Minimal representing measures arising from rank-increasing moment matrix extensions

Lawrence A. Fialkow

Abstract

If μ is a representing measure for $\gamma \equiv \gamma^{(2n)}$ in the Truncated Complex Moment Problem $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu$ $(0 \leq i + j \leq 2n)$, then *card supp* $\mu \geq$ *rank* $M(n)$, where $M(n) \equiv M(n)(\gamma)$ is the associated moment matrix. We present a concrete example of γ illustrating the case when *card supp* $\mu >$ *rank* $M(n)(\gamma)$ for every representing measure μ. This example is based on an analysis of moment problems in which some analytic column $\mathbf{Z^k}$ of $M(n)$ can be expressed as a linear combination of columns $\mathbf{\bar{Z}^i Z^j}$ of strictly lower degree.

1 Introduction

Given $n \geq 1$ and a sequence of complex numbers

$$\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \gamma_{1,2n-1}, \ldots, \gamma_{2n-1,1}, \gamma_{2n,0},$$

the Truncated Complex Moment Problem (TCMP) entails determining whether there exists a positive Borel measure μ on the complex plane such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \quad (0 \leq i + j \leq 2n).$$

Although no complete criterion for solubility of TCMP is known at present, in [CF2] [CF3] [CF4] R.E. Curto and the author developed various necessary or sufficient conditions for the existence of representing measures; these are expressed as positivity and extension properties of the *moment matrix* $M(n) \equiv M(n)(\gamma)$ associated to γ (see below for terminology and notation). If μ is a representing measure for γ, then *card supp* $\mu \geq$ *rank* $M(n)$ [CF2, Corollary 3.7] [F1]; moreover, there exists a *rank* $M(n)$-atomic (minimal) representing measure for γ if and only if $M(n) \geq 0$ and $M(n)$

[0]1991 *Mathematics Subject Classification.* Primary 47A57, 44A60.

admits an extension to a moment matrix $M(n+1)$ satisfying $rank\ M(n+1) = rank\ M(n)$ [CF2, Theorem 5.13]. The existence of such a *flat* (i.e., rank-preserving) extension has been established in a variety of cases [CF3] [CF4] (described below), but in 1995 J.E. McCarthy [McC] proved that there exist $\gamma \equiv \gamma^{(10)}$ for which every representing measure μ satisfies $card\ supp\ \mu > rank\ M(n)(\gamma)$. McCarthy further showed that such γ are in a sense generic among moment sequences having representing measures; nevertheless, the literature apparently contains no concrete example of a moment sequence displaying this feature. In the present note we provide such a concrete example. For $n = 3$, we exhibit a specific $\gamma^{(6)}$ with $rank\ M(3)(\gamma) = 8$ and $card\ supp\ \mu = 9$ for the unique minimal representing measure μ (Theorem 3.1). We establish this example within the framework of solving TCMP for moment matrices $M(n)$ in which some *analytic* column $\mathbf{Z^k}$ can be expressed as a linear combination of columns $\mathbf{\bar{Z}^i Z^j}$ of strictly lower degree. For a moment matrix $M(n)$ with such an *analytic constraint*, Section 2 provides an algorithmic procedure for determining whether or not $\gamma^{(2n)}$ admits a finitely atomic representing measure. If such a measure exists, there exists a unique minimal representing measure, which is not necessarily associated with a flat extension $M(n+1)$, but rather with finite sequence of successive extensions $M(n+1), \ldots, M(n+d), M(n+d+1)$, of which the first d are *rank-increasing* and the last is flat (cf. Theorem 1.2 below).

We devote the remainder of this section to terminology, notation, and a survey of some results that we require concerning extensions of moment matrices. The *moment matrix* $M(n)$ that we introduced in [CF2] [F1] has $m(n) \equiv (n+1)(n+2)/2$ rows and columns, labelled lexicographically by $\mathbf{1}, \mathbf{Z}, \mathbf{\bar{Z}}, \ldots, \mathbf{Z^n}, \mathbf{Z^{n-1}\bar{Z}}, \ldots, \mathbf{Z\bar{Z}^{n-1}}, \mathbf{\bar{Z}^n}$; the entry in row $\mathbf{\bar{Z}^i Z^j}$, column $\mathbf{\bar{Z}^k Z^l}$ is $\gamma_{j+k,i+l}$. Alternately, for $0 \le i, j \le n$, let B_{ij} denote the $(i+1)$ x $(j+1)$ matrix

$$B_{ij} = \begin{pmatrix} \gamma_{ij} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{ij} & \cdots & \gamma_{i+j-1,1} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{ji} \end{pmatrix}$$

(note that B_{ij} is constant on diagonals); then $M(n)$ admits the block decomposition $M(n) = (B_{ij})_{0 \le i,j \le n}$.

For $k \ge 0$, let $\mathcal{P}_k$ denote the complex polynomials $p(z,\bar{z}) = \sum a_{ij} \bar{z}^i z^j$ of total degree at most k. For $p \in \mathcal{P}_{2n}$, let $\Gamma(p) = \sum a_{ij}\gamma_{ij}$, and for $p \in \mathcal{P}_n$, let $\hat{p} \equiv (a_{ij}) \in \mathbf{C}^{m(n)}$ denote the coeffcient vector of p with respect to the basis $\{\bar{z}^i z^j\}$ of $\mathcal{P}_n$ (ordered lexicographically). $M(n)$ is uniquely determined by

$$\langle M(n)\hat{f}, \hat{g}\rangle = \Gamma(f\bar{g}) \quad (f, g \in \mathcal{P}_n);$$

moreover, if there exists a representing measure μ for γ, then $\langle M(n)\hat{f},\hat{f}\rangle = \Gamma(|f|^2) = \int |f|^2\ d\mu \geq 0 \ \ (f \in \mathcal{P}_n)$, so $M(n) \geq 0$ [CF2, Ch. 3, page 15]. For $p(\bar{z},z) = \sum a_{ij}\bar{z}^i z^j \in \mathcal{P}_n$, we define an element $p(\mathbf{Z},\bar{\mathbf{Z}})$ of $\mathcal{C}_{M(n)}$ (the column space of $M(n)$) by $p(\mathbf{Z},\bar{\mathbf{Z}}) = \sum a_{ij}\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}}$. If μ is a representing measure for γ, then *supp* $\mu \subset \mathcal{Z}(p) \equiv \{z : \ p(z,\bar{z}) = 0\}$ if and only if $p(\mathbf{Z},\bar{\mathbf{Z}}) = \mathbf{0}$ [CF2, Proposition 3.1], whence

$$rank\ M(n) \leq card\ supp\ \mu \leq card \bigcap_{p\in\mathcal{P}_n, p(\mathbf{Z},\bar{\mathbf{Z}})=\mathbf{0}} \mathcal{Z}_p$$

(cf. [CF4, Ch. 1, page 6]). The following result is the main tool for establishing the existence of representing measures in [CF2] [CF3] [CF4].

Theorem 1.1 *([CF2, Theorem 5.13]) There exists a rank $M(n)$-atomic representing measure for γ if and only if $M(n) \geq 0$ admits a flat extension $M(n+1)$.*

By combining Theorem 1.1 with a result of M. Putinar [P, Theorem 2], in [CF4] we established the following criterion for the existence of *finitely atomic* representing measures.

Theorem 1.2 *([CF4, Theorem 1.5]) The following are equivalent for $\gamma^{(2n)}$: i) There exists a representing measure with convergent moments of all orders; ii) There exists a representing measure having convergent moments up to (at least) order $2n+2$; iii) There exists a finitely atomic representing measure; iv) There exists $k \geq 0$ such that $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat extension $M(n+k+1)$.*

(The equivalence of ii) and iii) is due to M. Putinar [op. cit.]; it remains unknown whether the existence of a representing measure implies the existence of a finitely atomic representing measure.)

Can we always take $k = 0$ in Theorem 1.2-iv)? Equivalently, if there exists a finitely atomic representing measure for $\gamma^{(2n)}$, does there exist a representing measure whose support consists precisely of *rank* $M(n)(\gamma)$ atoms? The answer is affirmative for truncated moment problems on the real line [CF1] [F1]; for the Quadratic Moment Problem ($n = 1$) [CF2, Theorem 6.1]; for the case of Flat Data (when $M(n) \geq 0$ and *rank* $M(n) =$ *rank* $M(n-1)$) [CF2, Theorem 5.4]; for the case when there is a linear relation $\bar{\mathbf{Z}} = \alpha\mathbf{1} + \beta\mathbf{Z}$ in $\mathcal{C}_{M(n)}$ [CF3, Theorem 2.1]; and for the only known minimal quadrature rules of even degree for Lebesgue measure restricted to the square, disk, or triangle (rules of degree 2 or 4) [CR] [F2] [Str].

Despite this positive evidence, and in response to our affirmative conjecture [CF2], Prof. John E. McCarthy proved that there exist minimal representing measures which do *not* correspond to flat extensions, as follows.

Theorem 1.3 *(J.E. McCarthy [McC]) There exist moment sequences $\gamma^{(10)}$ admitting finitely atomic representing measures, but not admitting rank $M(5)$-atomic representing measures.*

McCarthy's proof of Theorem 1.3 (which appears in [CF4, Theorem 5.2]) depends on a topological embedding and dimensionality argument. This proof shows, moreover, that the sequences of Theorem 1.3 form a topologically "large" subset of the sequences $\gamma^{(10)}$ having representing measures; nevertheless, since the proof is nonconstructuve, it seems difficult to display such sequences or to construct their minimal representing measures.

Following [CF2] [F1], we say that a positive moment matrix $M(n)$ is *recursively generated* if it satisfies the following property:

$$(RG) \quad p,\ q,\ pq \in \mathcal{P}_n,\ p(\mathbf{Z}, \bar{\mathbf{Z}}) = \mathbf{0} \Longrightarrow (pq)(\mathbf{Z}, \bar{\mathbf{Z}}) = \mathbf{0}.$$

If γ admits a representing measure, then $M(n)$ is positive and recursively generated [CF2, Remark 3.15]. Although the converse is false [CF4], we have the following result of [CF3] which is the basis for Sections 2 and 3.

Theorem 1.4 *([CF3, Theorem 3.1]) Suppose $M(n)$ is positive and recursively generated. If $\mathbf{Z^k} = p(\mathbf{Z}, \bar{\mathbf{Z}})$ for some $p \in \mathcal{P}_{k-1}$, where $1 \leq k \leq [n/2]+1$, then $M(n)$ admits a unique flat extension $M(n+1)$.*

In the sequel we say that $M(n)$ admits an *analytic constraint* if there exists k, $1 \leq k \leq n$, such that $\mathbf{Z^k} = p(\mathbf{Z}, \bar{\mathbf{Z}})$ in $\mathcal{C}_{M(n)}$ for some $p \in \mathcal{P}_{k-1}$. [CF4, Example 4.4] illustrates $\gamma \equiv \gamma^{(6)}$ such that $M(3)(\gamma)$ (positive and recursively generated) admits an analytic constraint of the form $\mathbf{Z^3} = \alpha \bar{\mathbf{Z}}^{\mathbf{2}}$, but γ has no representing measure. In Section 2 we extend Theorem 1.4 so as to solve TCMP for moment matrices with analytic constraints. For the case when $k > [n/2]+1$, Algorithm 2.1 and Theorem 2.2 together show how to test (within the realm of elementary linear algebra) whether or not $M(n)$ admits a positive, recursively generated extension $M(2k-2)$; this is precisely the criterion for the existence of a finitely atomic representing measure. If this criterion is satisfied, then the unique minimal representing measure may be explicitly constructed using Theorem 1.6 (below).

To verify matrix positivity we will generally employ a criterion due to Smul'jan [Smu]. Consider a block matrix $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$.

Proposition 1.5 *$M \geq 0$ if and only if $A \geq 0$, $B = AW$ for some matrix W (equivalently, Ran $B \subset$ Ran A), and $C \geq W^*AW$. In this case, rank M = rank A if and only if $C = W^*AW$.*

We will have occasion to use the Extension Principle [F1, Proposition 2.4]: If $M \geq 0$, then each linear dependence relation in the column space of A extends to a corresponding relation in the column space of M.

The following Flat Extension Theorem is our main tool for explicitly constructing minimal representing measures in the presence of flat extensions. Observe that an extension of $M(n)$ to $M(n+1)$ is completely determined by a choice of "new moments" of degrees $2n+1$ and $2n+2$; moreover, all of these moments (or their conjugates) appear in column $\mathbf{Z^{n+1}}$ of $M(n+1)$, so $M(n+1)$ is determined by $M(n)$ and $\mathbf{Z^{n+1}}$. Let $[\mathbf{Z^{n+1}}]_n$ denote the truncation of $\mathbf{Z^{n+1}}$ through components indexed by monomials of degree at most n; this coincides with the left-most column vector of block $B \equiv (B_{i,n+1})_{0 \leq i \leq n}$. If $M(n+1)$ is a *positive* extension of $M(n)$, then (from Proposition 1.5) $[\mathbf{Z^{n+1}}]_n \in Ran\ M(n)$, so there exists $p \in \mathcal{P}_n$ such that

$$[\mathbf{Z^{n+1}}]_n = p(\mathbf{Z}, \bar{\mathbf{Z}}) \ in\ \mathcal{C}_{M(n)}. \tag{1.1}$$

Further, if $M(n+1)$ is a *flat* extension of $M(n) \geq 0$, then Proposition 1.5 implies

$$\mathbf{Z^{n+1}} = p(\mathbf{Z}, \bar{\mathbf{Z}}) \ in\ \mathcal{C}_{M(n+1)}. \tag{1.2}$$

A flat extension $M(n+1)$ of $M(n) \geq 0$ is thus completely determined from $M(n)$ by a choice of new moments of degree $2n+1$ via the equivalence of (1.1) and (1.2).

Theorem 1.6 *([CF2, Theorem 4.7, Corollary 5.12]) Suppose $M(n+1)$ is a flat extension of $M(n) \geq 0$, determined by a relation $\mathbf{Z^{n+1}} = p(\mathbf{Z}, \bar{\mathbf{Z}})$ for some $p \in \mathcal{P}_n$. Then $M(n+1)$ admits unique successive flat (positive) moment matrix extensions $M(n+2)$, $M(n+3)$, ..., where $M(n+d+1)$ is determined from $M(n+d)$ (via (1.1)-(1.2)) by the relation*

$$[\mathbf{Z^{n+d+1}}]_{n+d} = (z^d p)(\mathbf{Z}, \bar{\mathbf{Z}}) \ in\ \mathcal{C}_{M(n+d)} \quad (d \geq 1). \tag{1.3}$$

Let r = rank $M(n)$; in $M(r)$ there is a relation of the form $\mathbf{Z^r} = a_0\mathbf{1} + a_1\mathbf{Z} + \cdots + a_{r-1}\mathbf{Z^{r-1}}$. The r distinct roots of $z^r - (a_0 + \cdots + a_{r-1}z^{r-1})$, z_0, ..., z_{r-1}, comprise the support of an r-atomic minimal representing measure for γ, with densities $\rho_0, \ldots, \rho_{r-1}$ determined by the Vandermonde equation

$$V(z_0, \ldots, z_{r-1})(\rho_0, \ldots, \rho_{r-1})^t = (\gamma_{00}, \gamma_{01}, \ldots, \gamma_{0,r-1})^t. \tag{1.4}$$

We note that if $r > 2n+1$, then some of the analytic moments $\gamma_{0,j}$ in (1.4) are "new" moments, which are recursively generated for $M(r)$ via (1.3).

Acknowledgment Many of the calculations associated with Theorem 3.1 were carried out using the software tool *Mathematica* [Wol].

2 Solution of TCMP for moment matrices with analytic constraints

Let $M(n)$ be a positive, recursively generated moment matrix with an analytic constraint. Thus there exists k, $1 \leq k \leq n$, such that $\mathbf{Z^k} = p(\mathbf{Z}, \bar{\mathbf{Z}})$ in $\mathcal{C}_{M(n)}$ for some $p \in \mathcal{P}_{k-1}$. To solve TCMP for $\gamma^{(2n)}$, we first describe an algorithm which determines (in a finite sequence of steps) whether or not $M(n)$ admits a positive, recursively generated extension $M(n+1)$.

We denote a moment matrix extension $M(n+1)$ by the block decomposition

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}, \tag{2.1}$$

where $B = (B_{i,n+1})_{0 \leq i \leq n}$ and $C = B_{n+1,n+1}$. The Extension Principle implies that if $M(n+1) \geq 0$, then $\mathbf{Z^k} = p(\mathbf{Z}, \bar{\mathbf{Z}})$ in $\mathcal{C}_{M(n+1)}$. If, in addition, $M(n+1)$ is recursively generated, then

$$\mathbf{Z^{n+1}} = (z^{n+1-k}p)(\mathbf{Z}, \bar{\mathbf{Z}}) \ in \ \mathcal{C}_{M(n+1)}, \tag{2.2}$$

whence

$$[\mathbf{Z^{n+1}}]_n = (z^{n+1-k}p)(\mathbf{Z}, \bar{\mathbf{Z}}) \ in \ \mathcal{C}_{M(n)}. \tag{2.3}$$

Note also that $[\mathbf{Z^{n+1}}]_n$ is independent of the analytic constraint; indeed, if we also have $\mathbf{Z^j} = q(\mathbf{Z}, \bar{\mathbf{Z}})$ in $\mathcal{C}_{M(n)}$, where $1 \leq j \leq n$ and $q \in \mathcal{P}_{j-1}$, then the Extension Principle and property (RG) imply that $(z^{n-j+1}q)(\mathbf{Z}, \bar{\mathbf{Z}}) = \mathbf{Z^{n+1}} = (z^{n-k+1}p)(\mathbf{Z}, \bar{\mathbf{Z}})$ in $\mathcal{C}_{M(n+1)}$, whence $[(z^{n+1-j}q)(\mathbf{Z}, \bar{\mathbf{Z}})]_n = [(z^{n+1-k}p)(\mathbf{Z}, \bar{\mathbf{Z}})]_n$ in $\mathcal{C}_{M(n)}$.

Algorithm 2.1 *Test for existence of positive, recursively generated extensions of moment matrices with analytic constraints.*

i) To construct a positive, recursively generated extension $M(n+1)$, we begin by defining $[\mathbf{Z^{n+1}}]_n$ via (2.3). This definition uniquely determines (up to conjugation) all "new" moments of degree $2n+1$, and thus determines block $B_{n,n+1}$ and the corresponding block B of (2.1). If *Ran* B

is not contained in $Ran\ M(n)$, then, by Proposition 1.5, there is no positive, recursively generated extension $M(n+1)$.

ii) Suppose $Ran\ B \subset Ran\ M(n)$ and let W satisfy $B = M(n)W$ (cf. Proposition 1.5). Since $M(n+1)$ is to be positive and recursively generated, we use the Extension Principle and block B^* to define $\mathbf{Z^{n+1}}$ for $M(n+1)$ by $\mathbf{Z^{n+1}} = (z^{n+1-k}p)(\mathbf{Z}, \bar{\mathbf{Z}})$ in the column space of $\begin{pmatrix} M(n) \\ B^* \end{pmatrix}$. This relation uniquely determines (up to conjugation) all new moments of degree $2n+2$, and thus uniquely determines $C \equiv B_{n+1,n+1}$. From (2.1) and Proposition 1.5, the resulting extension $M(n+1)$ is positive if and only if $C \geq W^*M(n)W$. If $M(n+1)$ is not positive, then $M(n)$ admits no positive, recursively generated moment matrix extension.

iii) If $M(n+1)$ is positive, we then determine whether or not it is recursively generated. Since $M(n)$ is recursively generated, to make this determination, the Extension Principle implies that it suffices to consider relations in $\mathcal{C}_{M(n+1)}$ of the form

$$\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}} = s(\mathbf{Z}, \bar{\mathbf{Z}}),\ \textit{where } i+j = n \textit{ and } s \in \mathcal{P}_n. \tag{2.4}$$

In order that $M(n+1)$ satisfy (RG) it is necessary and sufficient to verify that for each such relation, $\bar{\mathbf{Z}}^{\mathbf{i+1}}\mathbf{Z}^{\mathbf{j}} = (\bar{z}s)(\mathbf{Z}, \bar{\mathbf{Z}})$ and $\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j+1}} = (zs)(\mathbf{Z}, \bar{\mathbf{Z}})$. Moreover, by considering a basis for $\mathcal{C}_{M(n+1)}$ consisting of a maximal independent subset of $\{\bar{\mathbf{Z}}^{\mathbf{i}}\mathbf{Z}^{\mathbf{j}}\}_{0 \leq i+j \leq n+1}$, it suffices to consider a relation as in (2.4) in which $s(\mathbf{Z}, \bar{\mathbf{Z}})$ is expressed as a linear combination of basis elements of degree at most n. Since there are at most finitely many such relations, it is possible to check for property (RG) in a finite number of steps. $\square$

If $M(n+1)$ (as just defined) is positive and recursively generated, then it is the unique such extension, and is called the *analytic extension* of $M(n)$; if such an extension does not exist, then γ admits no finitely atomic representing measure (cf. Theorem 1.2).

Theorem 2.2 *Suppose $M(n)$ is positive, recursively generated, and has an analytic constraint $\mathbf{Z^k} = p(\mathbf{Z}, \bar{\mathbf{Z}})$, where $1 \leq k \leq n$ and $p \in \mathcal{P}_{k-1}$. Then $\gamma^{(2n)}$ admits a finitely atomic representing measure if and only if $k \leq [n/2]+1$, or $k > [n/2]+1$ and $M(n)$ admits successive analytic extensions $M(n+1), \ldots, M(2k-2)$ as determined by Algorithm 2.1. In this case, $\gamma^{(2n)}$ has a unique minimal representing measure, which is rank $M(n)$-atomic if $k \leq [n/2]+1$ and is rank $M(2k-2)$-atomic if $k > [n/2]+1$.*

Proof. Suppose μ is a finitely atomic representing measure for $\gamma^{(2n)}$. Then μ has convergent moments of all orders, and the successive moment matrices $M(n+1)[\mu],\ M(n+2)[\mu], \ldots$ are positive and recursively generated (since μ is a representing measure for each). Since $M(n)$ has an analytic constraint, it follows from Algorithm 2.1 that the extensions $M(n+j)[\mu]$ $(j \geq 1)$ are the unique successive analytic extensions of $M(n)$.

For the converse, let $d = n$ if $k \leq [n/2]+1$ and let $d = 2k-2$ if $k > [n/2]+1$. The hypothesis implies that in $\mathcal{C}_{M(d)}$ we have $\mathbf{Z^k} = p(\mathbf{Z}, \bar{\mathbf{Z}})$ for $p \in \mathcal{P}_{k-1}$, where $k \leq [d/2]+1$. Since $M(d)$ is positive and recursively generated, Theorem 1.4 implies that $M(d)$ has a unique flat extension $M(d+1)$, and Theorem 1.1 yields a corresponding *rank* $M(d)$-atomic representing measure for $\gamma^{(2n)}$.

We next address size and uniqueness of minimal representing measures. For the case $k \leq [n/2]+1$, it follows directly from Theorem 1.4 that $M(n)$ has a unique flat extension, so Theorems 1.1 and 1.6 imply that there exists a unique *rank* $M(n)$-atomic minimal representing measure. Suppose next that $k > [n/2]+1$; the flat extension $M(2k-1)$ of $M(2k-2)$ corresponds to a representing measure μ for γ with $s \equiv$ *rank* $M(2k-2)$ atoms. Let ν denote a minimal respresenting measure for γ and let $r =$ *card supp* ν; we seek to show that $r = s$ and $\nu = \mu$. Since ν is r-atomic, *rank* $M(r-1)[\nu] = r$ (cf. [CF4, Theorem 4.7]). As above, $M(n+j)[\nu]$ $(j \geq 1)$ is a sequence of successive positive, recursively generated extensions of $M(n)$, so by the uniqueness of analytic extensions,

$$M(n+j)[\nu] = M(n+j) \quad (0 \leq j \leq 2k-2-n). \tag{2.5}$$

Suppose $r-1 \geq 2k-2$. Then $r =$ *rank* $M(r-1)[\nu] \geq$ *rank* $M(2k-2)[\nu] =$ *rank* $M(2k-2) = s \geq r$. Thus μ is a minimal representing measure; uniqueness follows from the uniqueness of analytic extensions and Theorem 1.6. Finally, suppose $r-1 < 2k-2$. Since ν is a representing measure for the moments of $M(r+i)[\nu]$ $(i \geq 0)$, (2.5) implies $r =$ *card supp* $\nu \geq$ *rank* $M(2k-2)[\nu] =$ *rank* $M(2k-2) = s \geq r$. Thus μ is minimal and uniqueness now follows readily from Theorem 1.6. □

3 Example of a minimal representing measure not corresponding to a flat extension.

In this section we use Theorem 2.2 to display a moment sequence $\gamma^{(6)}$ such that the minimal representing measure is 9-atomic, while *rank* $M(3)(\gamma) = 8$; the minimal measure therefore does not correspond to a flat extension of $M(3)(\gamma)$ (cf. Theorem 1.1).

Let $\delta > 0$, $y \in \mathbf{R}$ and define $\gamma_{00} = 1$, $\gamma_{11} = 1$, $\gamma_{22} = 1+\delta$, $\gamma_{14} = y$, $\gamma_{33} = \gamma_{06} = y^2/\delta$, $\gamma_{01} = \gamma_{02} = \gamma_{12} = \gamma_{03} = \gamma_{13} = \gamma_{04} = \gamma_{23} = \gamma_{05} = \gamma_{24} = \gamma_{15} = 0$. The corresponding moment matrix

$M(3)$ is of the form

$$M(3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1+\delta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1+\delta & 0 \\ 0 & 0 & 0 & 1+\delta & 0 & 0 & 0 & 0 & y & 0 \\ 1 & 0 & 0 & 0 & 1+\delta & 0 & y & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 1+\delta & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & y^2/\delta & 0 & 0 & y^2/\delta \\ 0 & 1+\delta & 0 & 0 & 0 & y & 0 & y^2/\delta & 0 & 0 \\ 0 & 0 & 1+\delta & y & 0 & 0 & 0 & 0 & y^2/\delta & 0 \\ 0 & 0 & 0 & 0 & y & 0 & y^2/\delta & 0 & 0 & y^2/\delta \end{pmatrix},$$

or, in block form,

$$M(3) = \begin{pmatrix} M(2) & B_3 \\ B_3^* & C_3 \end{pmatrix}, \tag{3.1}$$

where $B_3 = (B_{i,3})_{0 \le i \le 2}$ and $C_3 = B_{3,3}$.

Since $M(2)$ is positive and invertible, it follows from (3.1) and Proposition 1.5 that $M(3) \ge 0$ if and only if $y^2 \ge \delta(1+\delta)^3$, in which case *rank* $M(3) =$ *rank* $M(2)$ if and only if $y^2 = \delta(1+\delta)^3$. Thus, if $y^2 = \delta(1+\delta)^3$, then $M(3)$ has a unique flat extension, and $\gamma^{(6)}$ has a unique finitely atomic representing measure, which is 6-atomic (Case of Flat Data [CF2]).

Theorem 3.1 *If $y^2 > \delta(1+\delta)^3$, then rank $M(3) = 8$, and the unique minimal representing measure for $\gamma^{(6)}$ is 9-atomic.*

Proof. In $\mathcal{C}_{M(3)}$ we have

$$\mathbf{Z^3} = \mathbf{\bar{Z}^3} = \frac{-y}{\delta}\mathbf{1} + \frac{y}{\delta}\mathbf{Z\bar{Z}}; \tag{3.2}$$

a calculation shows that $\{\mathbf{1},\ \mathbf{Z},\ \mathbf{\bar{Z}},\ \mathbf{Z^2},\ \mathbf{Z\bar{Z}},\ \mathbf{\bar{Z}^2},\ \mathbf{Z^2\bar{Z}},\ \mathbf{Z\bar{Z}^2}\}$ forms a basis for $\mathcal{C}_{M(3)}$, so *rank* $M(3) = 8$ and $M(3)$ is recursively generated. Now (3.2) shows that $M(3)$ has an analytic constraint with $n = k = 3$. To complete the proof, using Theorem 2.2, it thus suffices to show that $M(3)$ admits an analytic extension $M(4)$ satisfying *rank* $M(4) = 9$, and to this end we follow Algorithm 2.1.

We denote the proposed extension $M(4)$ by $\begin{pmatrix} M(3) & B_4 \\ B_4^* & C_4 \end{pmatrix}$, with $B_4 = (B_{i,4})_{0 \le i \le 3}$ and $C_4 = B_{44}$. Using (3.2) and (2.3) we define $[\mathbf{Z^4}]_3$ in $\mathcal{C}_{M(3)}$ by

$$[\mathbf{Z^4}]_3 = \frac{-y}{\delta}\mathbf{Z} + \frac{y}{\delta}\mathbf{Z^2\bar{Z}}, \tag{3.3}$$

whence the B_4 block for $M(4)$ is given by

$$B_4 = \begin{pmatrix} 0 & 0 & 1+\delta & 0 & 0 \\ y & 0 & 0 & y & 0 \\ 0 & y & 0 & 0 & y \\ 0 & y^2/\delta & 0 & 0 & y^2/\delta \\ 0 & 0 & y^2/\delta & 0 & 0 \\ y^2/\delta & 0 & 0 & y^2/\delta & 0 \\ 0 & 0 & w & 0 & 0 \\ w & 0 & 0 & w & 0 \\ 0 & w & 0 & 0 & w \\ 0 & 0 & w & 0 & 0 \end{pmatrix}, \tag{3.4}$$

where $w = \frac{y(y^2/\delta - 1 - \delta)}{\delta}$. The conclusion that $Ran\ B_4 \subset Ran\ M(3)$ now results from the following relations in $\mathcal{C}_{M(3)}$:

$$[\mathbf{Z^4}]_3 = \frac{-y}{\delta}\mathbf{Z} + \frac{y}{\delta}\mathbf{Z^2\bar{Z}};\quad [\mathbf{\bar{Z}^4}]_3 = \frac{-y}{\delta}\mathbf{\bar{Z}} + \frac{y}{\delta}\mathbf{\bar{Z}^2 Z}; \tag{3.5}$$

$$[\mathbf{Z^3\bar{Z}}]_3 = \frac{-y}{\delta}\mathbf{\bar{Z}} + \frac{y}{\delta}\mathbf{Z\bar{Z}^2};\quad [\mathbf{\bar{Z}^3 Z}]_3 = \frac{-y}{\delta}\mathbf{Z} + \frac{y}{\delta}\mathbf{Z^2\bar{Z}}; \tag{3.6}$$

$$[\mathbf{Z^2\bar{Z}^2}]_3 = \frac{\delta(1+\delta)^2 - y^2}{\delta^2}\mathbf{1} + \frac{y^2 - \delta(1+\delta)}{\delta^2}\mathbf{Z\bar{Z}}; \tag{3.7}$$

moreover, these relations determine W satisfying $B_4 = M(3)W$.

Using (2.2), (3.2), the Extension Principle, and block B^*, we next define $\mathbf{Z^4}$ for $M(4)$ by $\mathbf{Z^4} = \frac{-y}{\delta}\mathbf{Z} + \frac{y}{\delta}\mathbf{Z^2\bar{Z}}$ (in the column space of $\begin{pmatrix} M(n) \\ B_4^* \end{pmatrix}$). Thus $\mathbf{Z^4} = (0, y, 0, 0, 0, y^2/\delta, 0, w, 0, 0, u, 0, 0, u, 0)^t$, where $u = \frac{y^2(y^2 - \delta - 2\delta^2)}{\delta^3}$. Moment matrix structure now dictates that block $C_4 (= B_{44})$ is of the form

$$C_4 = \begin{pmatrix} u & 0 & 0 & u & 0 \\ 0 & u & 0 & 0 & u \\ 0 & 0 & u & 0 & 0 \\ u & 0 & 0 & u & 0 \\ 0 & u & 0 & 0 & u \end{pmatrix}. \tag{3.8}$$

From (3.1), (3.4) and (3.8) we see that in $\mathcal{C}_{M(4)}$ we have

$$\mathbf{Z^4} = \frac{-y}{\delta}\mathbf{Z} + \frac{y}{\delta}\mathbf{Z^2\bar{Z}};\ \mathbf{\bar{Z}^4} = \frac{-y}{\delta}\mathbf{\bar{Z}} + \frac{y}{\delta}\mathbf{Z\bar{Z}^2};$$

$$\mathbf{Z^3\bar{Z}} = \frac{-y}{\delta}\mathbf{\bar{Z}} + \frac{y}{\delta}\mathbf{Z\bar{Z}^2};\ \mathbf{\bar{Z}^3 Z} = \frac{-y}{\delta}\mathbf{Z} + \frac{y}{\delta}\mathbf{\bar{Z}Z^2},$$

whence $M(4)$ is recursively generated and $8 = rank\ M(3) \le rank\ M(4) \le 9$.

Proposition 1.5 and (3.1) show that $M(4) \ge 0$ if and only if $C_4 \ge W^* M(3) W$, or

$$u \ge \frac{(\delta(1+\delta)^2 - y^2)(1+\delta)}{\delta^2} + \frac{(y^2 - \delta(1+\delta)y^2}{\delta^3},$$

which simplifies to $y^2 \ge \delta(1+\delta)^3$; moreover, $rank\ M(4) = rank\ M(3)$ if and only if $y^2 = \delta(1+\delta)^3$. Since $y^2 > \delta(1+\delta)^3$, it follows that $M(4)$ is positive, recursively generated and $rank\ M(4) = 9$. The result now follows from Theorem 2.2. □

We may compute the unique 9-atomic (minimal) representing measure for $\gamma^{(6)}$ as follows. Since the analytic extension $M(4)$ satisfies the conditions of Theorem 1.4 (with $n = 4$ and $k = 3$), $M(4)$ has a unique flat extension $M(5)$, and Theorem 1.6 implies that $M(5)$ admits unique successive flat extensions $M(6), \ldots, M(9)$. A calculation of $M(9)$ using (1.3) implies that in $\mathcal{C}_{M(9)}$ we have the relation

$$\mathbf{Z^9} = \frac{-y^3}{\delta^3}\mathbf{1} - 3\frac{y^2}{\delta^2}\mathbf{Z^3} + \frac{y^3 - 3\delta^2 y}{\delta^3}\mathbf{Z^6}. \tag{3.9}$$

The atoms of the minimal representing measure are the 9 distinct roots of the characteristic polynomial corresponding to (3.9) (cf. Theorem 1.6), and the densities may be computed via the Vandermonde equation (1.4). We illustrate with a numerical example.

Example 3.2 *Let $\delta = 1$, $y = 3$. From (3.9), the atoms of the minimal measure are the roots of $z^9 = -27 - 27z^3 + 18z^6$, i.e., $z_0 \approx 2.53209$, $z_1 \approx -0.879385$, $z_2 \approx 1.3473$, $z_3 \approx -1.26604 - 2.19285i$, $z_4 = \bar{z}_3$, $z_5 \approx -0.673648 - 1.16679i$, $z_6 = \bar{z}_5$, $z_7 \approx 0.439693 - 0.76157i$, $z_8 = \bar{z}_7$. The corresponding densities, determined from (1.4), are $\rho_0 = \rho_3 = \rho_4 \approx 0.0104859$, $\rho_1 = \rho_7 = \rho_8 \approx 0.307069$, $\rho_2 = \rho_5 = \rho_6 \approx 0.0157786$.*

References

[CR] R. Cools and P. Rabinowitz, Monomial cubature rules since "Stroud": a compilation, J. Comp. Appl. Math. 48(1993), 309-326.

[CF1] R. Curto and L. Fialkow, Recursiveness, positivity, and truncated moment problems, Houston J. Math. 17(1991), 603-635.

[CF2] R. Curto and L. Fialkow, Solution of the truncated complex moment problem for flat data, Memoirs Amer. Math. Soc. no. 568, Amer. Math. Soc., Providence, 1996.

[CF3] R. Curto and L. Fialkow, Flat extensions of positive moment matrices: Relations in analytic or conjugate terms, Operator Th.: Adv. Appl., to appear.

[CF4] R. Curto and L. Fialkow, Flat extensions of positive moment matrices: recursively generated relations, Memoirs Amer. Math. Soc., to appear.

[F1] L. Fialkow, Positivity, extensions and the truncated complex moment problem, Contemporary Math. 185(1995), 133-150.

[F2] L. Fialkow, Multivariable quadrature and extensions of moment matrices, preprint, 1996.

[McC] J.E. McCarthy, private communication, 1995.

[P] M. Putinar, On Tchakaloff's Theorem, preprint 1995.

[Smu] J.L. Smul'jan, An operator Hellinger integral (Russian), Mat. Sb. 91(1959), 381-430.

[Str] A.H. Stroud, Approximate Calculation of Multiple Integrals, Prentice-Hall, 1971.

[W] Wolfram Research, Inc., *Mathematica*, Version 2.1, Wolfram Research, Inc., Champaign, Illinois, 1992.

Department of Mathematics and Computer Science

State University of New York at New Paltz,

New Paltz, NY 12561

e-mail: fialkow@mcs.newpaltz.edu