde{B} = M(n)W \text{ for some } W);$$

(1.13) $$W^* M(n) W \text{ is constant on upper diagonals.}$$

We next provide a sufficient condition for $B[n, n+1]$ to be symmetric. Assume $\operatorname{Ran} \tilde{B} \subseteq \mathcal{C}_{M(n)}$; thus, for $i+j = n+1$, there exist $p_{ij} \in \mathcal{P}_n$ such that $\bar{Z}^i Z^j = p_{ij}(Z, \bar{Z}) \in \mathcal{C}_{M(n)}$.

Lemma 1.6. *If $p_{ij} = \overline{p_{ji}}$ for all $i + j = n+1$, then $B \equiv B[n, n+1]$ is symmetric.*

Proof. For $i + j = n+1$, $k + \ell = n$,

$$\begin{aligned}
\langle \bar{z}^i z^j, \bar{z}^k z^\ell \rangle_B &= (p_{ij}(Z, \bar{Z}), \widehat{\bar{z}^k z^\ell}) \\
&= \langle p_{ij}, \bar{z}^k z^\ell \rangle_{M(n)} \\
&= \langle z^k \bar{z}^\ell, \overline{p_{ij}} \rangle_{M(n)} \qquad \text{(by Theorem 1.5–2))} \\
&= \overline{\langle \overline{p_{ij}}, z^k \bar{z}^\ell \rangle_{M(n)} = (p_{ji}(Z, \bar{Z}), \widehat{z^k \bar{z}^\ell})} \\
&= \overline{\langle \bar{z}^j z^i, z^k \bar{z}^\ell \rangle_B}. \ \square
\end{aligned}$$

Let $[M(n)]_{n-1} := (B_{ij})_{0 \leq i \leq n-1, 0 \leq j \leq n}$, let $[\tilde{B}]_{n-1} := (B_{i,n+1})_{0 \leq i \leq n-1}$, and let $\{\bar{V}^i V^j\}_{0 \leq i+j \leq n+1}$ denote the columns of the block $S := ([M(n)]_{n-1} \ [\tilde{B}]_{n-1})$.

Lemma 1.7. *Suppose $M(n)$ is recursively generated. If $p,\ q \in \mathcal{P}_n$, $pq \in \mathcal{P}_{n+1}$ and $p(Z,\bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$, then $(pq)(V,\bar{V}) = 0$ in $\mathcal{C}_S$.*

Proof. Let $k := \deg p\ (\leq n)$ and denote $p(z,\bar{z}) = \sum_{0\leq r+s\leq k} a_{rs}\bar{z}^r z^s$. It suffices to prove that if $i+j \leq n+1-k$, then $(\bar{z}^i z^j p)(V,\bar{V}) = 0$. Since $M(n)$ satisfies (RG), if $i+j \leq n-k$, then $(\bar{z}^i z^j p)(Z,\bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$, so clearly $(\bar{z}^i z^j p)(V,\bar{V}) = 0$ in $\mathcal{C}_S$. Suppose $i+j = n+1-k$ and assume $j \geq 1$. Then (RG) $\Rightarrow (\bar{z}^i z^{j-1} p)(Z,\bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$, and an obvious adaptation of the proof of Theorem 1.3 shows that $(\bar{z}^i z^j p)(V,\bar{V}) = (z(\bar{z}^i z^{j-1} p))(V,\bar{V}) = 0$. The proof when $j = 0$, $i \geq 1$ is similar. □

Lemma 1.8. (cf. [CF4, Lemma 5.2–ii]) *Suppose $M(n+1)$ is a flat extension of $M(n) \geq 0$, and let $p \in \mathcal{P}_{n+1}$ be such that $p(Z,\bar{Z}) = 0$ in $\mathcal{C}_{[M(n+1)]_n}$. Then $p(Z,\bar{Z}) = 0$ in $\mathcal{C}_{M(n+1)}$.*

Lemma 1.9. *Suppose $M(n)$ is positive and recursively generated. If $M(n+1)$ is a flat extension of $M(n)$, then $M(n+1)$ is recursively generated.*

Proof. Suppose $p,\ q,\ pq \in \mathcal{P}_{n+1}$ and $p(Z,\bar{Z}) = 0$. We seek to show that $(pq)(Z,\bar{Z}) = 0$, and we may assume $p \in \mathcal{P}_n$, for otherwise q is a constant function and the result is immediate. Using [CF4, Lemma 3.10], it suffices to consider the case $q(z,\bar{z}) \equiv z^i$. When $\deg p + i \leq n$, a combination of $M(n)$ being recursively generated and Lemma 1.8 yields the desired result. Assume, therefore, $\deg p + i = n+1$. Since $p \in \mathcal{P}_n$ and $p(Z,\bar{Z}) = 0$ in $\mathcal{C}_{M(n+1)}$, the proof of Theorem 1.3 (applied to $M(n+1)$) shows that $[(z^i p)(Z,\bar{Z})]_n = 0$ (where $[\cdot]_n$ denotes truncation of a vector through rows corresponding to monomials of total degree n). The result now follows from Lemma 1.8. □

The authors wish to thank Professors Jim Agler, Edwin Franks, John McCarthy, Mihai Putinar and Mark Stankus for helpful discussions concerning the subject of this paper. Some of the calculations in Section 4 were obtained with the help of the software tool *Mathematica* [Wol].

2. The Case of $\bar{Z} = \alpha \mathit{1} + \beta Z$

In this section we focus on a positive, recursively generated moment matrix $M(n) \equiv M(n)(\gamma)$ in which the third column, $\bar{Z}$, is a linear combination of the first and second columns, $\mathit{1}$ and Z. We will show that $M(n)$ always admits a flat extension $M(n+1)$; thus there exists a rank $M(n)$-atomic representing measure μ for γ. As outlined in the Introduction, our method is to define a suitable block $\tilde{B}$ of the form $M(n)W$ so that properties (1.11)–(1.13) are satisfied. To motivate our construction, we consider first the quadratic moment problem ($n = 1$).

We are given six numbers, $\gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20}$, ($\gamma_{00} > 0$, $\gamma_{ij} = \overline{\gamma_{ji}}$) such that $M(1) \geq 0$ and $\bar{Z} = \alpha \mathit{1} + \beta Z$ for some $\alpha,\ \beta \in \mathbf{C}$, and we would like to find a 6×6 moment matrix $M(2)$ which is a flat extension of $M(1)$. Since the case rank $M(1) = 1$ is straightforward, we focus on rank $M(1) = 2$, that is, $\mathit{1}$ and Z are linearly independent. By Lemma 1.8, the relation $\bar{Z} = \alpha \mathit{1} + \beta Z$ and the flatness condition would imply $\bar{Z}Z = \alpha Z + \beta Z^2$ and $\bar{Z}^2 = \alpha \bar{Z} + \beta \bar{Z}Z$ in $\mathcal{C}_{M(2)}$. To describe $M(2)$ it thus remains to define column Z^2; we focus on the case $\alpha \neq 0$. By (1.2), we know that

$\langle z^2, 1\rangle_{M(2)}$ must equal γ_{02}, and by Theorem 1.5, a necessary condition for the existence of an extension $M(2)$ is

$$\begin{aligned}(2.1)\qquad \langle z^2, z\rangle_{M(2)} &= \langle z, \bar{z}z\rangle_{M(2)} = \langle z, (\alpha 1 + \beta z)z\rangle_{M(2)} = \bar{\alpha}\langle z, z\rangle_{M(1)} + \bar{\beta}\langle z, z^2\rangle_{M(2)} \\ &= \bar{\alpha}\langle z, z\rangle_{M(1)} + \bar{\beta}\overline{\langle z^2, z\rangle}_{M(2)}.\end{aligned}$$

Let us briefly pause to establish a relation between α and β : Since $\bar{Z} = \alpha 1 + \beta Z$, [CF4, Lemma 3.10] forces at once the relation $Z = \bar{\alpha}1 + \bar{\beta}\bar{Z} = \bar{\alpha}1 + \bar{\beta}(\alpha 1 + \beta Z) = (\bar{\alpha} + \bar{\beta}\alpha)1 + |\beta|^2 Z$. By the linear independence of 1 and Z, we must then have

$$(2.2)\qquad \begin{cases} \bar{\alpha} + \bar{\beta}\alpha = 0 \\ |\beta| = 1 \end{cases}.$$

Thus, (2.1) becomes

$$\alpha\langle z^2, z\rangle_{M(2)} = |\alpha|^2\langle z, z\rangle_{M(1)} + \alpha\bar{\beta}\overline{\langle z^2, z\rangle}_{M(2)} = |\alpha|^2\langle z, z\rangle_{M(1)} - \bar{\alpha}\langle z^2, z\rangle_{M(2)},$$

that is

$$2\,\mathrm{Re}(\alpha\langle z^2, z\rangle_{M(2)}) = |\alpha|^2\langle z, z\rangle_{M(1)}.$$

Therefore, $\alpha\langle z^2, z\rangle_{M(2)} = \frac{1}{2}|\alpha|^2\langle z, z\rangle_{M(1)} + it$, for some $t \in \mathbf{R}$. Observe also that

$$\langle z^2, \bar{z}\rangle_{M(2)} = \langle z^2, \alpha 1 + \beta z\rangle_{M(2)} = \bar{\alpha}\langle z^2, 1\rangle_{M(2)} + \bar{\beta}\langle z^2, z\rangle_{M(2)},$$

so the choice of $\langle z^2, z\rangle_{M(2)}$ (and the flatness requirement) fully determines the remaining entries of $M(2)$. We shall now extend this idea to the general case $n \geq 1$.

Theorem 2.1. *Assume that $M(n) \geq 0$ satisfies (RG) and that $\bar{Z} = \alpha 1 + \beta Z$. Then $M(n)$ admits a flat extension $M(n+1)$.*

We note that Theorem 2.1 is independent of Theorems 1.1 and 3.1. Indeed, [CF4, Section 6] contains the case when $M(1)$ is positive, recursively generated, $\bar{Z} = \alpha 1 + \beta Z$ and $\{1, Z\}$ is linearly independent; thus $M(1)$ is not flat (independence from Theorem 1.1) and $Z \notin \langle 1\rangle$ (independence from Theorem 3.1). A more ambitious example is contained in [Fia, Section 5], wherein $M(2)$ is positive, recursively generated, $\bar{Z} = \alpha 1 + \beta Z$, and $\{1, Z, Z^2\}$ is linearly independent.

First, let us show that the analytic columns of $M(n)$ can always be assumed to be linearly independent.

Proposition 2.2. *([CF4, Corollary 5.15]) Assume $M(n) \geq 0$ and that the analytic columns of $M(n)$ are linearly dependent. Let $r := \min\{k \geq 1 : Z^k \in \langle 1, Z, \ldots, Z^{k-1}\rangle\}$. Then γ has a representing measure if and only if $\{1, Z, ..., Z^{r-1}\}$ spans $\mathcal{C}_{M(n)}$. In this case the representing measure is unique, and is r $(= \mathrm{rank}\, M(n))$-atomic.*

In Theorem 2.1, (RG) and $\bar{Z} = \alpha 1 + \beta Z$ imply $\bar{Z}^i Z^j = ((\alpha + \beta z)^i z^j)(Z, \bar{Z})$ in $\mathcal{C}_{M(n)}$ $(0 \leq i + j \leq n,\ i \geq 1)$, whence $\{Z^i\}_{i=0}^n$ spans $\mathcal{C}_{M(n)}$. Since $M(n) \geq 0$, it follows from (RG), Proposition 2.2 and Theorem 1.2 that if $\{Z^i\}_{i=0}^n$ is dependent, then $M(n)$ has a flat extension $M(n+1)$. In the sequel we thus assume $\{Z^i\}_{i=0}^n$ is independent.

The proof of Theorem 2.1 will be a consequence of a series of lemmas. Our first goal is to define column Z^{n+1} of the block $\tilde{B}$. If $\alpha = 0$, then $\bar{Z} = \beta Z$ with $|\beta| = 1$, say $\beta = e^{i\psi}$. The requirement $\langle z^{n+1}, z^n\rangle_{M(n+1)} = \langle z^n, \bar{z}z^n\rangle_{M(n+1)} = \bar{\beta}\overline{\langle z^{n+1}, z^n\rangle}_{M(n+1)}$ forces us to define $\langle z^{n+1}, z^n\rangle_{\tilde{B}} := re^{i(2\pi j-\psi)/2}$ with $r > 0$ and $j \in \mathbf{Z}$. If $\alpha \neq 0$, then proceeding as in the $n = 1$ case we define $\langle z^{n+1}, z^n\rangle_{\tilde{B}} = \frac{1}{2}\bar{\alpha}\gamma_{nn} + \frac{it}{\alpha}$ for some fixed $t \in \mathbf{R}$. Let

$$\left\langle z^{n+1}, \bar{z}^i z^j\right\rangle_{\tilde{B}} := \begin{cases} \gamma_{j,n+i+1} & i+j \leq n-1 \\ \bar{\alpha}\gamma_{j,n+i} + \bar{\beta}\left\langle z^{n+1}, \bar{z}^{i-1}z^{j+1}\right\rangle_{\tilde{B}} & i \geq 1,\ j = n-i \end{cases} \quad (t \in \mathbf{R}), \tag{2.3}$$

and in $\mathcal{C}_{\tilde{B}}$ let

$$\bar{Z}^k Z^\ell := \alpha \bar{Z}^{k-1}Z^\ell + \beta \bar{Z}^{k-1}Z^{\ell+1} \ (k \geq 1,\ \ell = n+1-k) \tag{2.4}$$

(All of these columns have length equal to the size of $M(n)$, that is, $(n+1)(n+2)/2$.)

We may write $\tilde{B}$ as a block column matrix $\tilde{B} = (\tilde{B}_0, \ldots, \tilde{B}_n)^T$, where, for each j, the columns of $\tilde{B}_j$ are indexed lexicographically by Z^{n+1}, $\ldots$, $\bar{Z}^{n+1}$ and the rows by Z^j, $\ldots$, $\bar{Z}^j$. For $p \in \mathcal{P}_{n+1}$ and $q \in \mathcal{P}_n$ we define $\langle p, q\rangle_{\tilde{B}} := (p(Z, \bar{Z}), \hat{q})$, where $p(Z, \bar{Z})$ is defined in the usual way using the columns of $M(n)$ and of $\tilde{B}$; note that if p, $r \in \mathcal{P}_{n+1}$ and $p(Z, \bar{Z}) = r(Z, \bar{Z})$, then $\langle p, q\rangle_{\tilde{B}} = \langle r, q\rangle_{\tilde{B}}$.

Observe the following consequence of (RG): If $0 \leq \ell + m \leq 2n$, $\ell \geq 1$, then $\bar{Z}^\ell Z^m = \alpha\bar{Z}^{\ell-1}Z^m + \beta\bar{Z}^{\ell-1}Z^{m+1}$ in $\mathcal{C}_{M(n)}$; thus

$$\gamma_{\ell m} = \alpha\gamma_{\ell-1,m} + \beta\gamma_{\ell-1,m+1}. \tag{2.5}$$

Lemma 2.3. *$\tilde{B} = M(n)W$ for some W.*

Proof. From (2.4), it is enough to check that $Z^{n+1} \in Ran(M(n))$. Since $\{1, Z, \ldots, Z^n\}$ is independent and $M(n) \geq 0$, the Extension Principle [Fia] implies that the compression of $M(n)$ to the analytic rows and columns is invertible. Thus there exist complex numbers $a_0, ..., a_n$ such that

$$\sum_{i=0}^{n} a_i \left\langle z^i, z^\ell\right\rangle_{M(n)} = \left\langle z^{n+1}, z^\ell\right\rangle_{\tilde{B}} \quad (0 \leq \ell \leq n).$$

We shall show that the same relation holds for non-analytic rows, those determined by monomials of the form $\bar{Z}^k Z^\ell$, $k \geq 1$, $k + \ell \leq n$. We use induction on $k \geq 1$. For $k = 1$ we have

$$\sum_{i=0}^{n} a_i\langle z^i, \bar{z}z^\ell\rangle_{M(n)} = \bar{\alpha}\sum_{i=0}^{n} a_i\langle z^i, z^\ell\rangle_{M(n)} + \bar{\beta}\sum_{i=0}^{n} a_i\langle z^i, z^{\ell+1}\rangle_{M(n)};$$

since Z^ℓ and $Z^{\ell+1}$ are analytic, we have

$$\begin{aligned} \sum a_i\langle z^i\bar{z}, z^\ell\rangle_{M(n)} &= \bar{\alpha}\langle z^{n+1}, z^\ell\rangle_{\tilde{B}} + \bar{\beta}\langle z^{n+1}, z^{\ell+1}\rangle_{\tilde{B}} \\ &= \bar{\alpha}\gamma_{\ell,n+1} + \bar{\beta}\langle z^{n+1}, z^{\ell+1}\rangle_{\tilde{B}} \quad \text{(by (2.3), since } \ell \leq n-1). \end{aligned} \tag{2.6}$$

For $\ell < n-1$, the last expression in (2.6) equals

$$\begin{aligned} \bar{\alpha}\gamma_{\ell,n+1} + \bar{\beta}\gamma_{\ell+1,n+1} &= \gamma_{\ell,n+2} \qquad \text{(by (2.5))} \\ &= \langle z^{n+1}, \bar{z}z^\ell\rangle_{\tilde{B}} \text{ (by (2.3))}. \end{aligned}$$

For $\ell = n-1$, the final expression in (2.6) coincides with $\langle z^{n+1}, \bar{z}z^{\ell}\rangle_{\tilde{B}}$ by (2.3).

For $k \geq 1$, we have

$$\begin{aligned}
\textstyle\sum_{i=0}^{n} a_i \left\langle z^i, \bar{z}^k z^{\ell}\right\rangle_{M(n)} &= \textstyle\sum_{i=0}^{n} a_i \left\langle z^i, (\alpha+\beta z)\bar{z}^{k-1} z^{\ell}\right\rangle_{M(n)} \\
&= \bar{\alpha} \textstyle\sum_{i=0}^{n} a_i \left\langle z^i, \bar{z}^{k-1} z^{\ell}\right\rangle_{M(n)} + \bar{\beta} \textstyle\sum_{i=0}^{n} a_i \left\langle z^i, \bar{z}^{k-1} z^{\ell+1}\right\rangle_{M(n)} \\
&= \bar{\alpha} \left\langle z^{n+1}, \bar{z}^{k-1} z^{\ell}\right\rangle_{\tilde{B}} + \bar{\beta} \left\langle z^{n+1}, \bar{z}^{k-1} z^{\ell+1}\right\rangle_{\tilde{B}} \quad \text{(by induction)} \\
&= \bar{\alpha}\gamma_{\ell,n+k} + \bar{\beta} \left\langle z^{n+1}, \bar{z}^{k-1} z^{\ell+1}\right\rangle_{\tilde{B}} = \left\langle z^{n+1}, \bar{z}^{k} z^{\ell}\right\rangle_{\tilde{B}}
\end{aligned}$$

(by (2.3) if $k+\ell = n$ and by (2.3) and (2.5) if $k+\ell \leq n-1$), as desired. □

The next lemma shows that for $j \leq n-1$, $\tilde{B}_j = B_{j,n+1}(\gamma)$.

Lemma 2.4. *For $i+j = n+1$ and $p+q \leq n-1$,*

$$\left\langle \bar{z}^i z^j, \bar{z}^p z^q\right\rangle_{\tilde{B}} = \left\langle \bar{z}^i z^{j-1}, \bar{z}^{p+1} z^q\right\rangle_{M(n)} = \gamma_{i+q,j+p}.$$

Proof. We use induction on $i \geq 0$. For $i = 0$, (2.3) implies

$$\left\langle z^{n+1}, \bar{z}^p z^q\right\rangle_{\tilde{B}} = \gamma_{q,n+p+1} = \left\langle z^n, \bar{z}^{p+1} z^q\right\rangle_{M(n)}.$$

When $i \geq 1$,

$$\begin{aligned}
\left\langle \bar{z}^i z^{n+1-i}, \bar{z}^p z^q\right\rangle_{\tilde{B}} &= \alpha \left\langle \bar{z}^{i-1} z^{n+1-i}, \bar{z}^p z^q\right\rangle_{\tilde{B}} + \beta \left\langle \bar{z}^{i-1} z^{n+2-i}, \bar{z}^p z^q\right\rangle_{\tilde{B}} \\
&= \alpha \left\langle \bar{z}^{i-1} z^{n-i}, \bar{z}^{p+1} z^q\right\rangle_{M(n)} + \beta \left\langle \bar{z}^{i-1} z^{n+1-i}, \bar{z}^{p+1} z^q\right\rangle_{M(n)} \\
&\quad \text{(by Theorem 1.5 for the first term and} \\
&\quad \text{by induction for the second term)} \\
&= \left\langle \bar{z}^i z^{n-i}, \bar{z}^{p+1} z^q\right\rangle_{M(n)}. \ \square
\end{aligned}$$

The next lemma establishes normality between columns Z^{n+1} and $Z^n \bar{Z}$ of $\tilde{B}_n$.

Lemma 2.5. *For $p+q = n,\ p \geq 1$,*

$$\left\langle z^{n+1}, \bar{z}^{p-1} z^{q+1}\right\rangle_{\tilde{B}} = \langle \bar{z}z^n, \bar{z}^p z^q\rangle_{\tilde{B}}.$$

Proof. We use induction on $p \geq 1$. For $p = 1$,

$$\begin{aligned}
\left\langle \bar{z}z^n, \bar{z}z^{n-1}\right\rangle_{\tilde{B}} &= \langle \alpha z^n + \beta z^{n+1}, \bar{z}z^{n-1}\rangle_{\tilde{B}} \\
&= \alpha\langle z^n, \bar{z}z^{n-1}\rangle_{M(n)} + \beta\langle z^{n+1}, \bar{z}z^{n-1}\rangle_{\tilde{B}} \\
&= \alpha\gamma_{n-1,n+1} + \beta(\bar{\alpha}\gamma_{n-1,n+1} + \bar{\beta}\left\langle z^{n+1}, z^n\right\rangle_{\tilde{B}}) \\
&= \alpha\gamma_{n-1,n+1} - \alpha\gamma_{n-1,n+1} + \langle z^{n+1},\ z^n\rangle_{\tilde{B}} = \langle z^{n+1}, z^n\rangle_{\tilde{B}}.
\end{aligned}$$

The inductive step is a bit more complex: For $p \geq 2$,

$$\begin{aligned}
\left\langle z^{n+1}, \bar{z}^{p-1} z^{q+1}\right\rangle_{\tilde{B}} &= \bar{\alpha}\gamma_{q+1,n+p-1} + \bar{\beta} \left\langle z^{n+1}, \bar{z}^{p-2} z^{q+2}\right\rangle_{\tilde{B}} \quad \text{(by (2.3))} \\
&= \bar{\alpha} \left\langle \bar{z}z^n, \bar{z}^{p-1} z^q\right\rangle_{\tilde{B}} + \bar{\beta} \left\langle \bar{z}z^n, \bar{z}^{p-1} z^{q+1}\right\rangle_{\tilde{B}},
\end{aligned}$$

by Lemma 2.4 and the induction hypothesis; then

$$\begin{aligned}\left\langle z^{n+1}, \bar{z}^{p-1} z^{q+1}\right\rangle_{\tilde{B}} &= \bar{\alpha}\left\langle \alpha z^n + \beta z^{n+1}, \bar{z}^{p-1} z^q\right\rangle_{\tilde{B}} + \bar{\beta}\left\langle \alpha z^n + \beta z^{n+1}, \bar{z}^{p-1} z^{q+1}\right\rangle_{\tilde{B}} \quad \text{(by 2.4)} \\ &= \alpha(\bar{\alpha}\left\langle z^n, \bar{z}^{p-1} z^q\right\rangle_{M(n)} + \bar{\beta}\left\langle z^n, \bar{z}^{p-1} z^{q+1}\right\rangle_{M(n)}) + \beta(\bar{\alpha}\left\langle z^{n+1}, \bar{z}^{p-1} z^q\right\rangle_{\tilde{B}} + \bar{\beta}\left\langle z^{n+1}, \bar{z}^{p-1} z^{q+1}\right\rangle_{\tilde{B}}) \\ &= \alpha\left\langle z^n, \bar{z}^p z^q\right\rangle_{M(n)} + \beta\left\langle z^{n+1}, \bar{z}^p z^q\right\rangle_{\tilde{B}} = \left\langle \bar{z} z^n, \bar{z}^p z^q\right\rangle_{\tilde{B}} \qquad \text{(by (2.4)).} \ \square\end{aligned}$$

We next establish normality for $\tilde{B}_n$.

Lemma 2.6. *For $i+j=n+1$, $j \geq 1$, $p+q=n$, $p \geq 1$,*

$$\langle \bar{z}^i z^j, \bar{z}^{p-1} z^{q+1}\rangle_{\tilde{B}} = \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^p z^q\rangle_{\tilde{B}}.$$

Proof. The proof is by induction on $i \geq 0$; for $i = 0$, Lemma 2.5 implies

$$\langle z^{n+1}, \bar{z}^{p-1} z^{q+1}\rangle_{\tilde{B}} = \langle \bar{z} z^n, \bar{z}^p z^q\rangle_{\tilde{B}}.$$

For $i \geq 1$, we have

$$\begin{aligned}&\langle \bar{z}^i z^j, \bar{z}^{p-1} z^{q+1}\rangle_{\tilde{B}} \\ &= \alpha\langle \bar{z}^{i-1} z^j, \bar{z}^{p-1} z^{q+1}\rangle_{M(n)} + \beta\langle \bar{z}^{i-1} z^{j+1}, \bar{z}^{p-1} z^{q+1}\rangle_{\tilde{B}} \qquad \text{(by (2.4))} \\ &= \alpha\langle \bar{z}^i z^{j-1}, \bar{z}^p z^q\rangle_{M(n)} + \beta\langle \bar{z}^i z^j, \bar{z}^p z^q\rangle_{\tilde{B}} \qquad \text{(by Theorem 1.5-4) and by induction)} \\ &= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^p z^q\rangle_{\tilde{B}} \qquad \text{(by (2.4)).} \ \square\end{aligned}$$

To establish symmetry for $\tilde{B}_n$ we first show that the relationship between column Z^{n+1} and row Z^n of $\tilde{B}_n$ is compatible with the structure of a moment matrix block $B_{n,n+1}(\gamma)$.

Lemma 2.7. *For $k+\ell = n+1$, $k \geq 1$,*

$$\left\langle \bar{z}^k z^\ell, z^n\right\rangle_{\tilde{B}} = \overline{\langle z^{n+1}, \bar{z}^{k-1} z^\ell\rangle}_{\tilde{B}}.$$

Proof. We use induction on $k \geq 1$. For $k = 1$, the $\alpha = 0$ case is trivial so we assume $\alpha \neq 0$:

$$\begin{aligned}\langle \bar{z} z^n, z^n\rangle_{\tilde{B}} - \overline{\langle z^{n+1}, z^n\rangle}_{\tilde{B}} &= \alpha\langle\ z^n, z^n\rangle_{\tilde{B}} + \beta\left\langle z^{n+1}, z^n\right\rangle_{\tilde{B}} - \overline{\langle z^{n+1}, z^n\rangle}_{\tilde{B}} \\ &= \tfrac{1}{\bar{\alpha}}(|\alpha|^2\gamma_{nn} - \alpha\left\langle z^{n+1}, z^n\right\rangle_{\tilde{B}} - \overline{\alpha\left\langle z^{n+1}, z^n\right\rangle}_{\tilde{B}}) \\ &= \tfrac{1}{\bar{\alpha}}(|\alpha|^2\gamma_{nn} - 2Re(\alpha\left\langle z^{n+1}, z^n\right\rangle_{\tilde{B}}) = 0,\end{aligned}$$

by the definition of $\langle z^{n+1}, z^n\rangle_{\tilde{B}}$. As for the inductive step, consider $\overline{\langle z^{n+1}, \bar{z}^{k-1} z^\ell\rangle}_{\tilde{B}}$ with $k > 1$. By (2.3) and the induction hypothesis, this is equal to

$$\alpha\gamma_{n+k-1,\ell} + \beta\overline{\langle z^{n+1}, \bar{z}^{k-2} z^{\ell+1}\rangle}_{\tilde{B}} = \alpha\overline{\langle z^{n+1}, \bar{z}^{k-2}\ z^\ell\rangle}_{\tilde{B}} + \beta\left\langle \bar{z}^{k-1} z^{\ell+1}, z^n\right\rangle_{\tilde{B}},$$

which in turn is equal to $\alpha\overline{\langle z^n, \bar{z}^{k-1} z^\ell\rangle}_{M(n)} + \beta\left\langle \bar{z}^{k-1} z^{\ell+1}, z^n\right\rangle_{\tilde{B}}$, by Lemma 2.4. Thus,

$$\begin{aligned}\overline{\langle z^{n+1}, \bar{z}^{k-1} z^\ell\rangle}_{\tilde{B}} &= \alpha\overline{\langle z^n, \bar{z}^{k-1}\ z^\ell\rangle}_{M(n)} + \beta\left\langle \bar{z}^{k-1} z^{\ell+1},\ z^n\right\rangle_{\tilde{B}} \\ &= \alpha\langle \bar{z}^{k-1} z^\ell, z^n\rangle_{M(n)} + \beta\langle \bar{z}^{k-1} z^{\ell+1}, z^n\rangle_{\tilde{B}} = \left\langle \bar{z}^k z^\ell, z^n\right\rangle_{\tilde{B}},\end{aligned}$$

(by (2.4)), as desired. $\square$

We next establish symmetry for $\tilde{B}_n$.

Lemma 2.8. *For $i+j=n+1$, $k+\ell=n$,*

$$\langle\overline{\bar{z}^i z^j, \bar{z}^k z^\ell}\rangle_{\tilde{B}} = \langle z^i \bar{z}^j, z^k \bar{z}^\ell\rangle_{\tilde{B}}.$$

Proof.

We give the proof only for $k \geq i$ and leave the other case to the reader.

$$\begin{aligned}\langle\overline{\bar{z}^i z^j, \bar{z}^k z^\ell}\rangle_{\tilde{B}} &= \langle\overline{z^{n+1}, \bar{z}^{k-i} z^{\ell+i}}\rangle_{\tilde{B}} \qquad \text{(by Lemma 2.6)} \\ &= \langle \bar{z}^{k-i+1} z^{\ell+i}, z^n\rangle_{\tilde{B}} \qquad \text{(Lemma 2.7)} \\ &= \langle z^i \bar{z}^j, z^k \bar{z}^\ell\rangle_{\tilde{B}} \qquad \text{(Lemma 2.6).} \ \square\end{aligned}$$

For $i+j=2n+1$, we now define γ_{ij} as follows.

$$0 \leq i \leq n : \gamma_{ij} = \langle z^{n+1}, \bar{z}^{n-i} z^i\rangle_{\tilde{B}}$$
$$n < i \leq 2n+1 : \gamma_{ij} = \langle \bar{z}^{i-n} z^j, z^n\rangle_{\tilde{B}}.$$

It follows readily from normality and symmetry in $\tilde{B}_n$ that $\tilde{B}_n$ is of the form $B_{n,n+1}(\gamma)$. Since $\tilde{B}_n$ satisfies (1.11) and (1.12), to complete the proof of Theorem 2.1 we must show that $C \equiv W^* M(n) W$ is Toeplitz. Let $M := \begin{pmatrix} M(n) & \tilde{B} \\ \tilde{B}^* & C \end{pmatrix}$ denote the unique flat extension of $M(n)$ subordinate to $\tilde{B}$, and let $\langle\cdot,\cdot\rangle_M$ be the associated form. Since $M = M^*$, if p, r, $s \in \mathcal{P}_{n+1}$, with $r(Z,\bar{Z}) = s(Z,\bar{Z})$ in $\mathcal{C}_M$, then $\langle p, r\rangle_M = \langle p, s\rangle_M$. By flatness and Lemma 1.8, the columns of M of order $n+1$ satisfy the relations of (2.4).

Lemma 2.9. *For $i+j=p+q=n+1$, $j \geq 1$, $q \geq 1$,*

$$\langle \bar{z}^i z^j, \bar{z}^p z^q\rangle_M = \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{p+1} z^{q-1}\rangle_M.$$

Proof. We first consider $i=0$, $j=n+1$ and proceed by induction on $p \geq 0$. For $p=0$, the $\alpha=0$ case is trivial so we assume $\alpha \neq 0$.

$$\begin{aligned}&\langle \bar{z}z^n, \bar{z}z^n\rangle_M - \langle z^{n+1}, z^{n+1}\rangle_M \\ &\quad = \langle \alpha z^n + \beta z^{n+1}, \alpha z^n + \beta z^{n+1}\rangle_M - \langle z^{n+1}, z^{n+1}\rangle_M \\ &\quad = |\alpha|^2 \langle z^n, z^n\rangle_M + 2\,\mathrm{Re}(\bar{\alpha}\beta\langle z^{n+1}, z^n\rangle_M) \\ &\quad = |\alpha|^2 \gamma_{nn} - 2\,\mathrm{Re}(\alpha\langle z^{n+1}, z^n\rangle_M) = 0.\end{aligned}$$

For $p \geq 1$,

$$\begin{aligned}&\langle z^{n+1}, \bar{z}^p z^q\rangle_M \\ &\quad = \langle z^{n+1}, (\alpha + \beta z)\bar{z}^{p-1} z^q\rangle_M \\ &\quad = \bar{\alpha}\langle z^{n+1}, \bar{z}^{p-1} z^q\rangle_M + \bar{\beta}\langle z^{n+1}, \bar{z}^{p-1} z^{q+1}\rangle_M \\ &\quad = \bar{\alpha}\langle \bar{z}z^n, \bar{z}^p z^{q-1}\rangle_M + \bar{\beta}\langle \bar{z}z^n, \bar{z}^p z^q\rangle_M \\ &\qquad\quad \text{(by normality outside } C \text{ for the first term and by induction on } p \text{ for the second term)} \\ &\quad = \langle \bar{z}z^n, \bar{z}^{p+1} z^{q-1}\rangle_M.\end{aligned}$$

We now induct on $i \geq 0$. For $i \geq 1$, we use (2.4), normality in M outside C, and induction to obtain

$$\begin{aligned}\langle \bar{z}^i z^j, \bar{z}^p z^q \rangle_M &= \alpha \langle \bar{z}^{i-1} z^j, \bar{z}^p z^q \rangle_M + \beta \langle \bar{z}^{i-1} z^{j+1}, \bar{z}^p z^q \rangle_M \\ &= \alpha \langle \bar{z}^i z^{j-1}, \bar{z}^{p+1} z^{q-1} \rangle_M + \beta \langle \bar{z}^i z^j, \bar{z}^{p+1} z^{q-1} \rangle_M \\ &= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{p+1} z^{q-1} \rangle_M. \ \square\end{aligned}$$

The proof of Theorem 2.1 is now complete.

3. Flat extensions for $Z^k \in \langle \bar{Z}^i Z^j \rangle_{0 \leq i+j \leq k-1}$

In this section we study flat extensions of positive, recursively generated moment matrices $M(n)$ for which there is a relation $Z^k = p(Z, \bar{Z})$ for some $p \in \mathcal{P}_{k-1}$. In the case when $k \leq \left[\frac{n}{2}\right] + 1$, we prove the existence of a unique flat extension $M(n+1)$. For the case $\left[\frac{n}{2}\right] + 1 < k \leq n$, we describe a simple algorithm which can be used to determine the existence of flat extensions in numerical examples.

Theorem 3.1. *Suppose $M(n)$ is positive and recursively generated. If $1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and $Z^k = p(Z, \bar{Z})$ for some $p \in \mathcal{P}_{k-1}$, then $M(n)$ admits a unique flat extension $M(n+1)$.*

Remark 3.2. For n odd, or for n even and $k < \left[\frac{n}{2}\right] + 1$, Theorem 3.1 can be obtained as a consequence of Theorems 1.1 and 1.2, since in each of these cases $M(n)$ is actually flat. Indeed, since $Z^k = p(Z, \bar{Z})$, then $\bar{Z}^k = \bar{p}(Z, \bar{Z})$ [CF4, Lemma 3.10]; thus (RG) implies that for $i + j = n - k$, $\bar{Z}^i Z^{j+k} = (\bar{z}^i z^j p)(Z, \bar{Z})$ and $\bar{Z}^{i+k} Z^j = (\bar{z}^i \bar{p} z^j)(Z, \bar{Z})$. In the indicated cases the preceding relations imply that for $r + s = n$, $\bar{Z}^r Z^s = p_{rs}(Z, \bar{Z})$ for some $p_{rs} \in \mathcal{P}_{n-1}$. The proof of Theorem 3.1 that we present below is independent of Theorems 1.1 and 1.2 and uses a more direct argument.

Example 3.3. *The case when n is even and $k = \left[\frac{n}{2}\right] + 1$ does not follow from Theorems 1.1 and 1.2 since $M(n)$ need not be flat. For example, with $n = 2$, consider*

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \alpha & \bar{\alpha} & \bar{\beta} \\ 0 & 0 & 1 & \beta & \alpha & \bar{\alpha} \\ 0 & \bar{\alpha} & \bar{\beta} & |\alpha|^2 + |\beta|^2 & \bar{\alpha}^2 + \bar{\beta}\alpha & 2\bar{\alpha}\bar{\beta} \\ 1 & \alpha & \bar{\alpha} & \alpha^2 + \beta\bar{\alpha} & |\alpha|^2 + |\beta|^2 & \bar{\alpha}^2 + \bar{\beta}\alpha \\ 0 & \beta & \alpha & 2\alpha\beta & \alpha^2 + \beta\bar{\alpha} & |\alpha|^2 + |\beta|^2 \end{pmatrix}.$$

Note that $Z^2 = \alpha Z + \beta \bar{Z}$ and $\bar{Z}^2 = \bar{\alpha} \bar{Z} + \bar{\beta} Z$.

Since $\{1, Z, \bar{Z}\}$ is independent, $M(2)$ satisfies (RG). Now $M(2) \geq 0 \Leftrightarrow |\beta|^2 \geq 1 + |\alpha|^2$; moreover, if $|\beta|^2 > 1 + |\alpha|^2$, then $M(2)$ is positive and satisfies (RG), but it is *not flat* (since $\operatorname{rank} M(2) = 4 > \operatorname{rank} M(1)$). The existence of a unique flat extension for $M(2)$ follows from Theorem 3.1, whence Theorem 1.2 implies the existence of a unique 4-atomic representing measure, of the form $\mu = \sum_{i=0}^{3} \rho_i \delta_{z_i}$. Using the method of [CF4] and [Fia], we see that the atoms $\{z_i\}_{i=0}^{3}$ are the four distinct roots of

$$z^4 = 2\alpha z^3 + (\beta\bar{\alpha} - \alpha^2)z^2 + \beta(|\beta|^2 - |\alpha|^2)z,$$

and the densities $\{\rho_i\}_{i=0}^{3}$ may be obtained from the Vandermonde equation $V(z_0, \ldots, z_3)(\rho_0, \ldots, \rho_3)^T = (\gamma_{00}, \gamma_{01}, \gamma_{02}, \gamma_{03})^T$.

Proof of Theorem 3.1. Our first goal is to define a block $B \equiv B[n, n+1] \in M_{n+1,n+2}(\mathbf{C})$ to serve as $B_{n,n+1}$ in the extension. Denote p by $p(z, \bar{z}) = \sum_{0 \le r+s \le k-1} a_{rs} \bar{z}^r z^s$. For $i + j = n+1$, we define $p_{ij} \in \mathcal{P}_n$ as follows:

$$\begin{aligned} &i \ge k,\ j < k : p_{ij}(z, \bar{z}) = \bar{z}^{i-k} z^j \bar{p}(z, \bar{z}); \\ &i \ge k,\ j \ge k : p_{ij}(z, \bar{z}) = \bar{z}^{i-k} z^{j-k} |p|^2(z, \bar{z}). \\ &i < k,\ j \ge k : p_{ij}(z, \bar{z}) = \bar{z}^{i} z^{j-k} p(z, \bar{z}); \end{aligned} \tag{3.1}$$

(Note that since $k \le \left[\frac{n}{2}\right] + 1$, either $i \ge k$ or $j \ge k$, so p_{ij} is well-defined for all $i + j = n+1$.) Since $p(Z, \bar{Z}) = Z^k$ in $\mathcal{C}_{M(n)}$, then $\bar{p}(Z, \bar{Z}) = \bar{Z}^k$ in $\mathcal{C}_{M(n)}$ [CF4, Lemma 3.10]; since $M(n)$ satisfies (RG), Lemma 1.7 and (3.1) imply

$$\text{For } i + j = n+1,\ \bar{V}^i V^j = p_{ij}(V, \bar{V}) \text{ in } \mathcal{C}_S. \tag{3.2}$$

(We illustrate the case when n is odd and $i = j = k$. Since $Z^k = p(Z, \bar{Z})$, Lemma 1.7 implies $(\bar{z}^k z^k)(V, \bar{V}) = (\bar{z}^k p)(V, \bar{V})$. Also, since $\bar{p}(Z, \bar{Z}) = \bar{Z}^k$, (RG) implies $(p\bar{p})(Z, \bar{Z}) = (p\bar{z}^k)(Z, \bar{Z})$ in $\mathcal{C}_{M(n)}$; thus $\bar{V}^k V^k = (\bar{z}^k p)(V, \bar{V}) = |p|^2(V, \bar{V})$. The other cases of $i + j = n+1$ are somewhat simpler to analyze, so we omit the details.)

We now define $B \in M_{n+1,n+2}(\mathbf{C})$ by extending (3.2) through rows corresponding to degree n. Denote the columns of

$$\tilde{B} := \begin{pmatrix} B_{0,n+1} \\ \vdots \\ B_{n-1,n+1} \\ B \end{pmatrix}$$

by $\bar{Z}^i Z^j$ $(i + j = n+1)$. We define B implicitly via the relations

$$\bar{Z}^i Z^j = p_{ij}(Z, \bar{Z}) \qquad (i + j = n+1). \tag{3.3}$$

Note that (3.3) uniquely determines the candidate for a flat moment matrix extension $M(n+1)$. Indeed, since $M(n)$ satisfies (RG), the relation $Z^k = p(Z, \bar{Z})$ and Lemma 1.9 imply that (3.3) must hold in any flat moment matrix extension of $M(n)$; since B uniquely determines any flat extension of $M(n)$ containing this block, it follows that there is at most one flat extension $M(n+1)$; our goal is to prove that the flat extension determined by (3.3) is indeed of the form $M(n+1)$.

Since $\deg p \le k-1$, then $\deg p_{ij} \le n$, so $\operatorname{Ran} \tilde{B} \subseteq \mathcal{C}_{M(n)}$, which establishes (1.12). We next establish that B satisfies (1.9) (symmetric property) and (1.10) (normality).

<u>Symmetric property for B</u>. From (3.3) and Lemma 1.6, it suffices to show that for $i + j = n+1$, $p_{ij} = \overline{p_{ji}}$, but this is clear from (3.1).

Normality for B. For $i+j=n+1$, $\ell+m=n$, $m\geq 1$, $j\geq 1$ we must show

$$\langle \bar{z}^i z^j, \bar{z}^\ell z^m\rangle_B = \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_B.$$

We divide the proof into several cases.

Case B1.

$$\begin{aligned} i\geq k,\ j<k: \langle \bar{z}^i z^j, \bar{z}^\ell z^m\rangle_B &= \langle \bar{z}^{i-k} z^j \bar{p}, \bar{z}^\ell z^m\rangle_{M(n)} \qquad \text{(by (3.3))} \\ &= \langle \bar{z}^{i-k+1} z^{j-1} \bar{p}, \bar{z}^{\ell+1} z^{m-1}\rangle_{M(n)} \qquad \text{(Theorem 1.5)} \\ &= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_B \qquad \text{(by (3.3))}. \end{aligned}$$

Case B2. $j\geq k$, $i<k$: (3.3) implies

$$\langle \bar{z}^i z^j, \bar{z}^\ell z^m\rangle_B = \langle \bar{z}^i z^{j-k} p, \bar{z}^\ell z^m\rangle_{M(n)}.$$

Subcase B2a. $i+1<k$. Then $j-1\geq k$ (for otherwise, $n+1=i+j=(i+1)+(j-1)\leq (k-1)+(k-1)\leq n$); thus

$$\begin{aligned} \langle \bar{z}^i z^{j-k} p, \bar{z}^\ell z^m\rangle_{M(n)} &= \langle \bar{z}^{i+1} z^{j-k-1} p, \bar{z}^{\ell+1} z^{m-1}\rangle_{M(n)} \qquad \text{(by Theorem 1.5)} \\ &= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_B \qquad \text{(by (3.3))}. \end{aligned}$$

Subcase B2b. $i+1=k$, $j=k$. (This case occurs only when n is even, i.e., $n=2d$, $k=d+1$, $i=d$, $j=d+1$, $i+j=2d+1=n+1$.) We must show that

$$\langle \bar{z}^{k-1} z^k, \bar{z}^\ell z^m\rangle_B = \langle \bar{z}^k z^{k-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_B,$$

or, equivalently,

$$\langle \bar{z}^{k-1} p, \bar{z}^\ell z^m\rangle_{M(n)} = \langle \bar{p} z^{k-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_{M(n)}. \tag{3.4}$$

Since B is symmetric, it suffices to consider upper diagonals, with $m\geq k-1$ (see the remarks following (1.10)). Now

$$\langle \bar{z}^{k-1} p, \bar{z}^\ell z^m\rangle_{M(n)} = \sum_{0\leq r+s\leq k-1} a_{rs} \langle \bar{z}^{r+k-1} z^s, \bar{z}^\ell z^m\rangle_{M(n)}.$$

We have $r+k-1\leq k-1+k-1=n$ and $s+(r+k-1)\leq 2k-2=n$; moreover, $m\geq k-1\geq s$ and $r+k-1+m\geq r+k-1+k-1=r+n\geq n$. Thus

$$\begin{aligned} \langle \bar{z}^{r+k-1} z^s, \bar{z}^\ell z^m\rangle_{M(n)} &= \langle \bar{z}^{r+s+k-1}, \bar{z}^{\ell+s} z^{m-s}\rangle_{M(n)} \qquad \text{(by Theorem 1.5-4))} \\ &= \langle \bar{z}^n, \bar{z}^{\ell+s} z^{(m-s)-(n-(r+s+k-1))}\rangle_{M(n)} \qquad \text{(by Theorem 1.5-2)-3))}. \end{aligned}$$

Now

$$\begin{aligned} \langle \bar{z}^{k-1} p, \bar{z}^\ell z^m\rangle_{M(n)} &= \sum a_{rs} \langle \bar{z}^n, \bar{z}^{\ell+s} z^{r+k-1+m-n}\rangle_{M(n)} \\ &= \sum a_{rs} \langle \sum \overline{a_{tq}} z^t \bar{z}^{q+n-k}, \bar{z}^{\ell+s} z^{r+k-1+m-n}\rangle_{M(n)} \qquad \text{(by (RG))} \\ &= \sum a_{rs} \sum \overline{a_{tq}} \gamma_{q+r+m-1,t+s+\ell} \\ &= \sum \overline{a_{tq}} \sum a_{rs} \gamma_{q+r+m-1,t+s+\ell}. \end{aligned}$$

Also, since $M(n)$ is recursively generated, for $0 \le t+q \le k-1$, $Z^{t+k-1}\bar{Z}^q = (z^{t-1}\bar{z}^q p)(Z,\bar{Z}) = \sum a_{rs}\bar{Z}^{r+q}Z^{s+t-1}$; thus

$$\begin{aligned}\langle z^{t+k-1}\bar{z}^q, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)} &= \sum a_{rs}\langle \bar{z}^{r+q}z^{s+t-1}, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)} \\ &= \sum a_{rs}\gamma_{r+q+m-1,t+s+\ell}.\end{aligned}$$

Now

$$\begin{aligned}\langle \bar{z}^{k-1}p, \bar{z}^{\ell}z^m\rangle_{M(n)} &= \sum \overline{a_{tq}}\langle z^{t+k-1}\bar{z}^q, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)} \\ &= \langle z^{k-1}\bar{p}, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)},\end{aligned}$$

which establishes (3.4).

Subcase B2c. $i+1=k$, $j-1\ge k$. In $\mathcal{C}_{M(n)}$, $\bar{Z}^k = \bar{p}(Z,\bar{Z})$, so by (RG),

$$(z^{j-k-1}p\bar{z}^k)(Z,\bar{Z}) = (z^{j-k-1}|p|^2)(Z,\bar{Z});$$

thus

$$\begin{aligned}\langle \bar{z}^i z^{j-k}p, \bar{z}^{\ell}z^m\rangle_{M(n)} &= \langle \bar{z}^{i+1}z^{j-k-1}p, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)} \\ &= \langle z^{j-k-1}|p|^2, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)} \\ &= \langle \bar{z}^{i+1}z^{j-1}, \bar{z}^{\ell+1}z^{m-1}\rangle_B.\end{aligned}$$

Case B3. $i\ge k$, $j\ge k$. $\langle \bar{z}^i z^j, \bar{z}^{\ell}z^m\rangle_B = \langle \bar{z}^{i-k}z^{j-k}|p|^2, \bar{z}^{\ell}z^m\rangle_{M(n)}$. If $j-k>0$,

$$\begin{aligned}\langle \bar{z}^{i-k}z^{j-k}|p|^2, \bar{z}^{\ell}z^m\rangle_{M(n)} &= \langle \bar{z}^{i-k+1}z^{j-k-1}|p|^2, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)} \\ &= \langle \bar{z}^{i+1}z^{j-1}, \bar{z}^{\ell+1}z^{m-1}\rangle_B.\end{aligned}$$

Suppose $j=k$; since $Z^k = p(Z,\bar{Z})$, (RG) implies

$$(\bar{z}^{i-k}z^k\bar{p})(Z,\bar{Z}) = (\bar{z}^{i-k}|p|^2)(Z,\bar{Z}) = (\bar{z}^{i-k}z^{j-k}|p|^2)(Z,\bar{Z}).$$

Thus

$$\begin{aligned}\langle \bar{z}^{i-k}z^{j-k}|p|^2, \bar{z}^{\ell}z^m\rangle_{M(n)} &= \langle \bar{z}^{i-k}z^k\bar{p}, \bar{z}^{\ell}z^m\rangle_{M(n)} \\ &= \langle \bar{z}^{i-k+1}z^{k-1}\bar{p}, \bar{z}^{\ell+1}z^{m-1}\rangle_{M(n)} \\ &= \langle \bar{z}^{i+1}z^{j-1}, \bar{z}^{\ell+1}z^{m-1}\rangle_B.\end{aligned}$$

Following the plan outlined in Section 1 ((1.11)–(1.13)), we now define $B_{n,n+1} := B$, $\tilde{B} := (B_{i,n+1})_{0\le i\le n}$, we let

$$M := \begin{pmatrix} M(n) & \tilde{B} \\ \tilde{B}^* & C \end{pmatrix}$$

denote the unique flat extension of $M(n)$ subordinate to $\tilde{B}$, and we let $\langle\cdot,\cdot\rangle_M$ denote the associated form. If $W = (\widehat{p_{0,n+1}}, \widehat{p_{1,n}}, \ldots, \widehat{p_{n,1}}, \widehat{p_{n+1,0}})$, then $\tilde{B} = M(n)W$ and thus $C = W^*M(n)W$. Since (1.11) and (1.12) hold, to complete the proof it suffices to verify that C is constant on upper diagonals ((1.13)). By flatness and Lemma 1.8, the columns of M are defined by (3.3). Note that by the moment-matrix block structures of $M(n)$, $\tilde{B}$, and $\tilde{B}^*$, M satisfies the following properties which do not involve block C:

$$\langle \bar{z}^i z^j, \bar{z}^k z^{\ell}\rangle_M = \gamma_{i+\ell,j+k} \qquad \begin{aligned}&\text{for } 0\le i+j\le n,\ 0\le k+\ell\le n+1 \\ &\text{and for } 0\le i+j\le n+1,\ 0\le k+\ell\le n;\end{aligned} \tag{3.5}$$

$$(3.6)\qquad \begin{aligned} \langle pz, q\rangle_M &= \langle p, \bar{z}q\rangle_M \quad \text{and} \\ \langle p\bar{z}, q\rangle_M &= \langle p, zq\rangle_M \quad \text{for } p, q \in \mathcal{P}_n; \end{aligned}$$

$$(3.7)\qquad \langle pz, qz\rangle_M = \langle p\bar{z}, q\bar{z}\rangle_M \qquad \begin{aligned} &\text{for } p \in \mathcal{P}_{n-1},\ q \in \mathcal{P}_n \\ &\text{and for } p \in \mathcal{P}_n,\ q \in \mathcal{P}_{n-1}. \end{aligned}$$

We further note the following property of M:

$$(3.8)\qquad \begin{gathered} \text{If } p,\ q,\ pq \in \mathcal{P}_n \text{ and } p(Z, \bar{Z}) = 0 \text{ in } \mathcal{C}_M, \\ \text{then } (pq)(Z, \bar{Z}) \text{ in } \mathcal{C}_M. \end{gathered}$$

Indeed, since $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$, (RG) implies $(pq)(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(n)}$; since M is a positive extension of $M(n)$, [Fia] implies $(pq)(Z, \bar{Z})$ in $\mathcal{C}_M$.

To prove that C is constant on upper diagonals we must verify

$$(3.9)\qquad \begin{gathered} \text{For } i + j = \ell + m = n + 1,\ m \geq j \geq 1,\ i \geq \ell, \\ \langle \bar{z}^i z^j, \bar{z}^\ell z^m\rangle_M = \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_M. \end{gathered}$$

Case C1. $i \geq k$, $j < k$.

$$\begin{aligned} \langle \bar{z}^i z^j, \bar{z}^\ell z^m\rangle_M &= \langle \bar{z}^{i-k} z^j \bar{p}, \bar{z}^\ell z^m\rangle_M \\ &= \sum \overline{a_{rs}} \langle \bar{z}^{i-k+s} z^{j+r}, \bar{z}^\ell z^m\rangle_M \\ &= \sum \overline{a_{rs}} \langle \bar{z}^{i-k+s+1} z^{j+r-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_M \\ &\quad \text{(by (3.7), since } i - k + s + j + r - 1 \\ &\quad = (i + j) + (s + r) - k - 1 \\ &\quad \leq n + 1 + k - 1 - k - 1 = n - 1) \\ &= \langle \bar{z}^{i+1-k} z^{j-1} \bar{p}, \bar{z}^{\ell+1} z^{m-1}\rangle_M \\ &= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^m\rangle_M. \end{aligned}$$

Case C2. $i < k$, $j \geq k$.

Subcase C2a. $i < k - 1$. As in Case B2a, $j - 1 \geq k$, so

$$\begin{aligned} \langle \bar{z}^i z^j, \bar{z}^\ell z^m\rangle_M &= \langle \bar{z}^i z^{j-k} p, \bar{z}^\ell z^m\rangle_M \\ &= \sum a_{rs} \langle \bar{z}^{i+r} z^{j-k+s}, \bar{z}^\ell z^m\rangle_M \\ &= \sum a_{rs} \langle \bar{z}^{i+r+1} z^{j-k+s-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_M \\ &\quad \text{(by (3.7), since } i + r + j - k + s - 1 \leq n - 1) \\ &= \langle \bar{z}^{i+1} z^{j-k-1} p, \bar{z}^{\ell+1} z^{m-1}\rangle_M \\ &= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^{m-1}\rangle_M. \end{aligned}$$

Subcase C2b. $i+1=k$, $j=k$. (n even). Note that $m \geq j = k$.

$$\begin{aligned}
\langle \bar{z}^i z^j, \bar{z}^\ell z^m \rangle_M &= \langle \bar{z}^{k-1} p, \bar{z}^\ell z^m \rangle_M \\
&= \sum a_{rs} \langle \bar{z}^{r+k-1} z^s, \bar{z}^\ell z^m \rangle_M \\
&= \sum a_{rs} \langle \bar{z}^{r+s+k-1}, \bar{z}^{\ell+s} z^{m-s} \rangle_M \\
&\quad \text{(by (3.7), since } m-s \geq m-(k-1) > m-k \geq 0 \\
&\quad \text{and } r+s+k-1 \leq 2k-2 = n) \\
&= \sum a_{rs} \langle \bar{z}^n, \bar{z}^{\ell+s} z^{m-(n-(r+k-1))} \rangle_M \qquad \text{(by (3.6), since} \\
&\quad n-(r+k-1) \geq s \text{ and since } m \geq k \Rightarrow r+k-1+m \geq 2k-1 > n) \\
&= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^{m-1} \rangle_M
\end{aligned}$$

(exactly as in the proof of Case B2b, but replacing $M(n)$ by M; note only that if $r+s,\ t+q \leq k-1$, then $r+q+s+t-1 \leq n-1$, so (3.5) implies $\langle \bar{z}^{r+q} z^{s+t-1}, \bar{z}^{\ell+1} z^{m-1} \rangle_M = \gamma_{r+q+m-1,t+s+\ell}$).

Subcase C2c. $i+1=k$, $j-1 \geq k$. In $\mathcal{C}_M$ we have $\bar{Z}^k = \bar{p}(Z,\bar{Z})$ and, from (3.8), $(z^{j-k-1} p \bar{z}^k)(Z,\bar{Z}) = (z^{j-k-1}|p|^2)(Z,\bar{Z})$. Thus

$$\begin{aligned}
\langle \bar{z}^i z^j, \bar{z}^\ell z^m \rangle_M &= \langle \bar{z}^i z^{j-k} p, \bar{z}^\ell z^m \rangle_M \\
&= \langle \bar{z}^{i+1} z^{j-k-1} p, \bar{z}^{\ell+1} z^{m-1} \rangle_M \qquad \text{(by (3.7))} \\
&= \langle \bar{z}^{i+1} z^{j-1}, \bar{z}^{\ell+1} z^{m-1} \rangle_M .
\end{aligned}$$

Case C3. $i \geq k, j \geq k$. The proof is identical to that of Case B3 (replacing $M(n)$ by M and using (3.7)).

The proof of Theorem 3.1 is now complete. □

We conclude this section by considering the case when $Z^k = p(Z,\bar{Z})$ for $p \in \mathcal{P}_{k-1}$ and $\left[\frac{n}{2}\right]+1 < k \leq n$. Let $p(z,\bar{z}) \equiv \sum_{0 \leq i+j \leq k-1} a_{ij} \bar{z}^i z^j$. In constructing a flat extension $M(n+1)$, $B_{n,n+1}$ is uniquely determined by the relation $Z^{n+1} = \sum a_{ij} Z^{n+1-k+j} \bar{Z}^i$; indeed this relation uniquely determines $\gamma_{n,n+1}, \ldots, \gamma_{0,2n+1}$, and thus also $\gamma_{n+1,n} = \overline{\gamma_{n,n+1}}$, $\ldots$, $\gamma_{2n+1,0} = \overline{\gamma_{0,2n+1}}$. If the resulting block $\tilde{B} \equiv (B_{i,n+1})_{0 \leq i \leq n}$ satisfies $\operatorname{Ran} \tilde{B} \not\subseteq \mathcal{C}_{M(n)}$, then there is no flat extension $M(n+1)$. If $\operatorname{Ran} \tilde{B} \subseteq \mathcal{C}_{M(n)}$, let W be such that $\tilde{B} = M(n)W$; then $M(n)$ admits a flat extension $M(n+1)$ if and only if $W^* M(n) W$ is Toeplitz.

4. A Positive Invertible Moment Matrix Admitting No Representing Measure

Using results from algebraic geometry, D. Hilbert established in [Hil] the existence of a polynomial $q \in \mathbf{R}[x,y]$ of total degree 6 which is nonnegative on the real plane $\mathbf{R}^2$, but which cannot be expressed as a finite sum of squares of polynomials; an explicit such polynomial was later found by K. Schmüdgen [Sch1]. (Another concrete example is given in [BCJ].) We will use Schmüdgen's example to construct data $\gamma \equiv \{\gamma_{ij}\}_{0 \leq i+j \leq 6}$ whose associated moment matrix $M(3)(\gamma)$ is positive and invertible but admits no representing measure; in particular, $M(3)(\gamma)$ does not have a flat extension $M(4)$. This will disprove Conjecture 1.1 in [CF4], since invertible moment matrices satisfy property (RG) vacuously. We begin by recalling Schmüdgen's result. Let $\mathbf{C}[x,y]$ denote the polynomials in Hermitian variables x and

y with complex coefficients ($x = \bar{x}$, $y = \bar{y}$). Let $\mathcal{P}$ denote the cone in $\mathbf{C}[x, y]$ consisting of "sums of squares" $\sum \bar{p}_i p_i$. Let $\mathcal{C}_+ = \{p \in \mathbf{C}[x, y] : p(x, y) \geq 0$ for all real x, $y\}$. A linear functional $F : \mathbf{C}[x, y] \to \mathbf{C}$ is *positive* if $F|\mathcal{P}$ is non-negative; F is *strongly positive* if $F|\mathcal{C}_+$ is non-negative; F has a positive Borel representing measure if and only if F is strongly positive [ShT], [Sch1].

Theorem 4.1. ([Sch1, Theorem]) *(1) The polynomial*

$$q(x, y) := 200(x^3 - 4x)^2 + 200(y^3 - 4y)^2 + (y^2 - x^2)x(x + 2)[x(x - 2) + 2(y^2 - 4)]$$

is nonnegative on $\mathbf{R}^2$*, but cannot be written as a sum of squares.*

(2) There exists a positive linear functional F *on* $\mathbf{C}[x, y]$ *with* $F(q) < 0$*. Thus,* F *cannot be represented as integration with respect to a positive Borel measure with support in* $\mathbf{R}^2$*.*

F is defined first on the space $\mathbf{C}_6[x, y]$, the complex polynomials of total degree at most 6, as a linear combination of evaluation functionals (and is then extended to all of $\mathbf{C}[x, y]$):

$$F(p) := 32 \sum_{i=1}^{8} p(A_i) + p(B_1) + p(B_2) - p(A_9) \qquad (p \in \mathbf{C}_6[x, y]),$$

where

$$\begin{array}{llll} A_1 := (-2, -2), & A_2 := (0, -2), & A_3 := (2, -2), & A_4 := (-2, 0), \\ A_5 := (0, 0), & A_6 := (-2, 2), & A_7 := (0, 2), & A_8 := (2, 2), \\ A_9 := (2, 0), & B_1 := (\frac{1}{100}, 0), & B_2 := (0, \frac{1}{100}). & \end{array}$$

We define $\gamma_{k\ell} := F((x - iy)^k (x + iy)^\ell)$ $(0 \leq k + \ell \leq 6)$. Observe that

$$\gamma_{k\ell} = \sum_{r=0}^{k} \sum_{s=0}^{\ell} (-1)^{k-r} \, i^{k+\ell-r-s} \binom{k}{r} \binom{\ell}{s} F(x^{r+s} y^{k+\ell-r-s}),$$

and that

$$F(x^r y^s) = \begin{cases} 257 & r = 0 \text{ and } s = 0 \\ 96[(-2)^s + 2^s] + 10^{-2s} & r = 0 \text{ and } s \geq 1 \\ 32[3(-2)^r + 2^{r+1}] + 10^{-2r} - 2^r & r \geq 1 \text{ and } s = 0 \\ [1 + (-1)^r][1 + (-1)^s] 2^{r+s+5} & r \geq 1 \text{ and } s \geq 1 \end{cases}.$$

The associated matrix $M(3)$ is built using the following values:

$$\begin{array}{llll} \gamma_{00} = 257 & & & \\ \gamma_{01} = 10^{-2}(1{-}6599i) & & & \\ \gamma_{02} = 132 & \gamma_{11} = \frac{7020001}{5000} & & \\ \gamma_{03} = 10^{-6}(1{+}263999999i) & \gamma_{12} = \bar{\gamma}_{03} & & \\ \gamma_{04} = \frac{333599999999}{50000000} & \gamma_{13} = 528 & \gamma_{22} = \frac{485600000001}{50000000} & \\ \gamma_{05} = 10^{-10}(1{-}10559999999999i) & \gamma_{14} = \bar{\gamma}_{05} & \gamma_{23} = \gamma_{05} & \\ \gamma_{06} = 2112 & \gamma_{15} = -\frac{29727999999999999}{500000000000} & \gamma_{24} = \gamma_{06} & \gamma_{33} = \frac{35808000000000001}{500000000000} \end{array}$$

A straightforward calculation using the Nested Determinant Test now shows that $M(3) \geq 0$ and that $\det M(3) > 0$. Since the presence of a representing measure for γ would immediately give a corresponding measure for $F|\mathbf{C}_6[x,y]$, it follows from Theorem 4.1(2) that $M(3)$ cannot admit a representing measure.

In view of the preceding example we modify [CF4, Conjecture 1.1] as follows.

Conjecture 4.2. *The following are equivalent for a truncated moment sequence* $\gamma \equiv \gamma^{(2n)}$*:*

(i) γ *has a representing measure;*

(ii) γ *has a representing measure with moments of all orders;*

(iii) γ *has a compactly supported representing measure;*

(iv) γ *has a finitely atomic representing measure;*

(v) γ *has a* rank $M(n)$*-atomic representing measure;*

(vi) $M(n) \geq 0$ *admits a flat extension* $M(n+1)$.

Added in Proof. In recent work [CF5], we have adapted results of V. Tchakaloff [Tch] and I.P. Mysovskikh [Mys] to prove (i) $\Rightarrow$ (iv) in Conjecture 4.2; thus, conditions (i), (ii), (iii) and (iv) are all equivalent. Independently, M. Putinar [P5] has found a different proof of (i) $\Rightarrow$(iv), also based on extending results of [Tch]. (Somewhat earlier, we had obtained (iii) $\Rightarrow$(iv) by adapting [Tch], and J. McCarthy had communicated to us another proof of the same implication, using convexity theory.)

Theorem 1.2 shows that (v) and (vi) of Conjecture 4.2 are equivalent, and clearly (v) $\Rightarrow$(iv); however, J. McCarthy [McC], in response to Conjecture 4.2, has recently proved that there exist truncated moment sequences γ having representing measures, but such that $M(n)(\gamma)$ does *not* have a flat extension $M(n+1)$. Thus (i) $\not\Rightarrow$(v) and Conjecture 4.2 is false as stated. McCarthy's dimension-theoretic result actually shows that moment sequences γ admitting no flat extensions are in a sense generic: among moment sequences γ with representing measures, those with rank $M(n)(\gamma)$-atomic representing measures are rare. On the other hand, it follows from the equivalence of (i) and (iv) and from the equivalence of (v) and (vi) that a truncated moment sequence γ has a representing measure if and only if for some $k \geq 0$, $M(n)(\gamma)$ admits a positive extension $M(n+k)$ which in turn has a flat extension $M(n+k+1)$.

In [CF5] we continue to study concrete necessary or sufficient conditions for the existence of flat extensions. In particular, we exhibit several examples of positive, recursively generated moment matrices which do not admit representing measures and which are much easier to construct and analyze than the example of Theorem 4.1.

References

[AK] N.I. Ahiezer and M.G. Krein, Some Questions in the Theory of Moments, Transl. Math. Monographs 2(1962), Amer. Math. Soc., Providence.

[Akh] N.I. Akhiezer, The Classical Moment Problem, Hafner Publ. Co., New York, 1965.

[Atz] A. Atzmon, A moment problem for positive measures on the unit disc, Pacific J. Math. 59(1975), 317-325.

[Ber] C. Berg, The multidimensional moment problem and semigroups, Moments in Mathematics, Proc. Symposia Appl. Math. 37(1987), 110-124.

[BCJ] C. Berg, J.P.R. Christensen and C.U. Jensen, A remark on the multidimensional moment problem, Math. Ann. 223(1979), 163-169.

[BM] C. Berg and P.H. Maserick, Polynomially positive definite sequences, Math. Ann. 259(1982), 487-495.

[Cas] G. Cassier, Problème des moments sur un compact de $\mathbf{R}^n$ et décomposition des polynômes à plusieurs variables, J. Funct. Anal. 58(1984), 254-266.

[CF1] R. Curto and L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, Integral Equations and Operator Theory 17(1993), 202-246.

[CF2] R. Curto and L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, II, Integral Equations and Operator Theory 18(1994).

[CF3] R. Curto and L. Fialkow, Recursiveness, positivity, and truncated moment problems, Houston J. Math. 17(1991), 603-635.

[CF4] R. Curto and L. Fialkow, Solution of the truncated complex moment problem for flat data, Memoirs Amer. Math. Soc., to appear.

[CF5] R. Curto and L. Fialkow, Flat extensions of positive moment matrices, II: Recursively generated relations, preprint 1996.

[CP] R. Curto and M. Putinar, Nearly subnormal operators and moment problems, J. Funct. Anal. 115(1993), 480-497.

[Fia] L. Fialkow, Positivity, extensions and the truncated complex moment problem, Contemporary Math. 185(1995), 133-150.

[Fug] B. Fuglede, The multidimensional moment problem, Expo. Math. 1(1983), 47-65.

[Hau] F. Hausdorff, Momentprobleme für ein endliches Intervall, Math. Zeit. 16(1923), 220-248.

[Hav1] E.K. Haviland, On the momentum problem for distributions in more than one dimension, Amer. J. Math. 57(1935), 562-568.

[Hav2] E.K. Haviland, On the momentum problem for distributions in more than one dimension, Part II, Amer. J. Math. 58(1936), 164-168.

[Hil] D. Hilbert, Über die Darstellung definiter Formen als Summen von Formenquadraten, Math. Ann. 32 (1888), 342–350.

[KrN] M.G. Krein and A. Nudel'man, The Markov Moment Problem and Extremal Problems, Transl. Math. Monographs 50(1977), Amer. Math. Soc., Providence.

[Lan] H. Landau, Classical background of the moment problem, Moments in Mathematics, Proc. Symposia Appl. Math. 37(1987), 1-15.

[McC] J. McCarthy, private communication.

[McG] J.L. McGregor, Solvability criteria for certain N-dimensional moment problems, J. Approx. Theory 30(1980), 315-333.

[Mys] I.P. Mysovskikh, On Chakalov's Theorem, USSR Comp. Math. 15(1975), 221-227.

[P1] M. Putinar, A two-dimensional moment problem, J. Funct. Anal. 80(1988), 1-8.

[P2] M. Putinar, The L problem of moments in two dimensions, J. Funct. Anal. 94(1990), 288-307.

[P3] M. Putinar, Positive polynomials on compact semi-algebraic sets, Indiana Univ. Math. J. 42(1993), 969-984.

[P4] M. Putinar, Extremal solutions of the two-dimensional L-problem of moments, preprint 1994.

[P5] M. Putinar, On Tchakaloff's Theorem, preprint 1995.

[Rez] Reznick, e-mail communication.

[Sar] D. Sarason, Moment problems and operators on Hilbert space, Moments in Mathematics, Proc. Symposia Appl. Math. 37(1987), 54-70.

[Sch1] K. Schmüdgen, An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional, Math. Nachr. 88(1979), 385-390.

[Sch2] K. Schmüdgen, The K-moment problem for semi-algebraic sets, Math. Ann. 289(1991), 203-206.

[ShT] J. Shohat and J. Tamarkin,The Problem of Moments, Math. Surveys I, Amer. Math. Soc., Providence, 1943.

[Smu] J.L. Smul'jan, An operator Hellinger integral (Russian), Mat. Sb. 91(1959), 381-430.

[Sto] J. Stochel, private correspondence, 1994.

[StSz] J. Stochel and F. Szafraniec, Algebraic operators and moments on algebraic sets, Portugaliae Math. 51(1994), 1-21.

[Sza] F. Szafraniec, Moments on compact sets, in "Prediction Theory and Harmonic Analysis," V. Mandrekar and H. Salehi, eds., North-Holland, Amsterdam, 1983; pp. 379-385.

[Tch] V. Tchakaloff, Formules de cubatures mécaniques à coefficients non négatifs, Bull. Sc. Math. 81(1957), 123-134.

[Wol] Wolfram Research, Inc., *Mathematica*, Version 2.1, Wolfram Research, Inc., Champaign, Illinois, 1992.

Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242

Department of Mathematics
and Computer Science
SUNY at New Paltz
New Paltz, NY 12561