THE TRUNCATED K-MOMENT PROBLEM: A SURVEY

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ABSTRACT. Let K denote a nonempty closed subset of $\mathbb{R}^n$ and let $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\le m}$, $\beta_0 > 0$, denote a real n-dimensional multisequence of finite degree m. The Truncated K-Moment Problem (TKMP) concerns the existence of a positive Borel measure μ, supported in K, such that

$$\beta_i = \int_{\mathbb{R}^n} x^i d\mu \ (i \in \mathbb{Z}_+^n,\ |i| \le m).$$

In this survey we describe a number of interrelated techniques for establishing the existence of such *K-representing measures*. We discuss representing measures arising from *K-positivity* of the Riesz functional associated with β; measures arising from flat extensions of positive moment matrices; connections between TKMP and the classical *Full Multivariable K-Moment Problem*; Tchakaloff's Theorem and its generalizations and applications to TKMP; and connections between TKMP and optimization problems, via semidefinite programming.

1. Introduction

Let $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\le m}$, $\beta_0 > 0$, denote a real n-dimensional multisequence of finite degree m, and let K denote a closed subset of $\mathbb{R}^n$. The *Truncated K-Moment Problem* for β (TKMP) concerns the existence of a positive Borel measure μ, supported in K, such that

$$\beta_i = \int_K x^i d\mu \ (i \in \mathbb{Z}_+^n,\ |i| \le m). \tag{1.1}$$

(Here, for $x \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $i \equiv (i_1, \ldots, i_n) \in \mathbb{Z}_+^n$, we set $|i| = i_1 + \cdots + i_n$ and $x^i = x_1^{i_1} \cdots x_n^{i_n}$.) A measure μ as in (1.1) is a *K-representing measure* for β; for $K = \mathbb{R}^n$, we refer to TKMP simply as the *Truncated Moment Problem* (TMP) and to μ as a *representing measure*. In the sequel, we usually assume m is even, $m \equiv 2d$, so the moment data completely define a *moment matrix* M_d, as described below. By a *concrete solution* to the truncated K-moment problem we mean a set of necessary and sufficient conditions for K-representing measures that can be effectively applied in numerical examples. For a given K, concrete solutions, valid for all $d \ge 1$, are known only in a few cases. These include, for $n = 1$, $K = \mathbb{R}$, $[0, +\infty)$, and $[a, b]$ (cf. [6]), and for $n = 2$, when K is a curve $p(x, y) = 0$ with $deg\ p \le 2$ (cf. [11] [13] [15]), and for certain curves of higher degree [23]. Concrete solutions are also known for

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sequences whose moment matrices have special features, e.g., in the case of *flat data* (cf. Section 3).

TKMP is motivated in part by the much-studied *K-Moment Problem* (KMP) (or *Full K-Moment Problem*) for $\beta^{(\infty)} \equiv \{\beta_i\}_{i\in\mathbb{Z}_+^n}$ (cf. [1] [2] [5] [34] [43] [50] [51] [52] [57]). A result of J. Stochel [55] shows that a solution to TKMP actually implies a solution to KMP: $\beta^{(\infty)}$ has a K-representing measure if and only if $\beta^{(m)}$ has a K-representing measure for every $m \geq 1$ (cf. Section 4). Additional motivation for TKMP comes from its diverse applications, e.g., to subnormal operator theory, polynomial optimization and positive polynomials, control theory and Nevanlinna-Pick problems, and signal processing, among others. Some of these applications are described in several recent surveys and monographs (cf. [32] [36] [37]). The classical Riesz-Haviland Theorem solves KMP in terms of K-positivity of the Riesz functional associated to $\beta^{(\infty)}$ (cf. Section 2). In the sequel we discuss the extent to which K-positivity can be adapted to TKMP (Sections 2, 5, and 6). In many cases it is difficult to directly verify K-positivity in TKMP, so we discuss an alternate approach based on moment matrix extensions (Section 3). Several other recent approaches, based on semidefinite programming [31] [29], or on algebraic geometry and convex analysis [4], are largely beyond the scope of this survey, although we mention these directions briefly and include some relevent references. The paper is organized as follows:
Section 2: Representing measures from K-positivity
Section 3: Representing measures from flat extensions of positive moment matrices
Section 4: From the truncated moment problem to the full moment problem
Section 5: Tchakaloff's Theorem: generalizations and applications
Section 6: Strict K-positivity, the core variety, and positive definite moment matrices
Section 7: TKMP and optimization methods.

2. Representing measures from K-positivity.

In this section we discuss the role of K-positive Riesz functionals in moment problems. Let $\mathbb{R}[x] \equiv \mathbb{R}[x_1,\ldots,x_n]$ and for $0 < m < +\infty$, let $\mathcal{P}_m := \{p \in \mathbb{R}[x],\ deg\ p \leq m\}$. For $\beta \equiv \beta^{(m)}$, the *Riesz functional* $L_\beta : \mathcal{P}_m \longrightarrow \mathbb{R}$ is defined by

$$L_\beta(\sum_{i\in\mathbb{Z}_+^n,|i|\leq m} a_i x^i) = \sum a_i\beta_i.$$

If μ is a K-representing measure for β and $p \in \mathcal{P}_m$ satisfies $p|_K \geq 0$, then $L_\beta(p) = \int_K p\ d\mu \geq 0$, so in this sense L_β is *K-positive*; for $K = \mathbb{R}^n$, we say simply that L_β is *positive*. In the Full K-Moment Problem for $\beta \equiv \beta^{(\infty)}$, a classical theorem of M. Riesz ($n = 1$) [46] and E.K. Haviland ($n > 1$) [30] provides a solution to KMP expressed in terms of K-positivity of the associated functional $L_\beta : \mathbb{R}[x] \longrightarrow \mathbb{R}$.

Theorem 2.1. *(Riesz-Haviland Theorem) $\beta \equiv \beta^{(\infty)}$ has a K-representing measure if and only if the corresponding functional L_β is K-positive, i.e., for $p \in \mathbb{R}[x_1,\ldots,x_n]$, if $p|_K \geq 0$, then $L_\beta(p) \geq 0$.*

In the sequel we will be concerned with the extent to which Theorem 2.1 admits an analogue for TKMP. Our immediate concern, however, is to illustrate how Theorem 2.1 can be used in some cases to obtain "concrete" solutions to KMP, i.e., solutions expressed in terms of positivity of matrices closely associated with $\beta^{(\infty)}$. For $p \equiv \sum_{i\in\mathbb{Z}_+^n} a_i x^i \in \mathbb{R}[x]$, let $\widehat{p} \equiv (a_i)$ denote the coefficient vector of p relative to the basis $\mathcal{B}$ of monomials in $\mathbb{R}[x]$ in degree-lexicographic order. Following [7] [10] [14], we associate to $\beta \equiv \beta^{(\infty)}$ the *moment matrix* $M_\infty \equiv M_\infty(\beta)$, with rows and columns X^i indexed by the elements x^i of $\mathcal{B}$. The entry in row X^i, column X^j of M_∞ is β_{i+j} $(i, j \in \mathbb{Z}_+^n)$, so M_∞ is a real symmetric matrix characterized by $\langle M_\infty \widehat{p}, \widehat{q}\rangle = L_\beta(pq)$ $(p, q \in \mathbb{R}[x])$. If L_β is positive (in particular, if β has a representing measure), then $\langle M_\infty \widehat{p}, \widehat{p}\rangle = L_\beta(p^2) \geq 0$, and since M_∞ is real symmetric, it follows that M_∞ is positive semidefinite ($M_\infty \succeq 0$).

For a first application of Riesz-Haviland, consider the classical theorem of Hamburger for $K = \mathbb{R}$ (cf. [2]):

Theorem 2.2. *(Hamburger's Theorem) Let $n = 1$. $\beta \equiv \beta^{(\infty)}$ has a representing measure supported in $\mathbb{R}$ if and only if $M_\infty \succeq 0$.*

Proof. Suppose $M_\infty \succeq 0$. It is well-known that if $p \in \mathbb{R}[x]$ satisfies $p|_{\mathbb{R}} \geq 0$, then $p = u^2 + v^2$ for certain polynomials u and v [40]. Thus $L_\beta(p) = L_\beta(u^2) + L_\beta(v^2) = \langle M_\infty \widehat{u}, \widehat{u}\rangle + \langle M_\infty \widehat{v}, \widehat{v}\rangle \geq 0$, whence L_β is positive. The existence of a representing measure now follows from Theorem 2.1. $\square$

We next consider the theorem of Stieltjes for $K = [0, +\infty)$ (cf. [2]), but for this we need additional notation. Given $\beta \equiv \beta^{(\infty)}$, set $M \equiv M_\infty(\beta)$. For $p \in \mathbb{R}[x]$, we define the *localizing matrix* $M^{(p)}$, with rows and columns X^i indexed by the monomials in $\mathcal{B}$, as follows:

$$\langle M^{(p)} \widehat{f}, \widehat{g}\rangle = L_\beta(pfg) \ (f, g \in \mathbb{R}[x]).$$

Note that for $p \equiv 1$, $M^{(p)} = M$. If β has a representing measure μ supported in $\mathcal{S}_p := \{x \in \mathbb{R}^n : p(x) \geq 0\}$, then for $f \in \mathbb{R}[x]$,

$$\langle M^{(p)} \widehat{f}, \widehat{f}\rangle = L_\beta(pf^2) = \int_{\mathcal{S}_p} pf^2 \, d\mu \geq 0,$$

so $M^{(p)} \succeq 0$.

Theorem 2.3. *(Stieltjes' Theorem) Let $n = 1$ and $K = [0, +\infty)$. $\beta \equiv \beta^{(\infty)}$ has a K-representing measure if and only if $M \equiv M_\infty(\beta) \succeq 0$ and $M^{(x)} \succeq 0$.*

Proof. Since the necessity of the conditions is clear, we focus on sufficiency, and we suppose that $M \succeq 0$ and $M^{(x)} \succeq 0$. It is known that if p is a polynomial satisfying $p|_{[0,+\infty)} \geq 0$, then there exist polynomials $r,\ s\ , u,\ v$ such that $p = r^2 + s^s + x(u^2 + v^2)$ [40]. Now $L_\beta(p) = L_\beta(r^2) + L_\beta(s^2) + L_\beta(xu^2) + L_\beta(xv^2) = \langle M_\infty \widehat{r}, \widehat{r}\rangle + \langle M_\infty \widehat{s}, \widehat{s}\rangle + \langle M^{(x)} \widehat{u}, \widehat{u}\rangle + \langle M^{(x)} \widehat{v}, \widehat{v}\rangle \geq 0$. Thus L_β is K-positive, so the result follows from Theorem 2.1. $\square$

Our next application of Riesz-Haviland concerns the theorem of K. Schmüdgen [50] for the case when K is a compact basic semialgebraic subset of $\mathbb{R}^n$. Let $P \equiv \{p_1, \dots, p_k\} \subset \mathbb{R}[x]$ and consider the basic closed semialgebraic set $K_P \equiv \bigcap_{i=1}^{k} \mathcal{S}_{p_i}$. Let $\mathcal{I} = \{0,1\}^k$ and for $I \equiv (i_1, \dots, i_k) \in \mathcal{I}$, let $P_I := p_1^{i_1} \cdots p_k^{i_k}$. The following result of Schmüdgen provides a concrete solution to KMP in the case when K_P is compact.

Theorem 2.4. *(Schmüdgen [50]) Assume that $K \equiv K_P$ is compact. Then $\beta \equiv \beta^{(\infty)}$ has a K-representing measure if and only if the localizing matrix $M^{(P_I)}$ is positive semidefinite for each $I \in \mathcal{I}$.*

The original proof of Theorem 2.4 in [50] did not employ Riesz-Haviland. To prove Schmüdgen's Theorem via Riesz-Haviland, we require the following characterization of the polynomials that are strictly positive on K_P.

Theorem 2.5. *(Schmüdgen's Positivstellensatz [50]) Assume K_P is compact. If $p|_{K_P}$ is strictly positive, then*

$$p = \sum_{I \in \mathcal{I}} P_I (\sum_{j=1}^{j_I} (f_j^{(I)})^2),$$

where each $f_j^{(I)}$ is a polynomial, i.e., p is a weighted sum of squares with weights P_I.

Proof of Theorem 2.4. The necessity of the condition is clear. For sufficiency, suppose that a polynomial p is strictly positive on K_P. From Theorem 2.5 we have

$$L_\beta(p) = \sum_{I \in \mathcal{I}} L_\beta(P_I (\sum_j (f_j^{(I)})^2) = \sum_{I \in \mathcal{I}} \sum_j \langle M^{(P_I)} \widehat{f_j^{(I)}}, \widehat{f_j^{(I)}} \rangle \geq 0.$$

For $p|_K \geq 0$, we may apply the preceding to $p + \epsilon$ $(\epsilon > 0)$ to conclude that L_β is K-positive, whence Riesz-Haviland implies the existence of a K-representing measure. $\square$

Although Theorem 2.4 is concrete, it does entail 2^k positivity conditions. A result of M. Putinar [41] reduces the number of positivity conditions to just k under a mild additional hypothesis. Consider the *quadratic module* associated with P: $Q_P := \{s_0 + \sum_{i=1}^{k} s_i p_i\}$, where each s_i is a sum of squares of polynomials.

Theorem 2.6. *(Putinar [41]) Suppose K_P is a compact basic semialgebraic set such that*

$$\mathcal{S}_q \textit{ is compact for some } q \in Q_P. \tag{2.1}$$

Then each polynomial p that is strictly positive on K_P belongs to Q_P, i.e., p admits a representation $p = s_0 + \sum_{i=1}^{k} s_i p_i$ for certain sums of squares $s_0, \dots, s_k$.

Note that (2.1) holds, in particular, if some $\mathcal{S}_{p_i}$ is compact. Further, since K_P is compact, there exists N such that $||x||^2 < N$ for all $x \in K_P$. If we add to the presentation of K_P the polynomial $p_{k+1} := N - ||x||^2$, then, as a set, K_P is unchanged, but (2.1) now clearly holds. In view of Theorem 2.6, it is clear that if (2.1) holds, then the positivity of M and of the localizing matrices $M^{(p_i)}$ $(1 \le i \le k)$ are sufficient to imply the existence of a K_P representing measure via Riesz-Haviland.

We next begin to examine the extent to which the techniques related to Theorem 2.1 can be adapted to TKMP, and for this we require some additional notation. Following [7] [10] [14], we associate to $\beta \equiv \beta^{(2d)}$ the *moment matrix* $M_d \equiv M_d(\beta)$, with rows and columns X^i indexed by the monomials in $\mathcal{P}_d$ in degree-lexicographic order. The entry in row X^i, column X^j of M_d is β_{i+j} $(i, j \in \mathcal{Z}_+^n,\ |i|, |j| \le d)$, so M_d is a real symmetric matrix characterized by $\langle M_d \widehat{p}, \widehat{q} \rangle = L_\beta(pq)$ $(p, q \in \mathcal{P}_d)$. If L_β is positive, then $\langle M_d \widehat{p}, \widehat{p} \rangle = L_\beta(p^2) \ge 0$, so M_d is positive semidefinite $(M_d \succeq 0)$.

For $p(x) \equiv \sum a_i x^i \in \mathcal{P}_d$, we define an element $p(X)$ of *Col* M_d, the column space of M_d, by $p(X) := \sum a_i X^i$, and a calculation shows that $p(X) = M_d \widehat{p}$. We also set $\mathcal{Z}_p := \{x \in \mathbb{R}^n : \ p(x) = 0\}$.

Proposition 2.7. *([7, Prop. 3.1]) If $\beta \equiv \beta^{(2d)}$ admits a representing measure μ, then for $p \in \mathcal{P}_d$, supp $\mu \subseteq \mathcal{Z}_p \iff p(X) = 0$.*

Proof. $p(X) = 0 \iff M_d \widehat{p} = 0 \iff \langle M_d \widehat{p}, \widehat{p} \rangle = 0$ (since $M_d \succeq 0$) $\iff L_\beta(p^2) = 0 \iff \int p^2 d\mu = 0 \iff p|_{supp\ \mu} \equiv 0$. $\square$

Suppose μ is a representing measure for $\beta \equiv \beta^{(2d)}$ and $p(X) = 0$, so that $p|_{supp\ \mu} \equiv 0$. If q is a polynomial such that *deg* p+*deg* $q \le d$, then since $(pq)|_{supp\ \mu} \equiv 0$, it follows from Proposition 2.7 that $(pq)(X) = 0$ in *Col* M_d. We say that M_d is *recursively generated* if whenever $p, q, pq \in \mathcal{P}_d$ and $p(X) = 0$, then $(pq)(X) = 0$. The preceding shows that positivity and recursiveness of M_d are *necessary conditions* for representing measures.

Now let $n = 1$, $K = \mathbb{R}$, $d = 2$, and let $M_2(\beta)$ be given by

$$\begin{pmatrix} a & a & a \\ a & a & a \\ a & a & b \end{pmatrix} \quad (a < b) \tag{2.2}$$

[16, Example 2.1]. Since $M_2 \succeq 0$, and since each polynomial that is nonnegative on $\mathbb{R}$ is a sum of squares of polynomials, it follows exactly as in the proof of Hamburger's Theorem that L_β is positive. However, since M_2 is not recursively generated ($X = 1$, but $X^2 \ne X$), we see that there is no representing measure. Thus, the most direct analogue of Riesz-Haviland for TKMP is not valid: even in a case where the proof of K-positivity procedes exactly as in KMP (via sums of squares), the conclusion that a representing measure exists may fail.

In the preceding example we were able to apply sums of squares methods to establish K-positivity. However, in many truncated moment problems, even this step may be difficult to carry out. Let $n = 2$ and $\beta \equiv \beta^{(6)}$. Consider the case $K \equiv \overline{\mathbb{D}} = \mathcal{S}_{1-x^2-y^2}$, the closed unit disk in the plane. To establish K-positivity of

L_β, we consider $p \in \mathcal{P}_6$ satisfying $p|_K > 0$. According to Theorem 2.5, there exists a representation of p as

$$p = \sum f_i^2 + (1 - x^2 - y^2) \sum g_i^2, \tag{2.3}$$

where the f_i and g_i are polynomials. If we were to mimic the proof of Theorem 2.4, we would then compute $L_\beta(p) = \sum L_\beta(f_i^2) + \sum L_\beta((1-x^2-y^2)g_i^2)$. However, a result of C. Scheiderer [48] shows that for $d \geq 3$ and $N > 2d$, there exists $p \equiv p_{d,N} \in \mathcal{P}_{2d}$ such that $p|_K > 0$, but some f_i^2 or some g_i^2 has degree greater than N, so $L_\beta(f_i^2)$ or $L_\beta((1 - x^2 - y^2)g_i^2)$ is undefined. This lack of *degree-bounded* representations is a fundamental obstacle to applying the K-positvity techniques of KMP to TKMP.

Despite the preceding difficulties in adapting Riesz-Haviland to TKMP, there is an appropriate analogue, as we next describe.

Theorem 2.8. *(Truncated Riesz-Haviland Theorem [16]) Let $\beta \equiv \beta^{(2d)}$ or $\beta \equiv \beta^{(2d-1)}$. β has a K-representing measure if and only if β admits an extension to a sequence $\widetilde{\beta} \equiv \widetilde{\beta}^{(2d+2)}$ such that $L_{\widetilde{\beta}}$ is K-positive.*

Theorem 2.8 is not, by itself, a concrete solution to TKMP because, as discussed above, it may be very difficult to verify K-positivity. Several authors have addressed this issue from a variety of viewpoints. For K compact and semi-algebraic, Helton and Nie [31] developed an approach to K-positivity of L_β based on semidefinite programming. Extending this approach, the author and Nie showed in [29] that L_β is positive if and only if the optimal values for an infinite sequence of semidefinite programming problems associated to β are all nonnegative (see also [39]). In another direction, Vasilescu [59] has studied TMP using techniques from function spaces and C*-algebras. Recently, G. Blekherman [4] used methods of algebraic geometry and convex analysis to prove a remarkably general result, part of which we paraphrase as follows.

Theorem 2.9. *(cf. Blekherman [4]) If $d \geq 3$ and rank $M_d \leq 3d - 3$, then L_β is positive.*

Blekherman's result is actually stated in terms of existence of representing measures in the *Homogeneous Truncated Moment Problem* (HTMP) (cf. Theorem 6.13 below). By combining Theorem 2.9 with Theorem 2.8, we have the following concrete sufficient condition for representing measures.

Corollary 2.10. *If $d \geq 3$ and rank $M_d \leq 3d - 3$, then $\beta^{(2d-1)}$ has a representing measure.*

Finally, we note that although, in general, there is a gap between K-positivity and the existence of K-representing measures, this gap vanishes in the sense of approximation, as the following result shows.

Theorem 2.11. *(cf. [27]) Let $\beta \equiv \beta^{(m)}$. L_β is K-positive if and only if β (viewed as an element of $\mathbb{R}^{\dim\, \mathcal{P}_m}$) is the limit of multisequences $\beta^{(m)}[k]$ ($k \geq 1$) each of which has a K-representing measure μ_k. In this case, $\beta_i = \lim_{k\to\infty} \int x^i d\mu_k$ ($|i| \leq m$).*

3. Representing measures from flat extensions of positive moment matrices.

The results of Section 2 for TKMP that are based on K-positivity have two disadvantages. One is the difficulty in verifying K-positivity that we have previously discussed. The other is that the K-positivity results, based on convex analysis, yield the existence of representing measures, but provide no means for constructing them. In the present section we discuss some results which permit the construction of finitely atomic representing measures without recourse to K-positivity. Let $V \equiv V(M_d)$ denote the *algebraic variety* corresponding to M_d, i.e., $V = \bigcap_{p\in\mathcal{P}_d, p(X)=0} \mathcal{Z}_p$ (where $\mathcal{Z}_p := \{x \in \mathbb{R}^n : p(x) = 0\}$). It follows from Proposition 2.7 that that if μ is a representing measure for β, then $supp\ \mu \subseteq V(M_d)$, whence

$$r \equiv rank\ M_d \leq card\ supp\ \mu \leq v \equiv card\ V(M_d)\ (cf.\ [7, Cor.\ 3.7]). \tag{3.1}$$

We refer to the necessary condition $rank\ M_d \leq card\ V(M_d)$ as the *variety condition*. In the sequel we will cite the following basic existence theorem of [7] [14] for a "minimal" representing measure, a representing measure μ satisfying $card\ supp\ \mu = rank\ M_d$.

Theorem 3.1. *(cf. Flat Extension Theorem* [14, Theorems 1.1-1.2]*) $\beta \equiv \beta^{(2d)}$ has a rank M_d-atomic representing measure if and only if $M_d \succeq 0$ and M_d admits a flat moment matrix extension, i.e., a moment matrix extension M_{d+1} satisfying $rank\ M_{d+1} = rank\ M_d$. In this case, $\beta^{(2d+2)}$ admits a unique representing measure, $\mu \equiv \mu_{M_{d+1}}$, satisfying $supp\ \mu = V(M_{d+1})$ and $card\ supp\ \mu = rank\ M_d$. Further, M_{d+1} admits unique successive positive moment matrix extensions $M_{d+2}, M_{d+3}, \ldots$, and these are flat extensions.*

Note that for the case of *flat data* ($M_d \succeq 0$ and $rank\ M_d = rank\ M_{d-1}$), Theorem 3.1 (applied to M_{d-1}) implies the existence of a unique ($rank\ M_d$-atomic) representing measure for $\beta^{(2d)}$.

Suppose M_d is positive and admits a flat extension M_{d+1}. The unique representing measure for M_{d+1} referred to in Theorem 3.1 may be explicitly computed as follows (cf. [14, Theorem 1.2]). Let $r = rank\ M_d$, so that $card\ V(M_{d+1}) = r$ and $V(M_{d+1}) \equiv \{w_i\}_{i=1}^r$. Let $\mathcal{B} \equiv \{X^{i_1}, \ldots, X^{i_r}\}$ denote a basis for $Col\ M_d$, and consider the Vandermonde-type matrix

$$W \equiv W_{\mathcal{B}} := \begin{pmatrix} w_1^{i_1} & \ldots & w_r^{i_1} \\ \vdots & & \vdots \\ w_1^{i_r} & \ldots & w_r^{i_r} \end{pmatrix}. \tag{3.2}$$

Then W is invertible, and [14] shows that $\beta^{(2d+2)}$ has the unique representing measure $\mu \equiv \mu_{M_{d+1}}$, of the form $\mu = \sum_{i=1}^{r} \rho_i \delta_{w_i}$, where δ_{w_i} denotes the atomic measure with $supp\ \delta_{w_i} = \{w_i\}$, and $\rho \equiv (\rho_1, \ldots, \rho_r)$ is determined by $\rho^T = W^{-1}(\beta_{i_1}, \ldots, \beta_{i_r})^T$ (here and in the sequel, $\cdot^T$ denotes matrix transpose); in particular, μ is independent of basis $\mathcal{B}$. Of course, μ is also a representing measure for β.

We next recall some properties of positive moment matrix extensions that we will refer to in the sequel. As we discuss in Section 4, a result of Bayer and Teichmann [3] implies that if $\beta \equiv \beta^{(2d)}$ admits a K-representing measure, then β admits a finitely atomic K-representing measure ν. Since ν has convergent power moments of all orders, it follows that $M_d \equiv M_d[\nu]$ admits successive positive, recursively generated moment matrix extensions, namely, $M_{d+1}[\nu]$, $M_{d+2}[\nu], \ldots$. This leads to the following solution to TMP, expressed in terms of moment matrix extensions.

Theorem 3.2. *(Moment Matrix Extension Theorem* [14, Corollary 1.4]*) $\beta^{(2d)}$ has a representing measure if and only if there is an integer $k \geq 0$ such that M_d admits a positive moment matrix extension M_{d+k} which in turn admits a flat extension M_{d+k+1}.*

Upper bounds for k, derived from Theorems 5.1 and 5.3 (below) are given in [14], but these are not very useful in practice. In the application that we discuss below (Theorem 3.3), we have $0 \leq k \leq 1$. In Section 7 we will discuss an analogue of Theorem 3.2 for TKMP in the case when K is a basic closed semialgebraic set. Theorem 3.2 was proved for finitely atomic representing measures in [14]; the preceding formulation for general representing measures comes by combining the results of [14] and [3]. Theorem 3.2 is not, by itself, a concrete solution to the truncated moment problem, but it does provide a framework for obtaining concrete solutions in certain cases. We will describe several such cases below, but we require some preliminaries concerning positive moment matrices.

Consider a real symmetric block matrix $\widetilde{M} \equiv \begin{pmatrix} M & B \\ B^T & C \end{pmatrix}$. A result of Smul'jan [53] implies that $\widetilde{M} \succeq 0$ if and only if $M \succeq 0$, there exists a matrix W such that $B = MW$ (equivalently, $Ran\ B \subseteq Ran\ M$ [19]), and $C \succeq C^\flat \equiv W^T MW$ (note that $C^\flat$ is independent of W satisfying $B = MW$). In this case, the matrix $M^\flat \equiv [M; B] := \begin{pmatrix} M & B \\ B^T & C^\flat \end{pmatrix}$ is a positive *flat extension* of M, i.e., $M^\flat \succeq 0$ and $rank\ M^\flat = rank\ M$. Consider a moment matrix extension

$$M_{d+1} \equiv \begin{pmatrix} M_d & B_{d+1} \\ B_{d+1}^T & C_{d+1} \end{pmatrix}.$$

If $M_d \succeq 0$, then M_{d+1} is a flat (hence positive) extension of M_d if and only if $B_{d+1} = M_d W$ (for some W) and $C_{d+1} = C^\flat \equiv W^T M_d W$; equivalently, $M_{d+1} = [M_d; B_{d+1}]$.

Suppose $M_{d+1} \succeq 0$ and let $p \in \mathcal{P}_d$; [21] implies that if $p(X) = 0$ in $Col\ M_d$, then $p(X) = 0$ in $Col\ M_{d+1}$, i.e., column dependence relations in M_d extend to M_{d+1}. It follows that

$$M_{d+1} \succeq 0 \Longrightarrow V(M_{d+1}) \subseteq V(M_d). \tag{3.3}$$

In the sequel we will also require the following basic property of positive moment matrices:

$$M_{d+1} \succeq 0 \Longrightarrow M_d \textit{ is recursively generated } [7, \textit{Theorem } 3.14]. \tag{3.4}$$

Finally, for the planar case ($n = 2$), consider the block matrix decomposition $M_d \equiv (M[i,j])_{0 \le i,j \le d}$, where $M[i,j]$ is the matrix with $i+1$ rows and $j+1$ columns of the form

$$
M[i,j] \equiv \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & & \cdots & \beta_{i-1,j+1} \\ \beta_{i+j-2,2} & & & \cdots & \beta_{i-2,j+2} \\ \vdots & & & \cdots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \beta_{j-2,i+2} & \cdots & \beta_{0,i+j} \end{pmatrix}. \tag{3.5}
$$

Note that $M[i,j]$ has all of the moments in $\beta^{(2d)}$ of degree $i+j$ and has the Hankel-like property of being constant on cross-diagonals; in particular, in the extension M_{d+1}, block $C_{d+1} \equiv M[d+1,d+1]$ is a Hankel matrix. Further, blocks $B[d,d+1]$ and $B[d+1,d+1]$ are completely determined by the entries in columns X^{d+1} and Y^{d+1}.

To illustrate the moment matrix extension approach to TMP, we will discuss the following result of [15].

Theorem 3.3. *Let $n = 2$ and let $p(x,y)$ be a polynomial satisfying deg $p \le 2$. For $d \ge$ deg p and $\beta \equiv \beta^{(2d)}$, the following are equivalent:*
i) β has a representing measure supported in $\mathcal{Z}_p$;
ii) M_d has the column relation $p(X,Y) = 0$ and admits a positive, recursively generated extension M_{d+1};
iii) (concrete condition) M_d is positive semidefinite, recursively generated, $p(X,Y) = 0$ in Col M_d, and rank $M_d \le$ card $V(M_d)$;

For p as in Theorem 3.3 and $K = \mathcal{Z}_p$, the solution of the Full K-Moment Problem is due to J. Stochel [54]; we will discuss the connection between Theorem 3.3 and Stochel's result in the next section. The conditions in Theorem 3.3-iii) are concrete in the sense that they can be checked using elementary linear algebra or, in the case of the variety condition, checked using computer algebra software. Condition ii) is attractively simple, but without the aid of iii) it may be difficult to determine whether the desired extension exists. We note also that Theorem 3.3 does not extend to bivariate curves of degree 3; a detailed analysis in [23] of TKMP for the case when K is the curve $y = x^3$ shows that the conditions in iii) do not always imply the existence of a K-representing measure (cf. Example 5.9 below). Before sketching the proof of Theorem 3.3, we illustrate it with an example from [15].

Example 3.4. For $n = 2$, consider $\beta \equiv \beta^{(4)}$ given by

$$
M_2(\beta) = \begin{pmatrix} 1 & 1 & 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 9 \\ 2 & 4 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 9 & 0 & 0 & 28 \end{pmatrix}.
$$

M_2 is positive with $rank\ M_2 = 5$. The variety $V(M_2)$ is determined by the column relation $XY = 0$ and coincides with the degenerate hyperbola $xy = 0$. M_2 thus satisfies the conditions of Theorem 3.3. In any positive, recursively generated extension M_3, we have $X^2Y = XY^2 = 0$, so B_3 must be of the form

$$B_3 = \begin{pmatrix} 4 & 0 & 0 & 9 \\ 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 28 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

A calculation shows that $Ran\ B_3 \subseteq Ran\ M_2$, but $C^\flat$ in $[M_2; B_3]$ is *not* Hankel, e.g. $C^\flat_{41} = 1$ but $C^\flat_{32} = 0$. Thus M_2 has no flat extension M_3.

We claim that there exists a $rank$ 6 extension M_3 which admits a flat extension M_4. Actually, there are infinitely many such extensions, each dependent on arbitrary choices for p and q. For a definite example, set $p = 18$, $q = 84$. A calculation shows that $C^\flat_{11} = 42$. We now consider block C_3 (for an extension M_3) of the form

$$C_3 = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v \end{pmatrix}, \ u > 42.$$

With $u = 43$, a calculation shows that M_3 is positive and recursively generated, with $rank\ M_3 = 6$, if and only if $v = 263$. In the resulting M_3 there is a column relation $Y^3 = -5 \cdot 1 + 7X + 11Y - X^3$. Thus, in any recursively generated extension M_4 of M_3, we must have $Y^4 = -5Y + 11Y^2$, so Y^4 is uniquely determined. In B_4 we must also have $X^3Y = X^2Y^2 = XY^3 = 0$. Moment matrix structure now determines all of the elements of X^4 in B_4 except $r \equiv \beta_{70}$. A calculation shows that $r := 81$ is the unique value of r such that X^4 belongs to $Ran\ M_3$. With this value, we have $Ran\ B_4 \subseteq Ran\ M_3$, and a further calculation shows that $[M_3; B_4]$ is a flat moment matrix extension of M_3. From Theorem 3.1, this yields a 6-atomic representing measure μ for β, which may be computed as described following Theorem 3.1. The variety of M_4, which provides the support of μ, is the set of common zeros of $xy = 0$, $y^3 = -5 + 7x + 11y - x^3$, $y^4 = -5y + 11y^2$, and a new relation, $x^4 = -5x + 7x^2$: $(x_1, y_1) \approx (2.16601, 0)$, $(x_2, y_2) \approx (0.782816, 0)$, $(x_3, y_3) \approx (-2.94883, 0)$, $(x_4, y_4) \approx (0, 0.463604)$, $(x_5, y_5) \approx (0, 3.06043)$, $(x_6, y_6) \approx (0, -3.52404)$. The corresponding densities may be computed as described following (3.2): $\rho_1 \approx 0.303081$, $\rho_2 \approx 0.203329$, $\rho_3 \approx 0.00359018$, $\rho_4 \approx 0.0821253$, $\rho_5 \approx 0.316218$, $\rho_6 \approx 0.00165656$. $\square$

Proof of Theorem 3.3. We sketch a proof of Theorem 3.3 and in doing so we close a gap in the proof given in [15]. In [11], R.E. Curto and the author showed that the existence of representing measures in the *Truncated Complex Moment Problem* (TCMP) is stable under invertible degree one mappings, and also that TCMP is equivalent to TMP. As discussed in [25], in TMP, invertible degree one mappings $\tau : \mathbb{R}^2 \mapsto \mathbb{R}^2$ are of the form $\tau(x, y) = (a + \alpha x + \gamma y, b + \delta x + \lambda y)$, with $\alpha\lambda - \gamma\delta \neq 0$.

It is well-known that under such a mapping, the degree 2 curve $p(x, y) = 0$ may be transformed into one of the following nine basic varieties: $x^2 + y^2 = 1$, $y = x^2$, $xy = 1$, $xy = 0$, $x^2 = 1$, $x^2 = 0$, $x^2 = -1$, $x^2 + y^2 = 0$, and $x^2 + y^2 = -1$ (cf. [47, p. 405]). In a series of papers, we proved Theorem 3.3 for each of the first four varieties, corresponding to a circle [11], parabola [13], hyperbola [15], or degenerate hyperbola (intersecting lines) [15]. For each of these cases, we proved iii) implies i) by explicitly constructing either a flat extension M_{d+1}, or, in certain cases of $xy = 0$, a positive extension M_{d+1} followed by a flat extension M_{d+2} (as in Example 3.4). A careful examination of these proofs also shows that ii) implies i).

In [25] we noted that the preceding analysis is incomplete, since it does not consider all nine of the varieties. However, in [25] we showed that if M_d is positive, then the column relations $X^2 = -1$ and $X^2 + Y^2 = -1$ cannot occur. Further, we showed in [25] that if $M_d \succeq 0$ and either $X^2 = 0$ or $X^2 + Y^2 = 0$, then $rank\ M_2 \leq 3$. In this case, if $rank\ M_1 = 3$, then it follows from [14] that $rank\ M_d = rank\ M_1 = 3$ and that M_d has a unique representing measure. In the subcase when $rank\ M_1 < 3$, it follows from [8] that M_d has a measure if and only if it is recursively generated. Thus, to complete the proof of Theorem 3.3, it remains to consider the variety $x^2 = 1$, corresponding to two parallel lines. For this variety, we proved the equivalence of i) and iii) in [25], but we did not consider condition ii). In the sequel, we will therefore prove the equivalence of i) and ii) for $p(x, y) := x^2 - 1$.

Assume i) holds and suppose β has a representing measure μ supported in $\mathcal{Z}_p$. Then Proposition 2.7 implies that $p(X, Y) = 0$ in $Col\ M_d$. Also, since β has a representing measure, then Corollary 5.4 (below) implies that β has a finitely atomic representing measure ν. Since ν has finite moments of degree $2d + 2$, $M_{d+1}[\nu]$ is a positive, recursively generated extension of M_d, whence ii) holds.

For the converse, assume that M_{d+1} is a positive and recursively generated extension of M_d and $p(X, Y) = 0$ in $Col\ M_d$. Thus, in $Col\ M_{d+1}$, for each $q(x, y)$ with $deg\ q \leq d - 1$, we have

$$(x^2 q)(X, Y) = q(X, Y). \tag{3.6}$$

In particular, (3.6) recursively determines all columns of M_{d+1} except those of the form XY^i ($1 \leq i \leq d$) or Y^i ($1 \leq i \leq d + 1$). We consider several cases for the column structure of M_d. If M_d is *p-pure*, i.e., the column relations in M_d are precisely those derived from $p(X, Y) = 0$ via recursiveness, then it follows from [25] that M_d has a flat extension M_{d+1}, whence M_{d+1} has a representing measure by Theorem 3.1. Next, we consider the case when M_d is *recursively determinate*, i.e., in addition to $X^2 = 1$, there is a column relation of the form $Y^j = q(X, Y)$, where $0 \leq j \leq d$, $deg\ q(x, y) \leq j$, and q has no y^j term. Since we also have $X^i = 1$ with $i = 2$, we have $i + j - 2 = j \leq d$, so it follows from [17, Corollary 2.4] that M_d has a unique flat extension M_{d+1}, and thus β has a representing measure.

In the remaining case, there is a minimal i, $1 \leq i \leq d - 1$, such that M_d has a column relation of the form

$$XY^i = a_0 1 + b_1 X + a_1 Y + \cdots + b_j XY^{j-1} + a_j Y^j + \cdots + b_i XY^{i-1} + a_i Y^i. \tag{3.7}$$

Thus, by recursiveness in M_{d+1}, we see that XY^d is a linear combination of columns of M_d. The same is obviously also true for $X^{2+i}Y^j \equiv X^iY^j$ for $i+j=d-1$. Since $M_{d+1} \succeq 0$, by the definition of $[M_d; B_{d+1}]$, dependence relations in the columns of $\begin{pmatrix} M_d & B_{d+1} \end{pmatrix}$ extend to the columns of $\begin{pmatrix} B_{d+1}^T & C^\flat \end{pmatrix}$. It follows that the first $d+1$ columns of block C_{d+1} in M_{d+1} coincide with the corresponding columns of $C^\flat$ in $[M_d; B_{d+1}]$. Since both C_{d+1} and $C^\flat$ are positive (in particular, real symmetric), it follows that $C^\flat$ and C_{d+1} agree entrywise, except possibly in the lower right-hand corner (the moment for y^{2d+2}). Thus $C^\flat$ has the form of a moment matrix block $\widetilde{C}_{d+1}$, so $[M_d; B_{d+1}]$ is a flat moment matrix extension of M_d. The existence of a representing measure for β now follows from Theorem 3.1. $\square$

In the preceding result, we used moment matrix extensions to solve TKMP in the case where K is a planar curve of degree 1 or 2. Moment matrix extensions can sometimes be used to solve the truncated moment problem based not on a predefined K, but rather on the type of structures present in the moment matrix. Theorem 3.1 is such a result. We conclude this section with another illustration of this approach. For $n=2$, we say that M_d is *recursively determinate* if there exists a column relation $X^i = p(X,Y)$ with $i \le d$ and $deg\ p < i$, and also a relation $Y^j = q(X,Y)$ with $deg\ q \le j \le d$, where q has no y^j term (or with the roles of p and q interchanged). In this case, the only possible recursively generated moment matrix extension M_{d+1} is completely determined by setting $X^{d+1} := (x^{d+1-i}p)(X,Y)$ and $Y^{d+1} := (y^{d+1-j}q)(X,Y)$, first in $Col\ M_d$ to define B_{d+1}, then, if possible, in $Col\ B_{d+1}^T$, to define C_{d+1}. If this construction leads to a well-defined moment matrix M_{d+1}, then M_{d+1} is said to be a *recursively determined* extension of M_d. In this case, we can attempt to repeat this procedure so as to define M_{d+2}, and so on. This leads to the following algorithmic solution to the moment problem for recursively determinate moment matrices.

Theorem 3.5. *([17]) Let $n=2$ and suppose M_d is recursively determinate. $\beta \equiv \beta^{(2d)}$ has a representing measure if and only if M_d admits recursively determined positive, recursively generated moment matrix extensions $M_{d+1}, \dots, M_{2d-1}$. In this case, at least one of these extensions is a flat extension. Moreover, there exist cases in which the first flat extension is M_{2d-1}.*

Note that in Theorem 3.5, although there is no concrete positivity test directly related to β, there is a finite procedure for determining whether or not a representing measure exists, and, if so, for computing such a measure. In the worst case, however, it may require $d-1$ extension steps to resolve the existence of a measure.

4. From the truncated moment problem to the full moment problem

In this section we discuss the following result of J. Stochel [55], which provides an essential link between TKMP and the Full K-Moment Problem.

Theorem 4.1. *(Stochel [55]) $\beta \equiv \beta^{(\infty)}$ has a K-representing measure if and only if $\beta^{(m)}$ has a K-representing measure for each $m \ge 1$.*

Stochel's result provides a framework for solving the Full K-Moment Problem without explicitly using Riesz-Haviland, thereby circumventing the structure theory of K-positive polynomials. In this section, we illustrate this approach in several examples. We begin with a second proof of Hamburger's Theorem (cf. Theorem 2.2), concerning moment problems on the real line. This will be based on the following solution of TKMP for $K = \mathbb{R}$.

Theorem 4.2. *(*[6, Theorem 3.9]*) Let $n = 1$ and $K \equiv \mathbb{R}$. $\beta \equiv \beta^{(2d)}$ has a representing measure if and only if M_d is positive and recursively generated.*

Alternate proof of Hamburger's Theorem. Suppose $\beta \equiv \beta^{(\infty)}$ has a representing measure μ, and let $M \equiv M_\infty(\beta)$. For each $p \in \mathbb{R}[x]$, we have $\langle M\widehat{p}, \widehat{p}\rangle = L_\beta(p^2) = \int p^2 d\mu(x) \geq 0$, so the necessity of the condition follows. For sufficiency, suppose $M \succeq 0$. For each $d \geq 1$, $M_{d+1}(\beta) \succeq 0$, so it follows from (3.4) that M_d is positive and recursively generated. Theorem 4.2 now implies that $\beta^{(2d)}$ has a representing measure, so the result follows from Theorem 4.1. $\square$

We next consider the theorem of Stieltjes for the half-line $[0, \infty)$ (Theorem 2.3 above). For $n = 1$ and $\beta \equiv \beta^{(\infty)}$, let $M = M_\infty(\beta)$ and let J $(= M^{(x)})$ denote the "shifted" Hankel matrix $(\beta_{1+i+j})_{i,j\geq 0}$. Further, for $d \geq 1$, let $J_{d-1} \equiv (\beta_{1+i+j})_{0\leq i,j\leq d-1}$ and let $v_d = (\beta_{d+1}, \ldots, \beta_{2d})^T$. We recall the following solution to TKMP for $K = [0, +\infty)$.

Theorem 4.3. *(cf.* [6, Theorem 5.3]*) Let $n = 1$ and $\beta \equiv \beta^{(2d)}$. β has a representing measure supported in $K \equiv [0, +\infty)$ if and only if $M_d \succeq 0$, $J_{d-1} \succeq 0$, and $v_d \in$ Ran J_{d-1}.*

Alternate proof of Stieltjes' Theorem. Let $K = [0, +\infty)$ and suppose μ is a K-representing measure for $\beta \equiv \beta^{(\infty)}$. It follows as in the preceding proof that $M \succeq 0$. Moreover, for $p \in \mathbb{R}[x]$, $\langle J\widehat{p}, \widehat{p}\rangle = L_\beta(xp^2) = \int xp^2 d\mu(x) \geq 0$, so $J \succeq 0$. For the converse, suppose M and J are positive semidefinite. Let $d \geq 2$. Since $M \succeq 0$, then $M_d \succeq 0$. Since $J \succeq 0$, then $J_d \succeq 0$, so the discussion of positive block matrices in Section 3 implies that $J_{d-1} \succeq 0$ and that $v_d \in$ *Ran* J_{d-1}. It thus follows from Theorem 4.3 that $\beta^{(2d)}$ has a K-representing measure. Since $d \geq 2$ is arbitrary, Theorem 4.1 implies that $\beta^{(\infty)}$ has a representing measure supported in K. $\square$

For our final illustration of Theorem 4.1, let $p(x, y)$ denote a bivariate polynomial with *deg* $p(x, y) \leq 2$. In [54], Stochel solved the Full K-Moment Problem for $K \equiv \mathcal{Z}_p$; we may paraphrase Stochel's result as follows.

Theorem 4.4. *(Stochel) Let $n = 2$ and suppose deg $p(x, y) \leq 2$. $\beta \equiv \beta^{(\infty)}$ has a representing measure supported in $\mathcal{Z}_p$ if and only if $M \equiv M_\infty(\beta) \succeq 0$ and $p(X, Y) = 0$ in Col M.*

Proof. Let μ denote a representing measure supported in $K \equiv \mathcal{Z}_p$. Clearly, $M \succeq 0$. For each $q \in \mathbb{R}[x]$, since *supp* $\mu \subseteq \mathcal{Z}_p$, we have $\langle p(X, Y), \widehat{q}\rangle = L_\beta(pq) = \int pq\, d\mu = 0$. Since q is arbitrary, it follows that $p(X, Y) = 0$ in *Col* M. For the converse, suppose $M \succeq 0$ and $p(X, Y) = 0$. For $d \geq 2$, since $M_{d+2}(\beta) \succeq 0$, then (3.4) implies that $M_{d+1}(\beta)$ is a positive and recursively generated moment matrix extension of $M_d(\beta)$. Since we also have $p(X, Y) = 0$ in *Col* M_d, then it follows from Theorem 3.3 that $\beta^{(2d)}$ has a K-representing measure. Since $d \geq 2$ is arbitrary, it now follows from Theorem 4.1 that β has a K-representing measure. $\square$

Following Stochel [54], we say that a polynomial $p(x,y)$ is *type A* if the conditions $M_\infty(\beta) \succeq 0$ and $p(X,Y)=0$ in $Col\ M$ imply that β has a representing measure (supported in $\mathcal{Z}_p$). Theorem 4.4 shows that if $deg\ p \le 2$, then p is type A, and in [54] Stochel also proved that not every degree 3 polynomial is type A. In [56] Stochel and Szafraniec proved that there exist type A polynomials of arbitrarily large degree.

5. Tchakaloff's Theorem and TKMP

In this section we discuss Tchakaloff's Theorem and its generalizations, including the Bayer-Teichmann Theorem, and how these results relate to TKMP. Let μ denote a positive Borel measure and let $K = supp\ \mu$. Suppose μ has convergent moments up to at least degree m, i.e., $\int x^i d\mu$ is convergent for all i with $|i| \le m$. A *cubature rule* for μ of *degree m* is a *finitely atomic* K-representing measure for the sequence $\beta^{(m)}$ defined by $\beta_i := \int x^i d\mu$ $(|i| \le m)$. In [58], V. Tchakaloff used convex analysis to establish the following fundamental existence theorem for cubature rules.

Theorem 5.1. *(V. Tchakaloff* [58]*) Let K denote a compact subset of $\mathbb{R}^n$ with positive n-dimensional Lebesgue measure. Let μ denote the restriction of Lebesgue measure on $\mathbb{R}^n$ to K, and let m be a positive integer. There exist finitely many points in K, $w_1, \ldots, w_N$ ($N \le dim\ \mathcal{P}_m$), and positive weights $\alpha_1, \ldots, \alpha_N$, such that for each $p \in \mathcal{P}_m$, $L(p) := \int_K p(x) d\mu(x) = \sum_{i=1}^{N} \alpha_i p(w_i)$.*

A careful examination of [58] reveals that the role of μ is simply to establish that $L : \mathcal{P}_m \longmapsto \mathbb{R}$ is K-positive. Thus we may paraphrase Tchakaloff's Theorem as the analogue of Riesz-Haviland for TKMP in the compact case, as follows.

Theorem 5.2. *(cf.* [29, Theorem 2.2]*) Let $\beta \equiv \beta^{(m)}$, $\beta_0 > 0$, and let K be a compact subset of $\mathbb{R}^n$. β has a K-representing measure if and only if $L_\beta : \mathcal{P}_m \longmapsto \mathbb{R}$ is K-positive, in which case β admits a K-representing measure μ with card supp $\mu \le$ dim $\mathcal{P}_m$.*

The example in (2.2) shows that Theorem 5.2 cannot be extended to the case where K is non-compact. In that example, for $n=1$ and $d=2$, $K=\mathbb{R}$, and $\beta \equiv \beta^{(4)}$, we see that L_β can be positive although β has no representing measure. Nevertheless, the original cubature theorem of Tchakaloff does admit generalization. An extension of Theorem 5.1 to the case when K is unbounded was obtained by Mysovskikh [38]. For the compact case, Putinar [42] extended Theorem 5.1 to arbitrary positive Borel measures. For arbitrary closed K and a positive Borel measure μ having convergent moments up to at least degree $m \equiv 2d$, Putinar also established the existence of a cubature rule ν of degree $2d-1$, such that *card supp* $\nu \le$ *dim* $\mathcal{P}_{2d}$. Analogous results for the case $m \equiv 2d+1$ were subsequently obtained by Curto and the author in [12]. Finally, in 2006, Bayer and Teichmann proved the ultimate generalization of Tchakaloff's Theorem.

Theorem 5.3. *(Bayer and Teichmann* [3]*) Let μ be a positive Borel measure on $\mathbb{R}^n$ with convergent moments up to degree at least m, and let $\beta \equiv \beta^{(m)}[\mu]$. Then there* *exists a representing measure ν for β such that supp $\nu \subseteq$ supp μ and card supp $\nu \leq$ dim $\mathcal{P}_m$.*

Theorem 5.3 has the following significant consequence for the truncated moment problem.

Corollary 5.4. *If $\beta \equiv \beta^{(m)}$ has a K-representing measure μ, then β admits a finitely atomic K-representing measure.*

Proof. Since $\beta_i = \int x^i d\mu$ ($|i| \leq m$), Theorem 5.3 implies that there exists a degree-m cubature rule ν for μ. Thus ν is a finitely atomic K-representing measure for β. $\square$

Corollary 5.4 leads to the following important necessary condition for representing measures; we have already used this result in the proof of Theorem 3.3.

Proposition 5.5. *If $\beta \equiv \beta^{(2d)}$ has a representing measure, then M_d admits positive recursively generated moment matrix extensions of all orders, $M_{d+1},\ M_{d+2}, \ldots$.*

Proof. Since β has a representing measure, Corollary 5.4 implies that there exists a finitely atomic representing measure ν. Since ν has convergent moments of all orders, for each $k \geq 1$, $M_{d+k} := M_{d+k}[\nu]$ has a representing measure, namely ν, and is thus a positive, recursively generated extension of M_d. $\square$

Recall from Theorem 2.11 that $\Gamma := \{\beta \equiv \beta^{(m)} \in \mathbb{R}^{dim\ \mathcal{P}_m} :\ L_\beta \textit{ is positive}\}$ is the closure of the multisequences having representing measures. We devote the remainder of this section to an application of Proposition 5.5 which permits us to exhibit a nontrivial example of a sequence in *bdry* Γ, a positive Riesz functional $L_{\beta^{(2d)}}$ whose positivity does not arise from the existence of a representing measure for β or from the equivalence of positivity of L_β with positive semidefiniteness of $M_d(\beta)$. Following [44] [45], we refer to $p \in \mathcal{P}_{2d}$ as *positive semidefinite* (psd) if $p|_{\mathbb{R}^n} \geq 0$, and as a *sum of squares* (sos) if there exist $p_1, \ldots, p_k \in \mathcal{P}_d$ such that $p = \sum_{i=1}^{k} p_i^2$. For $\beta \equiv \beta^{(2d)}$, positivity of L_β is easily established if each psd polynomial is sos, since then L_β is positive if and only if $M_d \succeq 0$; indeed, in this case, if p is psd, then $p = \sum_{i=1}^{k} p_i^2$, so

$$L_\beta(p) = \sum L_\beta(p_i^2) = \sum \langle M_d \widehat{p_i}, \widehat{p_i} \rangle \geq 0. \tag{5.1}$$

A well-known theorem of Hilbert shows that each psd polynomial is sos if and only if $n = 1$, $d = 1$, or $n = d = 2$ (cf. [44] [45]). (For the purposes of this note, we refer to this result as "Hilbert's Theorem.") Thus, we seek an example of positivity of L_β in which β has no representing measure and in which (n, d) are beyond the scope of Hilbert's Theorem.

To this end, we first discuss an application of moment matrix extensions and the Bayer-Teichmann Theorem. This result characterizes the existence of representing measures in the bivariate truncated moment problem for $\beta \equiv \beta^{(2d)}$ in the case when the variety of M_d coincides with $y = x^3$. In [23] we associated to such a sequence β a certain computable rational function in the moment data denoted by $\psi(\beta)$.

Theorem 5.6. *([23]) Let $d \geq 3$. Suppose M_d is positive and $(y - x^3)$-pure, i.e., the variety $V(M_d)$ is completely determined (via recursiveness) by the column relation $Y = X^3$. The following are equivalent for $\beta \equiv \beta^{(2d)}$:*
i) β has a representing measure (necessarily supported in $y = x^3$);
ii) M_d has a flat extension M_{d+1} (and β has a corresponding (3d)-atomic minimal representing measure supported in $y = x^3$);
iii) (concrete condition) $\beta_{1,2d-1} > \psi(\beta)$;
iv) M_d admits a positive, recursively generated extension M_{d+1}.

We briefly outline the proof, merely to indicate the kinds of ingredients that are involved. The proof follows the steps *i)* $\Longrightarrow$ *iv)* $\Longrightarrow$ *iii)* $\Longrightarrow$ *ii)* $\Longrightarrow$ *i)*. The implication *i)* $\Longrightarrow$ *iv)* derives from Proposition 5.5- this is the step that depends on Bayer-Teichmann. The step *iv)* $\Longrightarrow$ *iii)* entails a detailed analysis of the structure of positive extensions of $(y - x^3)$-pure moment matrices. The implication *iii)* $\Longrightarrow$ *ii)* is based on the explicit construction of a flat extension M_{d+1}, and *ii)* $\Longrightarrow$ *i)* follows from the Flat Extension Theorem (Theorem 3.1).

Using the preceding result, we may exhibit an example, due to C. Easwaran and the author [20], concerning a positive Riesz functional L_β whose positivity does not arise from the existence of a representing measure or from sums of squares as in (5.1). In the sequel, for $n = 2$, we denote the successive rows and columns of the moment matrix $M \equiv M_3(\beta)$ by $1,\ X,\ Y,\ X^2,\ XY,\ Y^2, X^3,\ X^2Y,\ XY^2,\ Y^3$. We denote the elements of $\beta^{(6)}$ by β_{ij} $(i, j \geq 0,\ i + j \leq 6)$, where β_{ij} corresponds to the monomial $x^i y^j$. Let *Col* M denote the column space of M in $\mathbb{R}^{10}$. Under the conditions

$$M \equiv M_3(\beta) \succeq 0,\ Y = X^3 \textit{ in Col } M,\ \textit{rank}(M) = 9, \tag{5.2}$$

M_3 is $(y - x^3)$-pure, so we may compute $\psi(\beta)$, and a highly intensive computer algebra calculation reveals the following key property.

Proposition 5.7. *([20]) Under the conditions of (5.2), $\psi(\beta)$ is independent of β_{15} and β_{06}.*

In the sequel we say that $\beta \equiv \beta^{(2d)}$ is *consistent* if

$$p \in \mathcal{P}_{2d},\ p|V(M_d) \equiv 0 \Longrightarrow L_\beta(p) = 0. \tag{5.3}$$

Consistency is a necessary condition for representing measures which implies that M_d is recursively generated.

Theorem 5.8. *([20]) Let $n = 2$. For $\beta \equiv \beta^{(6)}$, suppose $M \equiv M_3(\beta) \succeq 0$, $Y = X^3$, and rank $M = 9$. If $\beta_{15} = \psi(\beta)$, then L_β is positive, but β has no representing measure, and positivity of L_β does not arise from sums of squares as in (5.1). Moreover, β is consistent (so M_3 is recursively generated) and the variety condition rank $M_3 \leq$ card $V(M_3)$ also holds.*

Example 5.9. For an example illustrating Theorem 5.8, consider

$$M \equiv M_3(\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & 1429 \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & 1429 & 4847 \end{pmatrix} \tag{5.4}$$

(cf. [20]). It is not difficult to verify all of the hypotheses; in particular, a calculation using [23] shows that $\psi(\beta) = 1429 = \beta_{15}$. $\square$

Proof. (*Sketch of proof of Theorem 5.8*) With β as in the hypothesis, M_3 is positive, and since $Y = X^3$ and *rank* $M_3 = 9$, it is clear that $V(M_3)$ coincides with the curve $y = x^3$, so the variety condition holds. Moreover, since $y - x^3$ is irreducible and has infinite variety, it follows from [23, Lemma 3.1] that β is consistent. We now claim that L_β is positive. Since $\beta_{15} = \psi(\beta)$, positivity for L_β cannot be derived from the existence of a representing measure, since Theorem 5.6-iii) shows that β has no representing measure. Moreover, as we discussed above, positivity for L_β cannot be derived from the positivity of M_3 via sums of squares arguments as in (5.1) because, by Hilbert's Theorem, there exist degree 6 bivariate polynomials that are psd but not sos.

To prove that L_β is positive, we employ a sequence of approximate representing measures. Observe that the matrix obtained from M by deleting row X^3 and column X^3 is positive definite. It follows that there exists $\delta > 0$ such that if we replace β_{15} $(= \psi(\beta))$ by $\beta_{15} + \frac{1}{m}$ (with $\frac{1}{m} < \delta$), then the resulting moment matrix, $M_3(\beta^{[m]})$, remains positive, with *rank* $M_3(\beta^{[m]}) = 9$ and $Y = X^3$ in *Col* $M_3(\beta^{[m]})$. Proposition 5.7 implies that $\psi(\beta^{[m]})$ is independent of $\beta_{15}[\beta^{[m]}]$ and $\beta_{06}[\beta^{[m]}]$, so we have $\psi(\beta^{[m]}) = \psi(\beta) = \beta_{15} < \beta_{15} + \frac{1}{m} = \beta_{15}[\beta^{[m]}]$. It now follows from Theorem 5.6-iii) that $\beta^{[m]}$ has a representing measure, whence $L_{\beta^{[m]}}$ is positive. Note that the convex cone $\{\beta \equiv \beta^{(6)} \in \mathbb{R}^{10} : L_\beta \textit{ is positive}\}$ is closed; since $\|\beta^{[m]} - \beta\| = \frac{1}{m} \longrightarrow 0$, we conclude that L_β is positive. $\square$

6. Strict K-positivity, the core variety, and positive definite moment matrices

In this section we consider the truncated moment problem for a positive definite moment matrix. This case of the truncated moment problem is largely unsolved, due to the lack of dependence relations in *Col* M_d. As we have seen, such relations, when present and combined with recursiveness, are useful in constructing positive extensions leading to flat extensions and representing measures (cf. Section 3). To study the positive definite case, we introduce refinements of K-positivity and of the variety $V(M_d)$, which may be of independent interest.

Following [27], for $\beta \equiv \beta^{(m)}$, we say that L_β is *strictly K-positive* if L_β is K-positive and the conditions $p \in \mathcal{P}_m$, $p|_K \geq 0$, and $p|_K \not\equiv 0$ imply $L_\beta(p) > 0$. If $K = \mathbb{R}^n$ and L_β is strictly K-positive, we say that L_β is *strictly positive.* Note that if β has a representing measure μ, then L_β is strictly K-positive for $K = supp\ \mu$. Further, K is a *determining set* for $\mathcal{P}_m$ if the conditions $p \in \mathcal{P}_m$ and $p|_K \equiv 0$ imply $p \equiv 0$. (If K has nonempty interior, then K is a determining set, but certain finite sets are also determining sets.) Strict positivity leads to the following existence criterion.

Theorem 6.1. *(*[27, Theorem 1.3]*) For $\beta \equiv \beta^{(m)}$, if K is a determining set for $\mathcal{P}_m$ and L_β is strictly K-positive, then β has a K-representing measure.*

The remainder of this section is based on [26]. Let $\beta \equiv \beta^{(2d)}$ and recall the variety $V(M_d) := \bigcap_{p \in \mathcal{P}_d,\ M_d\widehat{p}=0} \mathcal{Z}_p$, which we now designate by $\mathcal{V}^{(0)}$. For $i \geq 0$, let

$$\mathcal{V}^{(i+1)} := \bigcap_{p \in ker L_\beta,\ p|_{\mathcal{V}^{(i)}} \geq 0} \mathcal{Z}_p.$$

We define the *core variety* of β (or of $M_d(\beta)$) by $\mathcal{V} \equiv \mathcal{V}(\beta) := \bigcap_{i=0}^{\infty} \mathcal{V}^{(i)}$; we also denote this by $\mathcal{V}(M_d)$. The usefulness of the variety $V(M_d)$ lies in the fact that it contains the support of any representing measure. The core variety has the same inclusion property as $V(M_d)$, and since it is a subvariety of $V(M_d)$, it provides a better indication of the location of the support. In the sequel we set $\nu := card\ \mathcal{V}(\beta)$.

Proposition 6.2. [26, Prop. 2.1] *If μ is a representing measure for β, then $supp\ \mu \subseteq \mathcal{V}(\beta)$.*

Corollary 6.3. [26, Cor. 2.3] *i) If β has a representing measure, then $rank\ M_d \leq card\ \mathcal{V}(\beta)$.*
ii) If μ is a representing measure for β with $int(supp\ \mu) \neq \emptyset$, then $\mathcal{V}(\beta) = \mathbb{R}^n$.

Proof. i) (3.1) shows that if μ is a representing measure for β, then $rank\ M_d \leq card\ supp\ \mu$, so the result follows from Proposition 6.2.
ii) Since a proper affine variety has empty interior in $\mathbb{R}^n$, the result follows from Proposition 6.2. $\square$

We next turn to several results of [26] related to computing the core variety.

Lemma 6.4. [26, Lemma 2.6] *For $i \geq 0$, $\mathcal{V}^{(i+1)} \subseteq \mathcal{V}^{(i)}$.*

We note for future reference the following implications that are implicit in the proof of Lemma 6.4:

$$p \in kerL_\beta,\ p|_{\mathcal{V}^{(i)}} \geq 0 \Longrightarrow \mathcal{V}^{(i+1)} \subseteq \mathcal{Z}_p \bigcap \mathcal{V}^{(i)}\ [26, (2.1)] \tag{6.1}$$

$$p \in ker\ L_\beta,\ p|_{\mathcal{V}^{(i)}} > 0 \Longrightarrow \mathcal{V} = \emptyset\ [26, (2.2)] \tag{6.2}$$

$$\mathcal{V}^{(i)} = \mathcal{V}^{(i+1)} \Longrightarrow \mathcal{V} = \mathcal{V}^{(i)} \ [26, (2.3)] \tag{6.3}$$

We further note that there always exists i such that $\mathcal{V}^{(i)} = \mathcal{V}^{(i+1)}$ (as in (6.3)), whence $\mathcal{V} = \mathcal{V}^{(i)}$ [26, Prop. 2.7].

The core variety provides a tool for establishing strict K-positivity and furnishes a link to Theorems 2.8 and 6.1.

Theorem 6.5. *(*[26, Theorem 2.14]*) Let $\beta \equiv \beta^{(2d)}$. If the core variety $\mathcal{V} \equiv \mathcal{V}(\beta)$ is nonempty, then L_β is strictly $\mathcal{V}$-positive and $\beta^{(2d-1)}$ has a $\mathcal{V}$-representing measure. Moreover, if $\mathcal{V}$ is nonempty and is either compact or a determining set for $\mathcal{P}_{2d}$, then $\beta^{(2d)}$ has a $\mathcal{V}$-representing measure.*

We now return to the case of a positive definite moment matrix. Recall that Hilbert's theorem shows each psd polynomial is sos if and only if $n = 1$, $d = 1$, or $n = d = 2$ (cf. Section 1.2). In the cases of Hilbert's theorem, it is known that a positive definite moment matrix has a representing measure (cf. [6] for $n = 1$ and [27] for $d = 1$ and $n = d = 2$). We begin with a new proof of this result based on the core variety.

Proposition 6.6. [26, Prop. 4.1] *In the cases of Hilbert's Theorem, if $M_d(\beta) \succ 0$, then $\mathcal{V}(\beta) = \mathbb{R}^n$ and β has a representing measure.*

Proof. Since $M_d \succ 0$, then $V(M_d) = \mathbb{R}^n$. From (6.3), to show $\mathcal{V} = \mathbb{R}^n$, it suffices to verify that $\mathcal{V}^{(1)} = \mathbb{R}^n$. Suppose $q \in ker\ L_\beta$ and q is psd. Then q is of the form $q = \sum q_i^2$ for certain $q_i \in \mathcal{P}_d$, and thus $0 = L_\beta(q) = \sum L_\beta(q_i^2) = \sum \langle M_d \hat{q}_i, \widehat{q_i} \rangle$. Since $M_d \succ 0$, it follows that each $\widehat{q_i} = 0$, whence $q = 0$ and $\mathcal{Z}_q = \mathbb{R}^n$. We thus have $\mathcal{V}^{(1)} = \bigcap_{q \in ker\ L_\beta, q|_{\mathbb{R}^n} \geq 0} \mathcal{Z}_q = \mathbb{R}^n = \mathcal{V}^{(0)}$, whence (6.3) implies $\mathcal{V} = \mathcal{V}^{(1)} = \mathcal{V}^{(0)} = \mathbb{R}^n$. Since $\mathbb{R}^n$ is a determining set, Theorem 6.5 implies that β has a representing measure. $\square$

Remark 6.7. For the cases $d = 1$ or $n = 1$, it is known that if $M_d \succ 0$, then M_d admits a flat extension M_{d+1} (cf. [7] [27]). For the case $n = d = 2$, it was an open question for several years as to whether a positive definite M_2 admits a flat extension M_3. This has recently been answered in the affirmative in [18], using the *proof* of Theorem 3.3 and a "reduction of rank" technique.

Relatively little is known concerning the positive definite case beyond the scope of Hilbert's Theorem. However, we do have the following general result.

Proposition 6.8. *(*[26, Prop. 4.3]*) The following are equivalent for $\beta \equiv \beta^{(2d)}$:*
i) L_β is strictly positive;
ii) $M_d \succ 0$ and $\mathcal{V} = \mathbb{R}^n$;
iii) $\mathcal{V} = \mathbb{R}^n$.

Proof. We begin with i) $\Longrightarrow$ ii). Suppose L_β is strictly positive. For $p \in \mathcal{P}_d$ with $p \not\equiv 0$, we have $\langle M_d \widehat{p}, \widehat{p} \rangle = L_\beta(p^2) > 0$, so $M_d \succ 0$. It follows that $\mathcal{V}^{(0)} = V(M_d) = \mathbb{R}^n$. From (6.3), to show $\mathcal{V} = \mathbb{R}^n$, it suffices to prove that $\mathcal{V}^{(1)} = \mathbb{R}^n$. Suppose $p \in ker\ L_\beta$ and $p|_{\mathcal{V}^{(0)}} \geq 0$. Then since p is psd and L_β is strictly positive, we have $p \equiv 0$, whence $\mathcal{Z}_p = \mathbb{R}^n$. It follows that $\mathbb{R}^n = \mathcal{V}^{(1)} \subseteq \mathcal{V}^{(0)} = \mathbb{R}^n$, so the conclusion $\mathcal{V} = \mathbb{R}^n$ follows. Clearly ii) implies iii), and the implication iii) $\Longrightarrow$ i) follows immediately from Theorem 6.5. $\square$

For several of the results and examples in the sequel we require some additional notation. Let $\mathcal{P}_{2d}^+$ and Σ_{2d} denote, respectively, the positive cones in $\mathcal{P}_{2d}$ consisting of the psd and sos polynomials, and let $\Delta \equiv \Delta_{n,2d} := \mathcal{P}_{2d}^+ \setminus \Sigma_{2d}$. Concrete examples of polynomials in Δ were discovered beginning some 60 years after Hilbert's work. We note two such examples from $\Delta_{3,6}$ that are discussed by Reznick [44] [45]: the Motzkin form

$$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2;$$

the Robinson form

$$R(x,y,z) = x^6 + y^6 + z^6 - x^4y^2 - x^2y^4 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2.$$

It is well-known that a homogeneous form $F(x,y,z)$ is psd (respectively, sos) if and only if its de-homogenization $f(x,y) := F(x,y,1)$ is psd (respectively, sos). In the sequel we will denote the de-homogenizations of M and R by m and r.

For the case $n = 2$, $d = 3$, and $M_3 \succ 0$, we may characterize the existence of representing measures in terms of the core variety, as follows.

Theorem 6.9. *(*[26, Theorem 4.4]*) For $n = 2$ and $M_3 \succ 0$, exactly one of the following holds:*
i) $\mathcal{V} = \mathbb{R}^2$ and there is a representing measure;
ii) $\nu = 10$ and there is a representing measure;
iii) $\nu = 0$ and there is no representing measure.

Example 6.10. ([26, Example 4.5]) We illustrate case ii) of Theorem 6.9. It is known that the Robinson plynomial $r(x,y)$ has precisely the following eight affine zeros $w_i \equiv (x_i, y_i)$ $(1 \le i \le 8)$: $w_1 = (-1,-1)$, $w_2 = (0,-1)$, $w_3 = (1,-1)$, $w_4 = (-1,0)$, $w_5 = (1,0)$, $w_6 = (-1,1)$, $w_7 = (0,1)$, $w_8 = (1,1)$. Corresponding to these are eight projective zeros of the Robinson form R: $\widetilde{w}_i := (x_i, y_i, 1)$ $(1 \le i \le 8)$. The Robinson form has two additional projective zeros, $\widetilde{w}_9 \equiv (x_9, y_9, z_9) := (1,1,0)$ and $\widetilde{w}_{10} \equiv (x_{10}, y_{10}, z_{10}) := (1,-1,0)$. We now define $T(x,y,z)$ to be the Robinson form composed with a linear change of variables in $\mathbb{R}^3$, as follows:

$$T(x,y,z) := R(3x - 3y + z, -3x + 5y - 2z, x - 2y + x).$$

Since $R \in \Delta_{3,6}$, so is T. Then the dehomogenization of T, defined by $t(x,y) := T(x,y,1)$, is in $\Delta_{2,6}$. For each projective zero (x,y,z) of R, $(u,v,w) := (x+y+z, x+2y+3z, x+3y+6z)$ is a projective zero of T, and if $w \neq 0$, then $(\frac{u}{w}, \frac{v}{w})$ is an affine zero of t. A calculation now shows that t has 10 distinct affine zeros, $u_i \equiv (a_i, b_i)$ $(1 \le i \le 10)$, as follows: $u_1 = (-\frac{1}{2}, 0)$, $u_2 = (0, \frac{1}{3})$, $u_3 = (\frac{1}{4}, \frac{1}{2})$, $u_4 = (0, \frac{2}{5})$, $u_5 = (\frac{2}{7}, \frac{4}{7})$, $u_6 = (\frac{1}{8}, \frac{1}{2})$, $u_7 = (\frac{2}{9}, \frac{5}{9})$, $u_8 = (\frac{3}{10}, \frac{3}{5})$, $u_9 = (\frac{1}{2}, \frac{3}{4})$, $u_{10} = (0, \frac{1}{2})$. Setting $\mu := \sum_{i=1}^{10} \delta_{(a_i,b_i)}$, a straightforward calculation with nested determinants shows that $M_3[\mu] \succ 0$, and a calculation with the moments $\beta \equiv \beta^{(6)}[\mu]$ shows that $L_\beta(t) = 0$. Since t is psd, we have $\mathcal{V}^{(1)} \subseteq \mathcal{Z}_t$, whence $\nu \equiv card\ \mathcal{V}(\beta) \leq 10$. Since β has the representing measure μ, we also have $10 = card\ supp\ \mu \leq \nu$, so we see that $\nu = 10$ and $\mathcal{V} = \mathcal{Z}_t$. $\square$

Concerning case iii) of Theorem 6.9, the first example of a positive definite moment matrix not having a representing measure appears in [8, Section 4], for the case $n = 2$ and $d = 3$. This example is based on Schmüdgen's construction of a polynomial $s(x, y)$ that is psd but not sos [49]. This polynomial is used in [49] to explicitly construct a linear functional L on $\mathcal{P}_6$ such that $L|_{\Sigma_6} \geq 0$ but L is not positive. The corresponding moment matrix $M_3(\beta)$ is positive definite and thus illustrates case iii) in Theorem 6.9. The following result uses the core variety to prove the existence of a large family of positive definite moment matrices which do not have representing measures.

Proposition 6.11. *(*[26, Prop. 4.6]*) Let $p \in \Delta_{n,2d}$ with card $\mathcal{Z}_p <$ dim $\mathcal{P}_d$. Then there exists $M_d \equiv M_d(\beta_p)$ such that $M_d \succ 0$, $L_{\beta_p}(p) = 0$, and $\mathcal{V}(M_d) = \emptyset$, whence β_p has no representing measure.*

Remark 6.12. i) The Robinson polyomial $r(x, y)$ satisfies the hypothesis of Proposition 6.11, since $r \in \Delta_{2,6}$ and $card\ \mathcal{Z}_r = 8 < 10 = dim\ \mathcal{P}_3$. Similarly, the Motzkin polynomial $m(x, y)$ satisfies $card\ \mathcal{Z}_m = 6$, and Schmüdgen's polynomial $s(x, y)$ satisfies $card\ \mathcal{Z}_s = 9$.
ii) We do not know whether, in general, L_{β_p} is $\mathcal{V}$-positive, or even positive.

If L_β is strictly positive, then $M_d \succ 0$, and Theorem 6.1 or Theorem 6.5 implies that β has a representing measure. In [27, Question 1.2] we asked whether the same conclusion holds if $M_d \succ 0$ and L_β is merely positive (cf. [16, Question 2.9]). We will answer this question below, but we require some preliminaries.

In [29] we studied the connection between TMP for an n-dimensional sequence $\beta \equiv \beta^{(2d)}$ and the moment problem with respect to homogeneous polynomials of degree $2d$ in $n+1$ variables $x_0, x_1, \ldots, x_n$, with moment data $\widetilde{\beta} \equiv \widetilde{\beta}^{(=2d)}$ defined by

$$\widetilde{\beta}_{(2d-|\alpha|,\alpha)} := \beta_\alpha$$

for every $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq 2d$. We refer to this problem as the *Homogeneous Truncated Moment Problem* (HTMP). The moment problem for β and the homogeneous moment problem for $\widetilde{\beta}$ are not equivalent, but are closely related.

Theorem 6.13. *(*[29, Theorem 3.1]*) If β has a representing measure in TMP, then $\widetilde{\beta}$ has a representing measure in HTMP. Moreover, L_β is positive if and only if $\widetilde{\beta}$ has a representing measure in HTMP.*

We are now prepared to resolve [27, Question 1.2].

Theorem 6.14. *(*[26, Theorem 4.8]*) For $n = 2$, there exists M_3 such that $M_3 \succ 0$ and L_β is positive, but $\mathcal{V} = \emptyset$, so β does not have a representing measure.*

Proof. As noted in Example 6.10, the de-homogenized Robinson polynomial $r(x, y)$ has the following eight affine zeros $w_i \equiv (x_i, y_i)$ $(1 \le i \le 8)$: $w_1 = (-1, -1)$, $w_2 = (0, -1)$, $w_3 = (1, -1)$, $w_4 = (-1, 0)$, $w_5 = (1, 0)$, $w_6 = (-1, 1)$, $w_7 = (0, 1)$, $w_8 = (1, 1)$. Corresponding to these are eight projective zeros of the Robinson form R, $\widetilde{w}_i := (x_i, y_i, 1)$ $(1 \le i \le 8)$, and there are two additional projective zeros for R, $\widetilde{w}_9 \equiv (x_9, y_9, z_9) := (1, 1, 0)$ and $\widetilde{w}_{10} \equiv (x_{10}, y_{10}, z_{10}) := (1, -1, 0)$. Define the measure ω on $\mathbb{R}^3$ by $\omega := \sum_{i=1}^{10} \delta_{\widetilde{w}_i}$. Let $\widetilde{\beta} \equiv \widetilde{\beta}^{(=6)}$ denote the ω-moments of degree 6, and let $L_{\widetilde{\beta}}$ denote the corresponding functional on homogeneous forms of degree 6 in $\mathbb{R}[x, y, z]$. Now define $L : \mathbb{R}[x, y]_6 \mapsto \mathbb{R}$ by $L(p(x, y)) := L_{\widetilde{\beta}}(\widetilde{p}(x, y, z))$ (where $\widetilde{p}$ denotes the homogenization of p, i.e., $\widetilde{p}(x, y, z) = z^6 p(\frac{x}{z}, \frac{y}{z})$). Let $\beta_{ij} := L(x^i y^j)$ $(i, j \ge 0,\ i + j \le 6)$, so that $L_\beta = L$. (Note that $\beta_{ij} = \widetilde{\beta}_{i,j,6-i-j}$, so $\widetilde{\beta}$ is the *homogenization* of β in the language of [29] (cf. Theorem 6.13).) The moment matrix corresponding to β, $M \equiv M_3(\beta)$, is given by

$$M = \begin{pmatrix} 8 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 6 \\ 6 & 0 & 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 8 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 8 \end{pmatrix}.$$

Using nested determinants, it is easy to check that $M \succ 0$. Since $\widetilde{\beta}$ has the representing measure ω in HTMP, it follows from Theorem 6.13 that L_β is positive.

We next compute $\mathcal{V}(M)$. Clearly, $\mathcal{V}^{(0)} = \mathbb{R}^2$. A calculation with the moments of M shows that $L_\beta(r) = 0$, and since r is psd, it follows that $\mathcal{V}^{(1)} \subseteq \mathcal{Z}_r$. Thus, $\nu \le card\ \mathcal{Z}_r = 8 < 10 = rank\ M$, so Corollary 6.3-i) already implies that $\nu = 0$ and that β has no representing measure. We will also verify this conclusion explicitly. If $t(x, y) \in \mathcal{P}_6$ is psd and $L_\beta(t) = 0$, then $\int \widetilde{t}(x, y, z) d\omega = L_{\widetilde{\beta}}(\widetilde{t}) = L_\beta(t) = 0$. Since t is psd, so is $\widetilde{t}$, and thus $\widetilde{t}|_{supp\ \omega} \equiv 0$. It follows that $t|_{\mathcal{Z}_r} \equiv 0$, which implies $\mathcal{Z}_r \subseteq \mathcal{V}^{(1)}$. Since, from above, we also have $\mathcal{V}^{(1)} \subseteq \mathcal{Z}_r$, then $\mathcal{V}^{(1)} = \mathcal{Z}_r$. Now let $f(x, y) := 2 - x^2 - y^2$ and $g(x, y) := \frac{3}{2}x^2y^2 - x^2y^4$. Then $L_\beta(f) = L_\beta(g) = 0$ and f and g are nonnegative on $\mathcal{Z}_r$. Moreover, for $1 \le i \le 8$, either $f(w_i) = 0$ and $g(w_i) = \frac{1}{2}$, or $f(w_i) = 1$ and $g(w_i) = 0$, so $\mathcal{V} \subseteq \mathcal{V}^{(2)} \subseteq \mathcal{Z}_r \bigcap \mathcal{Z}_f \bigcap \mathcal{Z}_g = \emptyset$. $\square$

7. TKMP and optimization methods

In this section we briefly describe some connections between TKMP and optimization theory. We begin with J.-B. Lasserre's application of TKMP to the problem of minimizing a polynomial over a basic closed semialgebraic set in $\mathbb{R}^n$ [35] [36]. To discuss Lasserre's method, we first require an analogue for TKMP of the Moment Matrix Extension Theorem (Theorem 3.2). We begin by defining the appropriate notion of localizing matrix for a truncated multisequence $\beta \equiv \beta^{(2d)}$. Let $1 \le k \le d$ and suppose $p \in \mathbb{R}[x]$ satisfies $deg\ p = 2k$ or $2k-1$. Let $\eta := dim\ \mathcal{P}_{d-k}$. We define the *localizing matrix* $M_d^{(p)} \equiv M_d^{(p)}(\beta)$ to be the $\eta \times \eta$ matrix characterized by

$$\langle M_d^{(p)} \widehat{f}, \widehat{g} \rangle = L_\beta(pfg)\ (f, g \in \mathcal{P}_{d-k}). \tag{7.1}$$

If β has a representing measure supported in $\mathcal{S}_p$, then $M_d^{(p)} \succeq 0$. Now let $P \equiv \{p_1, \ldots, p_m\} \subset \mathbb{R}[x]$ and suppose $deg\ p_i = 2k_i$ or $2k_i - 1$ $(1 \le i \le m)$. Consider the basic closed semialgebraic set $K_P \equiv \bigcap_{i=1}^{m} \mathcal{S}_{p_i}$. The following result characterizes the existence of K_P-representing measures in terms of moment matrix extensions; note that in this result, we do not assume that K_P is compact. Recall from Theorem 3.1 that if $M_d \succeq 0$ has a flat extension M_{d+1}, then M_{d+1} admits unique successive (positive) flat extensions $M_{d+2},\ M_{d+3}, \ldots$. In this case, each localizing matrix $M_{d+k_i}^{(p_i)}$ is well-defined.

Theorem 7.1. *(cf. [14]) $\beta \equiv \beta^{(2d)}$ has a K_P-representing measure if and only if M_d admits a positive extension M_{d+j} (with $j \le 2\binom{2d+n}{n} - d$), which in turn admits a flat extension M_{d+j+1} for which $M_{d+j+k_i}^{(p_i)} \succeq 0$ $(1 \le i \le m)$. The unique representing measure for M_{d+j+1} has precisely rank M_{d+j} − rank $M_{d+j+k_i}^{(p_i)}$ atoms in $\mathcal{Z}_{p_i}$ $(1 \le i \le m)$.*

For $p \in \mathbb{R}[x_1, \ldots, x_n]$, the Optimization Problem entails estimating

$$p_* := \inf_{x \in K_P} p(x). \tag{7.2}$$

Suppose $deg\ p = 2k_0$ or $2k_0 - 1$ and let $\kappa := \max_{0 \le j \le m} k_j$. Fix $t \ge \kappa$ and consider the t-th *Lasserre moment relation* for (7.2) given by

$$p_t := inf\{L_\beta(p) : \beta \equiv \beta^{(2t)},\ \beta_{\mathbf{0}} = 1, M_t(\beta) \succeq 0,\ M_t^{(p_j)}(\beta) \succeq 0\ (j = 1, \ldots, m)\}. \tag{7.3}$$

The Lasserre relaxations may be computed within the framework of semidefinite programming [35] [36] [37]. It is not difficult to verify that $p_t \le p_*$ and that for $t' \ge t$, $p_{t'} \ge p_t$; thus $\{p_t\}$ is convergent, and $p^{mom} \equiv \lim_{t \to \infty} p_t \le p_*$. A result of Lasserre [35] (cf. [37, Theorem 6.8]) shows that $p^{mom} = p_*$ if the quadratic module associated to K_P is Archimedean, i.e., the quadratic module contains $N - ||x||^2$ for some $N > 0$ (cf. Theorem 2.6).

In some cases there is even *finite convergence* to p_*. In the general case, for fixed t, the infimum in (7.2) is not necessarily attained. Assuming that the infimum *is* attained, at some optimal sequence $\beta \equiv \beta^{\{t\}}$, we seek conditions which imply that $L_\beta(p) = p_*$, so that we have finite convergence of $\{p_s\}_{s \ge \kappa}$ to p_* at stage $s = t$. A basic result of [33] shows that this is the case if $rank\ M_t(\beta) = rank\ M_{t-\kappa}(\beta)$ (cf. [37, Theorem 6.18]). Indeed, in this case, Theorem 7.1 and the conditions of (7.3) imply that β has a K_P representing measure, which always implies convergence at stage t. To see this last point, note that if μ is a K_P-representing measure for $\beta \equiv \beta^{\{t\}}$, then

$$p_* = p_*\beta_0 = p_* \int_{K_P} 1 d\mu \leq \int_{K_P} p d\mu = L_\beta(p) = p_t \leq p_*.$$

We note also that when $rank\ M_t(\beta) = rank\ M_{t-\kappa}(\beta)$ holds, then the support of the unique representing measure for $M_t(\beta)$ consists of minimizers for (7.2). Further, using Theorem 2.11, it follows that whether or not an optimizing sequence $\beta \equiv \beta^{\{t\}}$ at stage t has a K_P-representing measure, if the functional L_β is K_P-positive, then we still have finite convergence, i.e., $L_\beta(p) = p_*$ [28, Theorem 1.5].

The preceding discussion shows how TKMP may be adapted to the problem of polynomial optimization. Conversely, optimization techniques based on semidefinite programming can be used to analyze TKMP (cf. [36, Chapter 4]). This approach was recently studied by Helton and Nie for the case when K_P is compact [31], was extended to the noncompact case in [29], and was further generalized by Nie in [39]. Finally, consider the finite-variety case of TMP [22]. Instead of developing sufficient conditions for representing measures based on properties of the moment matrix, if one is content to solve the problem on a sequence-by-sequence basis, then if the variety is finite and is explicity known, this problem can be posed as a standard linear programming problem. This unpublished observation is due to J. Nie.

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