

# POSITIVITY OF POLYNOMIALS

ROSEMARY HOLGUIN, MATH/SECONDARY EDUCATION, CLASS OF 2008 AND  
CYNDHIA SILVERIO, ELECTRICAL ENGINEERING, CLASS OF 2009.

## CONTENTS

1. Introduction	1
2. Positivity of Polynomials on the Real Line	4
3. Positivity of Polynomials on the Half Line	11
4. Positivity of Polynomials on the Interval $[0,1]$	17
5. Appendix	29
6. Graphs	31
References	32

## 1. INTRODUCTION

The main purpose of this project is to study representations of positive polynomials. Let  $p(x)$  be a real polynomial of degree  $n$ :

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where  $a_0, \dots, a_n$  are real numbers and  $a_n \neq 0$ . Let  $K \subseteq \mathbb{R}$ . A polynomial  $p(x)$  is *positive* on  $K$  if  $p(x) \geq 0$  for all  $x \in K$ . Positive polynomials play an important role in the study of Moment Theory[1][2]. In the early 20th century positive polynomials were studied using methods of Complex Analysis. Polya and Szego [6] includes representation theorems for polynomials that are positive on the real line  $\mathbb{R}$ , the half line  $\mathbb{R}^+ \equiv [0, +\infty)$  and the closed interval  $[-1,1]$ . In the content of this project we will discuss the same results by using simpler methods, and we will illustrate the representation theorems through examples.

The three basic representation theorems that we will study are the following:

---

This project was carried out during the Spring, 2007 and Summer I, 2007 semesters with Professor Lawrence A. Fialkow as the mentor. During Spring, 2007 the project was sponsored by SUNY/NSF Alliance for Minority Participation at SUNY New Paltz. The New Paltz AMP Director is Professor Stacie Nunes from the Department of Physics. During Summer I, 2007 the project was sponsored under National Science Foundation Grant DMS-0457138. The authors thank the sponsors for their support.

(i)  $p(x) \geq 0$  for all  $x \in \mathbb{R} \implies p(x) = g(x)^2 + h(x)^2$  for certain polynomials  $g, h$ .  
 (See Polya and Szegő [6, Part VI, Sec. 6, 44, page 77].)

(ii)  $p(x) \geq 0$  for all  $x \geq 0 \implies p(x) = r(x)^2 + s(x)^2 + x(u(x)^2 + v(x)^2)$  for certain polynomials  $r(x), s(x), u(x)$  and  $v(x)$  [6, Part VI, Sec. 6, 45, page 78].

(iii)  $p(x) \geq 0$  for all  $x \in [0, 1] \implies p(x) = r(x)^2 + s(x)^2 + x(1-x)(u(x)^2 + v(x)^2)$  for certain polynomials  $r(x), s(x), u(x)$  and  $v(x)$  [6, Part VI, Sec. 6, 46, page 78].

The theory of positive polynomials goes beyond polynomials in one variable. Consider a polynomial in two real variables,

$$p(x, y) = \sum_{i+j=0}^n a_{ij} x^i y^j.$$

A famous result of David Hilbert [5] shows that theorem (i) (above) cannot be generalized to two variables.

**Theorem 1.1.** (Hilbert [5])

If  $p(x, y) \geq 0$  for all  $x, y$  and  $\deg p \leq 4$ , then  $p(x, y)$  is a sum of squares of polynomials. However, there exists a polynomial  $p(x, y)$  of  $\deg 6$  such that  $p(x, y) \geq 0$  for all  $x, y$ , but  $p(x, y)$  is not a sum of squares.

Although theorem (i) cannot be generalized to higher dimensions, theorem (iii) (for a closed interval) can be generalized to any number of variables. Here we state a simplified version of a famous theorem of Konrad Schmudgen [7]:

**Theorem 1.2.** (Schmudgen [7]) Let  $d \geq 1$  and let  $x = (x_1, \dots, x_d)$ . Suppose  $q(x)$  is a polynomial such that  $K \equiv \{x \in \mathbb{R}^d : q(x) \geq 0\}$  is closed and bounded. If  $p(x) > 0$  for every  $x$  in  $K$ , then  $p(x) = \sum p_i(x)^2 + q(x) \sum q_i(x)^2$  (for certain finite families of polynomials  $p_i(x)$  and  $q_i(x)$ ).

In theorem (iii) we have  $d = 1$ ,  $K = [0, 1]$  and  $q(x) = x(1-x)$ .

For two or more variables, we sometimes get representations for positive polynomials when  $K$  is unbounded. Here is a sample result for the case when  $K$  is a parabola,  $y = x^2$ .

**Theorem 1.3.** (R. Curto, L. Fialkow [3]) Let  $q(x, y) = y - x^2$  and let  $K = \{(x, y) : q(x, y) = 0\}$ . If  $p(x, y) > 0$  for all  $(x, y)$  with  $y = x^2$ , then  $p(x, y) = \sum p_i(x, y)^2 + q(x, y) \sum q_i(x, y)^2$ , for all  $x, y$  (for certain polynomials  $p_i$  and  $q_i$ ). Here,  $\deg p_i^2, \deg q_i^2 \leq \deg p$ .

Let  $\mathbb{R}$  denote the set of real numbers and let  $\mathbb{R}^+$  be the set of non-negative real numbers, i.e.,  $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ . We will use the following properties of  $\mathbb{R}^+$ : if  $x$  and  $y \in \mathbb{R}^+$ , then

- 1)  $xy \in \mathbb{R}^+$
- 2)  $x + y \in \mathbb{R}^+$

3)  $y \in \mathbb{R}^+ \iff y = x^2$  for some  $x \in \mathbb{R}$ .

Let  $P$  denote the set of real polynomials  $p : \mathbb{R} \rightarrow \mathbb{R}$ , and let  $K$  be a subset of  $\mathbb{R}$ . Let  $P_K^+$  be the set of polynomials that are non-negative on  $K$  :  $P_K^+ = \{p \in P : p(x) \geq 0 \text{ for every } x \text{ in } K.\}$  For  $K = \mathbb{R}$ , we let  $P^+ = P_{\mathbb{R}}^+$ . If  $f, g \in P_K^+$  then  $f+g$  and  $fg$  are in  $P_K^+$ , where

- 1)  $(f + g)(x) = f(x) + g(x)$ .
- 2)  $(fg)(x) = f(x)g(x)$ .

Let  $\deg p$  denote the degree of a polynomial  $p$ , and let  $P_n = \{p \in P : \deg p \leq n\}$ . As an application, in an appendix we will use theorem (i) to describe the positive linear functionals  $L : P_{2n} \rightarrow \mathbb{R}$ .

## 2. POSITIVITY OF POLYNOMIALS ON THE REAL LINE

Let  $p(x)$  be a real polynomial,

$$p(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots + a_nx^n = \sum_{i=0}^n a_ix^i,$$

where  $a_0, \dots, a_n \in \mathbb{R}$ ,  $a_n \neq 0$  and  $n$  is the *degree* of  $p(x)$ . Let  $P$  be the set of polynomials and let  $P^+$  be the set of positive polynomials. Therefore,

$$p, q \in P^+ \implies p + q, pq \in P^+ \text{ and } p \in P \implies p^2 \in P^+.$$

For  $x \in \mathbb{R}^+$ , there exists  $y \in \mathbb{R}$  such that  $y = x^2$ . Does a similar property hold for polynomials? In other words, if  $p \in P^+$ , is there a polynomial  $q \in P$  such that  $p = q^2$ ?

**Example 2.1.**

$$\text{Let } p(x) = x^2 + 1.$$

Is  $p = q^2$  for some  $q \in P$ ? (See graph 1. on page 31.a) If so, then  $2 = \deg p = 2 \cdot \deg q$ , so  $\deg q = 1$ . Let  $q = a + bx$  for some  $a, b \in \mathbb{R}$  and suppose that  $p = q^2$ .

$$\begin{aligned} \text{Then, } x^2 + 1 &= (a + bx)^2 \\ &= a^2 + 2abx + b^2x^2, \end{aligned}$$

$$\text{so } a^2 = 1, b^2 = 1 \text{ and } 2ab = 0.$$

But  $2ab = 0$  yields the conclusion that either  $a = 0$  or  $b = 0$ , which is a contradiction. So  $x^2 + 1$  is a strictly positive polynomial and is not the square of any polynomial.

Nevertheless,  $x^2 + 1 = x^2 + 1^2$ , which means that we can write  $x^2 + 1$  as the sum of the squares of *two* polynomials,

$$p(x) = g^2 + h^2,$$

where in this case  $g(x) = x$  and  $h(x) = 1$ .

We will study the following classical representation theorem:

**Theorem 2.2.** (See *Polya and Szego [6, Part VI, Sec. 6, 44, page 77].*)

$$p(x) \in P^+ \iff p(x) = g(x)^2 + h(x)^2 \text{ for certain } g, h \in P.$$

To prove Theorem 2.2, we will use the following well-known result (see [6]):

**Theorem 2.3.** (*Factorization Theorem*) If  $p(x) = a_n x^n + \dots + a_1 x + a_0$  for  $a_i \in \mathbb{R}$ , then  $p$  can be factored in the form  $p(x) = q_1(x) \cdots q_r(x) (x - x_1)^{n_1} \cdots (x - x_s)^{n_s}$ , where:

1)  $q_i(x)$  is a quadratic with no real roots (irreducible quadratic), i.e.,  $q_i(x) = A_i x^2 + B_i x + C_i$  with  $B_i^2 - 4A_i C_i < 0$ .

2)  $x_1, \dots, x_s$  are distinct real numbers; each  $n_i$  is a non-negative integer.

(In Theorem 2.3, either the  $q_i(x)$  or the  $(x - x_i)^{n_i}$  may be absent.)

**Example 2.4.** Let

$$p(x) = x^{10} - 27x^9 + 308x^8 - 1800x^7 + 4350x^6 + 9918x^5 \\ - 107320x^4 + 354600x^3 - 665375x^2 + 928125x - 937500.$$

Then,

$$p(x) = (x^2 + 3)(x^2 - 8x + 25)^2(x - 5)^3(x + 4).$$

Here,  $x^2 - 8x + 25$  is an irreducible quadratic;  $x^2 + 3 = x^2 + \sqrt{3}^2$  is an irreducible quadratic that is clearly the sum of two squares;  $(x - 5)^3$  and  $(x + 4)^1$  have odd degrees. The linear polynomials raised to odd powers, i.e.,  $(x + x_i)^{2k-1}$ , are not positive for every  $x$ ; they change from negative to positive at their roots  $x_i$ . (See graph 2. on page 31.a).□

*Proof of Theorem 2.2* Suppose  $p(x) \geq 0$  for all  $x \in \mathbb{R}$ . This implies that in the Factorization Theorem 2.3, if  $n_i > 0$ , then  $n_i$  is even, i.e.  $n_i = 2k_i$ . This is because if  $n_i$  is odd, then  $p(x)$  changes sign at  $x_i$ , a contradiction. So,  $(x - x_i)^{n_i} = (x - x_i)^{2k_i} = [(x - x_i)^k]^2$ .

Suppose first that the degree of  $p(x)$  equals 2, i.e.,

$$p(x) = ax^2 + bx + c,$$

with  $a \neq 0$ . Now  $c \geq 0$  because  $c = p(0) \geq 0$ . We claim that  $a > 0$ . For  $x \neq 0$ , we have

$$p(x) = x^2 \left( a + \frac{b}{x} + \frac{c}{x^2} \right).$$

As  $x \rightarrow +\infty$ ,  $\left( a + \frac{b}{x} + \frac{c}{x^2} \right) \rightarrow a$ . So if  $a < 0$ , then  $\left( a + \frac{b}{x} + \frac{c}{x^2} \right) < 0$  for large positive  $x$ . But then

$$p(x) = x^2 \left( a + \frac{b}{x} + \frac{c}{x^2} \right) < 0$$

for large positive  $x$ , which is a contradiction; therefore we have proved that  $a > 0$ .

Next we consider two cases, based on whether or not the quadratic  $p(x) \in P^+$  has any real roots. Recall that the real roots of  $p(x) = 0$  can be found with the quadratic equation:

$$(2.1) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Also, if  $p(x)$  has two distinct real roots  $r$  and  $s$  with  $r < s$ , then since  $a > 0$ ,  $p(x) < 0$  for  $r < x < s$ , which is a contradiction; so we can assume either that  $p$  has one real root or no real roots.

**Case I.**  $p(x)$  has exactly one root. From the quadratic formula (2.1),  $p(x)$  has exactly one real root if and only if  $b^2 - 4ac = 0$ . If  $b^2 - 4ac = 0$ , then  $r = \frac{-b}{2a}$  is the only real root, so,

$$p(x) = a(x - r)(x - r).$$

Let  $h(x) = \sqrt{a}(x - r) = \sqrt{a}x + \frac{b}{2\sqrt{a}}$ . Then we can conclude by saying that if  $b^2 - 4ac = 0$ , then

$$p(x) = h(x)^2 = \left( \sqrt{a}x + \frac{b}{2\sqrt{a}} \right)^2,$$

so  $p(x)$  is a square.

**Case II.**  $p(x)$  has no real roots, i.e.,  $b^2 - 4ac < 0$ . In this case, since  $b^2 < 4ac$ , then  $c > 0$ , and so  $\frac{b^2}{4c} < a$ . Now let  $a' = \frac{b^2}{4c} (< a)$  and let  $a'' = a - a' (> 0)$ , so that  $a = a' + a''$ . Then

$$\begin{aligned} p(x) &= ax^2 + bx + c \\ &= (a' + a'')x^2 + bx + c \\ &= a''x^2 + a'x^2 + bx + c. \end{aligned}$$

Note that  $a'' = a - a' = a - \frac{b^2}{4c} > 0$ , so  $a''x^2$  is a square, namely  $a''x^2 = r(x)^2$ , where  $r(x) = \sqrt{a - a'}x$ . Now we will show that  $q(x) \equiv a'x^2 + bx + c$  is also a square. In fact, the discriminant of  $q(x)$  equals  $b^2 - 4a'c = 0$ . Now  $q(x)$  has exactly one real root and  $a' = \frac{b^2}{4c} > 0$ , so by Case I,  $q(x)$  is a square, say  $q(x) = s(x)^2$ , where  $s(x) = \sqrt{a'}x + \frac{b}{2\sqrt{a'}}$ . So we can conclude by saying that if  $b^2 - 4ac < 0$ , then

$$p(x) = r(x)^2 + s(x)^2 = \left( \sqrt{a - a'}x \right)^2 + \left( \sqrt{a'}x + \frac{b}{2\sqrt{a'}} \right)^2,$$

so  $p(x)$  is a sum of two squares.

This completes the proof when  $\deg p = 2$ .

Now consider

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \text{ where } a_n \neq 0,$$

and assume  $p(x) \geq 0$  for all  $x$ . By the remarks at the beginning of the proof, the degree of  $p$  is even and in the factorization of  $p$ , every factor is either an irreducible quadratic or a linear factor that repeats an even number of times. Therefore, by the first part of the proof,  $p$  is a product of factors where each factor is either a square or a sum of two squares. To complete the proof, we need the following fact:

**Lemma 2.5.** *If  $p = p_1 p_2 \cdots p_m$  and each  $p_i$  is a sum of two squares, then  $p$  is a sum of two squares.*

*Proof.* By using induction, it is enough to prove this for  $m = 2$ . Let  $p = f^2 + g^2$  and  $q = h^2 + k^2$ . Then  $(fh + gk)^2 = f^2 h^2 + 2ghfk + g^2 k^2$ . Also,  $(fk - gh)^2 = f^2 k^2 - 2ghfk + g^2 h^2$ . When we add these equations together, the result is  $(fh + gk)^2 + (fk - gh)^2 = f^2 k^2 + g^2 h^2 + f^2 h^2 + g^2 k^2$ . Now we take  $(f^2 + g^2)(h^2 + k^2)$  and multiply; the result is  $f^2 h^2 + g^2 h^2 + f^2 k^2 + g^2 k^2$ . This proves that  $(f^2 + g^2)(h^2 + k^2) = (fh + gk)^2 + (fk - gh)^2$ .  $\square$

The proof of Theorem 2.2 is now complete.  $\square$

**Example 2.6.** Consider

$$p(x) = x^4 - 14x^3 + 86x^2 - 254x + 325.$$

(See graph 3. on page 31.b).

Now

$$p(x) = \underbrace{(x^2 - 8x + 25)}_{p_1} \underbrace{(x^2 - 6x + 13)}_{p_2}.$$

The discriminant of  $p_1$  is  $b^2 - 4ac = (-8)^2 - 4(1)(25) = -36 < 0$ . Therefore,  $p_1(x)$  has no real roots, so  $p_1(x) > 0$  for all  $x$ .

The discriminant of  $p_2$  is  $b^2 - 4ac = (-6)^2 - 4(1)(13) = -16 < 0$ . Therefore,  $p_2(x)$  has no real roots, so  $p_2(x) > 0$  for all  $x$ .

We will represent  $p(x)$  as  $p(x) = R(x)^2 + S(x)^2$ .

Note: Before beginning the proof, we use the notation of Case II the proof of Theorem 2.2.

For  $p_1 = x^2 - 8x + 25$ , we have  $a = 1$ ,  $a' = \frac{b^2}{4c} = \frac{64}{100} = \frac{16}{25}$ . Now,

$$s(x)^2 = \left( \sqrt{a'} \left( x + \frac{b}{2a'} \right) \right)^2$$

$$\begin{aligned}
&= \left( \sqrt{\frac{16}{25}} \left( x + \frac{-8}{\frac{2 \cdot 16}{25}} \right) \right)^2 \\
&= \frac{16}{25} \left( x + \frac{-8}{\frac{32}{25}} \right)^2 \\
&= \left[ \left( \frac{4}{5}x - 5 \right) \right]^2
\end{aligned}$$

Also,  $a'' = a - a' = 1 - \frac{16}{25} = \frac{9}{25}$ , so  $r(x) = \sqrt{a - a'x} = \frac{3}{5}x$ . Therefore, as in Case II,

$$p_1 = \left( \frac{4}{5}x - 5 \right)^2 + \left( \frac{3}{5}x \right)^2.$$

Next,  $p_2 = x^2 - 6x + 13$ , so we have:

$$a = 1, \quad a' = \frac{b^2}{4c} = \frac{36}{4(13)} = \frac{36}{52} = \frac{9}{13}, \quad a'' = 1 - \frac{9}{13} = \frac{4}{13}.$$

Using Case II again, we get

$$\left( \sqrt{\frac{9}{13}} \left( x + \frac{-6}{2\left(\frac{9}{13}\right)} \right) \right)^2 + \left( \frac{2}{\sqrt{13}}x \right)^2, \text{ so}$$

$$\begin{aligned}
p_2 &= \left( \sqrt{\frac{9}{13}} \left( x + \frac{-6}{2\left(\frac{9}{13}\right)} \right) \right)^2 + \left( \frac{2}{\sqrt{13}}x \right)^2 \\
&= \left( \frac{3}{\sqrt{13}}x - \frac{13}{\sqrt{13}} \right)^2 + \left( \frac{2}{\sqrt{13}}x \right)^2.
\end{aligned}$$

Now let  $f = \frac{4}{5}x - 5$ ,  $g = \frac{3}{5}x$ ,  $h = \frac{3}{\sqrt{13}}x - \frac{13}{\sqrt{13}}$ , and  $k = \frac{2}{\sqrt{13}}x$ . So we can use Lemma 2.5 to represent  $p = p_1 \cdot p_2$  as a sum of two squares:

$$\begin{aligned}
p(x) &= (fh + gk)^2 + (fk - gh)^2 \\
&= \left[ \left( \frac{4}{5}x - 5 \right) \left( \frac{3}{\sqrt{13}}x - \frac{13}{\sqrt{13}} \right) + \left( \frac{3}{5}x \right) \left( \frac{2}{\sqrt{13}}x \right) \right]^2 + \left[ \left( \frac{4}{5}x - 5 \right) \left( \frac{2}{\sqrt{13}}x \right) - \left( \frac{3}{5}x \right) \left( \frac{3}{\sqrt{13}}x - \frac{13}{\sqrt{13}} \right) \right]^2
\end{aligned}$$

$$= \left( \frac{18x^2 - 127x + 325}{5\sqrt{13}} \right)^2 + \left( \frac{-x(11+x)}{5\sqrt{13}} \right)^2. \quad \square$$

Now let's do another example:

**Example 2.7.** Let

$$p(x) = 45 + 37x + 20x^2 + 5x^3 + x^4$$

(See graph 4. on page 31.b).

$$= \underbrace{(x^2 + 3x + 5)}_{p_1} \underbrace{(x^2 + 2x + 9)}_{p_2}.$$

Using Case II of Theorem 2.2, we obtain

$$p_1 = \left[ \frac{3}{2\sqrt{5}} \left( x + \frac{10}{3} \right) \right]^2 + \left[ \sqrt{\frac{11}{20}} x \right]^2$$

and

$$p_2 = \left[ \frac{1}{3} (x + 9) \right]^2 + \left[ \sqrt{\frac{8}{9}} x \right]^2$$

Now we will use the Lemma 2.5 to represent  $p$  as a sum of 2 squares:

Let  $f = \frac{3}{2\sqrt{5}} \left( x + \frac{10}{3} \right)$ ,  $g = \sqrt{\frac{11}{20}} x$ ,  $h = \frac{1}{3} (x + 9)$ , and  $k = \sqrt{\frac{8}{9}} x$ . Then  $p = (f^2 + g^2)(h^2 + k^2) = (fh + gk)^2 + (fk - gh)^2$ . So,

$$\begin{aligned} p &= p_1 \cdot p_2 = \left[ \frac{3}{2\sqrt{5}} \left( x + \frac{10}{3} \right) \right]^2 + \left[ \sqrt{\frac{11}{20}} x \right]^2 \left[ \left[ \frac{1}{3} (x + 9) \right]^2 + \left[ \sqrt{\frac{8}{9}} x \right]^2 \right] \\ &= \left[ \frac{3}{2\sqrt{5}} \left( x + \frac{10}{3} \right) \frac{1}{3} (x + 9) + \sqrt{\frac{11}{20}} x \sqrt{\frac{8}{9}} x \right]^2 + \left[ \frac{3}{2\sqrt{5}} \left( x + \frac{10}{3} \right) \sqrt{\frac{8}{9}} x - \sqrt{\frac{11}{20}} x \frac{1}{3} (x + 9) \right]^2 \\ &= \underbrace{\left[ \frac{(9+x)(10+3x)}{6\sqrt{5}} + \frac{\sqrt{22}x^2}{3\sqrt{5}} \right]^2}_{r^2} + \underbrace{\left[ \frac{(10x+3x^2)\sqrt{2}}{3\sqrt{5}} - \sqrt{\frac{11}{5}} x \left( \frac{9+x}{6} \right) \right]^2}_{s^2}. \quad \square \end{aligned}$$

To show a more general application of Theorem 2.2, we will next demonstrate how the product of three polynomials can be represented as the sum of two squares, we will use this representation in a later example.

**Example 2.8.**

$$\text{Let } P(x) = P_1 \cdot P_2 \cdot P_3$$

where  $P_1 = r(x)^2 + s(x)^2$ ,  $P_2 = t(x)^2 + u(x)^2$ , and  $P_3 = v(x)^2 + y(x)^2$ .

Then, using the Lemma 2.5,

$$P(x) = \left( r(x)^2 + s(x)^2 \right) \cdot \left( t(x)^2 + u(x)^2 \right) \cdot \left( v(x)^2 + y(x)^2 \right) \\ = \left[ \left( r(x)t(x) + s(x)u(x) \right)^2 + \left( r(x)u(x) - s(x)t(x) \right)^2 \right] \cdot \left( v(x)^2 + y(x)^2 \right), \text{ and using}$$

Lemma 2.5 again, the last expression equals

$$\left[ \left( r(x)t(x) + s(x)u(x) \right) \cdot v(x) + \left( r(x)u(x) - s(x)t(x) \right) \cdot y(x) \right]^2 + \left[ \left( r(x)t(x) + s(x)u(x) \right) \cdot y(x) - \left( r(x)u(x) - s(x)t(x) \right) \cdot v(x) \right]^2$$

$$\text{So, } P(x) = P_1 \cdot P_2 \cdot P_3 = B(x)^2 + C(x)^2,$$

which is also the sum of two squares.  $\square$

### 3. POSITIVITY OF POLYNOMIALS ON THE HALF LINE

Now we turn to a representation theorem for polynomials that are non-negative for  $x \geq 0$ .

**Definition 3.1.** *In what follows, we say that  $p(x)$  is a Class I polynomial if  $p(x) = r(x)^2 + s(x)^2 + x(u(x)^2 + v(x)^2)$ , where  $r(x)$ ,  $s(x)$ ,  $u(x)$ ,  $v(x)$  are polynomials.*

**Theorem 3.2.** *[6, Part VI, Sec. 6, 45, page 78] A polynomial  $p(x)$  satisfies  $p(x) \geq 0$  for all  $x \geq 0$  if and only if  $p(x)$  is a Class I polynomial, i.e., there exist polynomials  $r(x)$ ,  $s(x)$ ,  $u(x)$  and  $v(x)$  such that:  $p(x) = r(x)^2 + s(x)^2 + x(u(x)^2 + v(x)^2)$ .*

Assume that  $p(x) \geq 0$  for all  $x \geq 0$ . Recall from Theorem 2.3 (the Factorization Theorem) that

$$p(x) = Aq_1(x)q_2(x)\dots q_r(x) (x - x_1)^{n_1}\dots(x - x_p)^{n_p},$$

where  $A$  is a constant, each  $q_i$  is an irreducible quadratic, and the  $x_i$ 's are distinct.

Replacing  $q_i$  by  $-q_i$  (and  $A$  by  $-A$ ) if necessary, we can assume that for each  $i$ ,  $q_i(x) > 0$  for all  $x$ . Since  $(x - x_i)^{n_i} > 0$  for  $x > x_i$ , it follows that  $A > 0$ .

Now consider whether  $n_i$  is even or odd.

1) Suppose  $n_i = 2k_i$  ( $n_i$  is even). Then  $(x - x_i)^{n_i} = (x - x_i)^{2k_i} = [(x - x_i)^{k_i}]^2$ .

2) Suppose  $n_i = 2k_i + 1$  ( $n_i$  is odd): Then  $x_i \leq 0$ . This is because  $(x - x_i)^{n_i}$  changes sign at  $x_i$ , so  $x_i$  cannot be positive.

**Lemma 3.3.** *Suppose  $p(x) = r(x)^2 + s(x)^2 + x(u(x)^2 + v(x)^2)$ . If  $x_i \leq 0$ , then  $p(x)(x - x_i) = \underbrace{A^2 + B^2 + X(C^2 + D^2)}_{\text{also Class I}}$  for certain polynomials  $A(x)$ ,  $B(x)$ ,  $C(x)$ , and  $D(x)$ .*

*Proof.*

$$\begin{aligned} p(x)(x - x_i) &= [r(x)^2 + s(x)^2 + x(u(x)^2 + v(x)^2)](x - x_i) \\ &= x[r(x)^2 + s(x)^2] + x^2[u(x)^2 + v(x)^2] - x_i[r(x)^2 + s(x)^2] - x \cdot x_i[u(x)^2 + v(x)^2] \\ &= \underbrace{[x^2[u(x)^2 + v(x)^2] - x_i[r(x)^2 + s(x)^2]]}_{G^2+H^2} + x \underbrace{[r(x)^2 + s(x)^2 - x_i[u(x)^2 + v(x)^2]]}_{L^2+M^2}. \end{aligned}$$

In more detail, by Theorem 2.2, we can write the first expression in braces as  $G(x)^2 + H(x)^2$ , and the second one as  $L(x)^2 + M(x)^2$ , because  $x_i \leq 0$ , which makes both expressions in braces non-negative.  $\square$

Now let's complete the proof of Theorem 3.2 by applying the above remarks about the Factorization Theorem.

*Proof of Theorem 3.2.*

We have  $p(x) = Aq_1(x)q_2(x)\dots q_r(x) \cdot (x - x_1)^{n_1}\dots(x - x_p)^{n_p}$  (where, as above,  $A > 0$  and each  $q_i$  satisfies  $q_i(x) > 0$  for all  $x$ .)

**Case I.** *Every  $n_i$  is even.* Then  $p(x) \geq 0$  for all  $x \in \mathbb{R}$ , so according to Theorem 2.2,  $p(x) = r(x)^2 + s(x)^2 = r(x)^2 + s(x)^2 + x(0^2 + 0^2)$ .

**Case II.**  $p(x) = Aq_1(x)q_2(x)\dots q_r(x) \cdot (x - x_1)^{n_1}\dots(x - x_m)^{n_m} \cdot (x - y_1)^{k_1}\dots(x - y_q)^{k_q}$ , where  $n_1\dots n_m$  are even,  $k_1\dots k_q$  are odd, and every  $y_i < 0$ .

Let  $L(x) = Aq_1(x)q_2(x)\dots q_r(x)(x - x_1)^{n_1}\dots(x - x_m)^{n_m}$ . Since each  $n_i$  is even,  $L(x) \geq 0$  for all  $x$ . So as in Case I,  $L(x)$  is in Class I. Lemma 3.1 now implies that  $L(x)(x - y_1)$  is in Class I. Similarly,  $[L(x)(x - y_1)](x - y_2)$  is in Class I. By repeated application of Lemma 3.1 we see that  $p(x) = L(x)(x - y_1)\dots(x - y_q)$  is in Class I. Therefore,  $p(x)$  is in Class I. This completes the proof of Theorem 3.2.  $\square$

**Example 3.4.** Consider Example 2.7 in Section 2:

In Example 2.7,

$$p(x) = p_1 \cdot p_2 = 45 + 37x + 20x^2 + 5x^3 + x^4$$

$$= \underbrace{\left[ \frac{(9+x)(10+3x)}{6\sqrt{5}} + \frac{\sqrt{22}x^2}{3\sqrt{5}} \right]^2}_{f^2} + \underbrace{\left[ \frac{(10x+3x^2)\sqrt{2}}{3\sqrt{5}} - \sqrt{\frac{11}{5}}x\left(\frac{9+x}{6}\right) \right]^2}_{g^2}$$

Let's multiply the result in that example by  $x + 3$ :

$$\begin{aligned} \text{Let } q(x) &= p_1 \cdot p_2 \cdot p_3 = \\ &= (f^2 + g^2)(x + 3), \\ &= 135 + 156x + 97x^2 + 35x^3 + 8x^4 + x^5, \end{aligned}$$

so that  $q(x) \geq 0$  for  $x \geq 0$ . (See graph 5. on page 31.c). Then

$$\begin{aligned} q(x) &= (f^2 + g^2)x + (f^2 + g^2)(\sqrt{3})^2 \\ &= (f^2 + g^2)x + [(f\sqrt{3})^2 + (g\sqrt{3})^2], \end{aligned}$$

which is in the form of Theorem 3.2. Now let's replace  $f$  and  $g$  by their corresponding values:

$$\begin{aligned}
p(x) &= p_1 \cdot p_2 \cdot p_3 = \\
&= \left[ \frac{2\sqrt{22}x^2 + 3x^2 + 37x + 90}{6\sqrt{5}} \right]^2 + \left[ -\frac{\sqrt{11}x^2 - 6\sqrt{2}x^2 + 9\sqrt{11}x - 20\sqrt{2}x}{6\sqrt{5}} \right]^2 x \\
&+ \left[ \left( \frac{2\sqrt{22}x^2 + 3x^2 + 37x + 90}{2\sqrt{15}} \right) \right]^2 + \left[ \frac{-\sqrt{11}x^2 + 6\sqrt{2}x^2 - 9\sqrt{11}x + 20\sqrt{2}x}{2\sqrt{15}} \right]^2. \square
\end{aligned}$$

**Example 3.5.** Consider  $p(x) = x^6 + 2x^5 - 5x^3 - 4x^2 + 24x + 24$ . (See graph 6. on page 31.c). Let's factor this polynomial:

$$p(x) = \underbrace{(x^2 + 2x + 4)}_{p_1} \underbrace{(x^2 - 3x + 3)}_{p_2} \underbrace{(x + 2)}_{p_3} \underbrace{(x + 1)}_{p_4}$$

Let's take  $p_1$  and find the discriminant:

$$p_1 = x^2 + 2x + 4,$$

so

$$b^2 - 4ac = 4 - 4(1)(4) < 0.$$

Therefore,  $p_1$  has no real roots, so  $p_1(x) > 0$  for all  $x$ .

$$\text{Let } a = 1, \quad a' = \frac{b^2}{4c} = \frac{4}{16} = \frac{1}{4}, \quad \sqrt{a - a'} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

So as in Case I of Theorem 2.2,  $p_1(x) = (\sqrt{a'}(x + \frac{b}{2a'}))^2 + (\sqrt{a - a'}x)^2$

$$= \left[ \sqrt{\frac{1}{4}} \left( x + \frac{2}{2(\frac{1}{4})} \right) \right]^2 + \left( \frac{\sqrt{3}}{2} x \right)^2$$

$$= \left[ \underbrace{\frac{1}{2}(x + 4)}_f \right]^2 + \left[ \underbrace{\frac{\sqrt{3}}{2}x}_g \right]^2$$

Let's take  $p_2$  and repeat the same steps as above:

$$p_2 = x^2 - 3x + 3,$$

so,

$$b^2 - 4ac = 9 - 4(1)(3) = 9 - 12 < 0.$$

Therefore,  $p_2$  has no real roots, so  $p_2(x) > 0$  for all  $x$ .

Let  $a = 1$ ,  $a' = \frac{9}{4(3)} = \frac{3}{4}$ , so

$$\begin{aligned} p_2 &= \left[ \frac{\sqrt{3}}{2} \left( x + \frac{-3}{2(\frac{3}{4})} \right) \right]^2 + \left[ \frac{1}{2}x \right]^2 \\ &= \underbrace{\left[ \frac{\sqrt{3}}{2}(x-2) \right]^2}_h + \underbrace{\left[ \frac{1}{2}x \right]^2}_k \end{aligned}$$

Now that we have the information needed, using the Lemma 2.5, we can write  $p_1 p_2$  as a sum of two squares:

$$\begin{aligned} p_1 \cdot p_2 &= (fh + gk)^2 + (fk - gh)^2 \\ &= \left[ \frac{1}{2}(x+4) \frac{\sqrt{3}}{2}(x-2) + \frac{\sqrt{3}}{4}x^2 \right]^2 + \left[ \frac{1}{4}(x+4)x - \frac{\sqrt{3}}{2}x \cdot \frac{\sqrt{3}}{2}(x-2) \right]^2 \\ &= \left[ \frac{\sqrt{3}}{4}(x+4)(x-2) + \frac{\sqrt{3}}{4}x^2 \right]^2 + \left[ \frac{x}{4}(x+4) - \frac{3}{4}x(x-2) \right]^2. \end{aligned}$$

Now multiply by  $p_3$  :

$p_1 \cdot p_2 \cdot p_3 = \left[ \left[ \frac{\sqrt{3}}{4}(x+4)(x-2) + \frac{\sqrt{3}}{4}x^2 \right]^2 + \left[ \frac{x^2+4x}{4} - \frac{3x^2-6x}{4} \right]^2 \right] \cdot (x+2)$ , which gives us  $(p_1 \cdot p_2)x + (p_1 \cdot p_2)2 = (p_1 \cdot p_2)x + ((\sqrt{2})^2 \cdot p_1 \cdot p_2)$ . Finally, we multiply the result by  $p_4$ , so that

$$\begin{aligned} p(x) &= p_1 \cdot p_2 \cdot p_3 \cdot p_4 = (p_1 \cdot p_2 \cdot p_3)(x+1) \\ &= (p_1 \cdot p_2 \cdot p_3)x + (p_1 \cdot p_2 \cdot p_3). \end{aligned}$$

Therefore, by calculating the formula in Lemma 3.1 we get:

$$p(x) = \left[ (p_1 p_2)x + ((\sqrt{2})^2 p_1 p_2) \right]x + \left[ (p_1 p_2)x + ((\sqrt{2})^2 p_1 p_2) \right],$$

where

$$p_1 = \left[ \frac{1}{2}(x+4) \right]^2 + \left[ \frac{\sqrt{3}}{2}x \right]^2 \text{ and } p_2 = \left[ \frac{\sqrt{3}}{2}(x-2) \right]^2 + \left[ \frac{1}{2}x \right]^2.$$

Now we multiplied out, so that  $p(x) = p_1 \cdot p_2 x^2 + (\sqrt{2})^2 \cdot p_1 \cdot p_2 x + p_1 p_2 x + (\sqrt{2})^2 \cdot p_1 \cdot p_2$ .

Then group all the expressions that are multiples of  $x^1$ , and group the ones that are not:  
 $p(x) = (p_1 \cdot p_2 x^2 + (\sqrt{2})^2 \cdot p_1 \cdot p_2) + x((\sqrt{2})^2 \cdot p_1 \cdot p_2 + p_1 \cdot p_2)$ . Now substitute for  $p_1$  and  $p_2$ . Since  $p_1$  and  $p_2$  are non-negative, each expression in parentheses is non-negative. If we apply Theorem 2.2 to each of these expressions, we get the representation we want:

$$p(x) = \left(\frac{\sqrt{3}x^3 + \sqrt{3}x^2 - \sqrt{2}x^2 - 4\sqrt{3}x + 5\sqrt{2}x}{2}\right)^2 + \left(\frac{-x^3 - \sqrt{6}x^2 + 5x^2 - \sqrt{6}x + 4\sqrt{6}}{2}\right)^2 + x \left[\left(\frac{3x^2 + 3x - 12}{2}\right)^2 + \left(\frac{-\sqrt{3}x^2 + 5\sqrt{3}x}{2}\right)^2\right]. \square$$

The following corollaries are inferences that follow from the proof of Theorem 3.2.

**Corollary 3.6.** *If  $p(x) \geq 0$  for  $x \geq \alpha$ , then  $p(x) = A(x)^2 + B(x)^2 + (x - \alpha)(C(x)^2 + D(x)^2)$ .*

*Proof.* Let  $q(s) = p(s + \alpha)$ ; if  $s \geq 0$ , then  $s + \alpha \geq \alpha$ , so  $q(s) \geq 0$  for  $s \geq 0$ .

By Theorem 3.2:

$$q(s) = a(s)^2 + b(s)^2 + s(c(s)^2 + d(s)^2).$$

Now let's go back in terms of  $x$ . Let  $x = s + \alpha$  so  $s = x - \alpha$ . Then  $p(x) = p(s + \alpha) = q(s)$

$$q(s) = a(s)^2 + b(s)^2 + s(c(s)^2 + d(s)^2)$$

$$= a(x - \alpha)^2 + b(x - \alpha)^2 + (x - \alpha)(c(x - \alpha)^2 + d(x - \alpha)^2).$$

Since all of the terms  $a(x - \alpha)$ ,  $b(x - \alpha)$ ,  $c(x - \alpha)$ ,  $d(x - \alpha)$  are still in the form of polynomials, we can then write the last expression as follows:

$$p(x) = A(x)^2 + B(x)^2 + (x - \alpha)(C(x)^2 + D(x)^2). \square$$

By Corollary 3.6, we have a representation for a positive polynomial for  $x \geq \alpha$ . We can also represent a polynomial that is non-negative for  $x \leq \alpha$ .

**Corollary 3.7.** *If  $p(x) \geq 0$  for  $x \leq \alpha$ , then  $p(x) = A(x)^2 + B(x)^2 + (\alpha - x)(C(x)^2 + D(x)^2)$ .*

*Proof.* Let  $q(s) = p(-s)$ . If  $s \geq -\alpha$ , then  $\alpha \geq -s$ ; so  $q(s) = p(-s) \geq 0$ .

Remark that according to the previous corollary,

$$p(-s) = q(s) = a(s)^2 + b(s)^2 + (s + \alpha)(c(s)^2 + d(s)^2).$$

Now let  $x = -s$ , so  $s = -x$ . Thus,  $p(x) = a(-x)^2 + b(-x)^2 + (-x + \alpha)(c(-x)^2 + d(-x)^2)$ .

Since all of the terms  $a(-x)$ ,  $b(-x)$ ,  $c(-x)$ ,  $d(-x)$  are still in the form of polynomials, we can write the last expression as follows:

$$p(x) = A(x)^2 + B(x)^2 + (\alpha - x)(C(x)^2 + D(x)^2).$$

□

#### 4. POSITIVITY OF POLYNOMIALS ON THE INTERVAL $[0,1]$

Now we turn to a representation for the polynomials that are non-negative on the interval  $[0,1]$ .

**Theorem 4.1.** *If  $p(x) \geq 0$  for all  $x \in [0,1]$ , then there exist polynomials  $r(x)$ ,  $s(x)$ ,  $u(x)$ , and  $v(x)$  such that  $p(x) = r(x)^2 + s(x)^2 + x(1-x)(u(x)^2 + v(x)^2)$ .*

Note that  $x(1-x) > 0 \iff x > 0$  and  $(1-x) > 0$ , or  $x < 0$  and  $(1-x) < 0 \iff 0 < x < 1$ . So,  $x(1-x) \geq 0 \iff 0 \leq x \leq 1$ .

**Definition 4.2.** *In what follows, we say that  $p(x)$  is a Class II polynomial if  $p(x) = r(x)^2 + s(x)^2 + x(1-x)(u(x)^2 + v(x)^2)$ , where  $r$ ,  $s$ ,  $u$ ,  $v$  are polynomials.*

*Proof of Theorem 4.1.*

We will derive Theorem 4.1 from Theorem 3.2. Suppose  $p(x) \geq 0$  for  $0 \leq x \leq 1$ .

**Case I.** Assume first that  $p$  has even degree,  $\deg p = 2k$ , i.e.,

$$p(x) = a_0 + a_1x + \cdots + a_{2k}x^{2k}.$$

Let  $h(x) = p(\frac{1}{x})$ , i.e.,

$$h(x) = a_0 + a_1\frac{1}{x} + \cdots + a_{2k}\frac{1}{x^{2k}}.$$

If  $x \geq 1$ ,  $0 < \frac{1}{x} \leq 1$ ; therefore,  $p(\frac{1}{x}) \geq 0$ . This means that  $h(x) \geq 0$  for  $x \geq 1$ . Let

$$g(x) = x^{2k}h(x) = a_0x^{2k} + \cdots + a_{2k}.$$

So  $g(x) \geq 0$  for  $x \geq 1$ . Now let  $s = x - 1$ , so that  $x = s + 1$ . Note that  $x \geq 1$  if and only if  $s \geq 0$ . Let  $q(s) = g(s + 1) = (s + 1)^{2k}h(s + 1)$ , so  $q(s) \geq 0$  for  $s \geq 0$ . By Theorem 3.2, there are polynomials  $R(s), S(s), U(s), V(s)$  such that

$$q(s) = R(s)^2 + S(s)^2 + s(U(s)^2 + V(s)^2).$$

We must have  $\deg R$ ,  $\deg S \leq k$ , and  $\deg U$ ,  $\deg V \leq k - 1$ . Note that

$$q(s) = g(s + 1) = g(x) = x^{2k}h(x) = x^{2k}p(\frac{1}{x}).$$

So,

$$x^{2k}p(\frac{1}{x}) = R(x-1)^2 + S(x-1)^2 + (x-1)(U(x-1)^2 + V(x-1)^2).$$

Let  $u = \frac{1}{x}$ . Then

$$\frac{1}{u^{2k}}p(u) = R\left(\frac{1}{u} - 1\right)^2 + S\left(\frac{1}{u} - 1\right)^2 + \left(\frac{1}{u} - 1\right)\left(U\left(\frac{1}{u} - 1\right)^2 + V\left(\frac{1}{u} - 1\right)^2\right).$$

Now multiply both sides by  $u^{2k}$ :

$$p(u) = u^{2k}\left(R\left(\frac{1}{u} - 1\right)^2 + S\left(\frac{1}{u} - 1\right)^2\right) + u^2\left(\frac{1}{u} - 1\right)u^{2k-2}\left(U\left(\frac{1}{u} - 1\right)^2 + V\left(\frac{1}{u} - 1\right)^2\right).$$

Since  $\deg R \leq k$ ,  $u^k R\left(\frac{1-u}{u}\right)$  is a polynomial in  $u$ , say  $R'(u) = u^k R\left(\frac{1-u}{u}\right)$ . Similarly, there are polynomials  $S' = u^k S\left(\frac{1-u}{u}\right)$ ,  $U' = u^{k-1}U\left(\frac{1-u}{u}\right)$ ,  $V' = u^{k-1}V\left(\frac{1-u}{u}\right)$ , such that:

$$p(u) = R'(u)^2 + S'(u)^2 + u(1-u)\left(U'(u)^2 + V'(u)^2\right).$$

Now just re-write this formula using  $x = u$ ,

$$p(x) = R'(x)^2 + S'(x)^2 + x(1-x)\left(U'(x)^2 + V'(x)^2\right).$$

This completes the proof when  $\deg p$  is even. □

**Example 4.3.** Let  $p(x) = 1 - x^4$ , so that  $p(x) \geq 0$  for  $0 \leq x \leq 1$ . (See graph 7. on page 31.d). We will follow the proof of Theorem 4.1. Let  $h(x) = p\left(\frac{1}{x}\right) = 1 - \left(\frac{1}{x}\right)^4 = 1 - \frac{1}{x^4}$ . Let  $g(x) = x^4 h(x) = x^4 - 1$ , so that  $g(x) \geq 0$  for  $x \geq 1$ .

Now, let  $s = x - 1$ , so that  $x = s + 1$ , and consider

$$q(s) := g(s + 1) = (s + 1)^4 - 1$$

$$= s^4 + 4s^3 + 6s^2 + 4s$$

$$= s^4 + 6s^2 + s(4s^2 + 4).$$

For  $s \geq 0$ , we have  $x \geq 1$ , so  $q(s) = g(s+1) \geq 0$ . In this case we can represent  $q(s)$  as a type I polynomial very easily. Let  $R(s) = s^2, S(s) = \sqrt{6}s, U(s) = 2s, V(s) = 2$ . Then,

$$q(s) = R(s)^2 + S(s)^2 + s\left(U(s)^2 + V(s)^2\right).$$

If we replace  $s$  by  $x-1$ , we get:

$$q(s) = q(x-1) = R(x-1)^2 + S(x-1)^2 + (x-1)[U(x-1)^2 + V(x-1)^2].$$

Now recall that

$$\begin{aligned} q(x-1) &= g(x-1+1) \\ &= g(x) \\ &= x^4 h(x) \\ &= x^4 p\left(\frac{1}{x}\right). \end{aligned}$$

Let  $v = \frac{1}{x}$ , so that

$$\frac{1}{v^4} p(v) = q\left(\frac{1}{v} - 1\right) = R\left(\frac{1}{v} - 1\right)^2 + S\left(\frac{1}{v} - 1\right)^2 + \left(\frac{1}{v} - 1\right) \left[ U\left(\frac{1}{v} - 1\right)^2 + V\left(\frac{1}{v} - 1\right)^2 \right].$$

Now  $p(v) = v^4 q\left(\frac{1}{v} - 1\right)$

$$\begin{aligned} &= v^4 \cdot \left[ R\left(\frac{1}{v} - 1\right)^2 + S\left(\frac{1}{v} - 1\right)^2 + \left(\frac{1}{v} - 1\right) \left( U\left(\frac{1}{v} - 1\right)^2 + V\left(\frac{1}{v} - 1\right)^2 \right) \right] \\ &= v^4 \cdot \left[ \frac{(1-v)^4}{v^4} + \frac{6(1-v)^2}{v^2} \right] + v^4 \left(\frac{1-v}{v}\right) \left[ \left(2\left(\frac{1}{v} - 1\right)\right)^2 + 2^2 \right] \\ &= (1-v)^4 + 6v^2(1-v)^2 + v(1-v)v^2 \left[ 2^2 \frac{(1-v)^2}{v^2} + 4 \right]. \end{aligned}$$

We now have the Class I representation

$$p(v) = \underbrace{\left((1-v)^2\right)^2}_{R(s)} + \underbrace{\left[\sqrt{6}v(1-v)\right]^2}_{S(s)} + \underbrace{v(1-v)}_{x(1-x)} \left[ \underbrace{\left(2(1-v)\right)^2}_{U(s)} + \underbrace{(2v)^2}_{V(s)} \right].$$

Now replace  $p(v)$  by  $p(x)$ :

$$\begin{aligned} p(x) &= ((1-x)^2)^2 + [\sqrt{6}x(1-x)]^2 + x(1-x) \left[ (2(1-x))^2 + (2x)^2 \right] \\ &= [r(x)^2 + s(x)^2] + x(1-x)[u(x)^2 + v(x)^2]. \quad \square \end{aligned}$$

Now let's work with a more complicated polynomial.

**Example 4.4.** Let

$$p(x) = -x^4 + x^3 + x^2 + x + 2.$$

From the graph of  $p(x)$  (See page 31.e), we have  $p(x) \geq 0$  for  $0 \leq x \leq 1$  (We can also see this from the factorization  $p(x) = -(x^2+1)(x+1)(x-2)$ .) As in the proof of Theorem 4.1, let

$$H(x) = p\left(\frac{1}{x}\right) = -\frac{1}{x^4} + \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + 2$$

and let

$$g(x) = x^4 H(x) = -1 + x + x^2 + x^3 + 2x^4.$$

Now let's replace  $x$  by  $s+1$ , and define

$$q(s) := g(s+1) = (s+1)^4 H(s+1) = -1 + (s+1) + (s+1)^2 + (s+1)^3 + 2(s+1)^4.$$

With the use of Mathematica, we expanded  $p(x)$  and then factored it:

$$q(s) = \underbrace{(s^2 + 2s + 2)}_{p_1} \underbrace{(2s + 1)}_{p_2} \underbrace{(s + 2)}_{p_3}.$$

Let  $p_1(s) = s^2 + 2s + 2$ , an irreducible positive polynomial. According to the proof of Theorem 2.2, we have:  $p_1(s) = \left[ \frac{1}{\sqrt{2}}(s+2) \right]^2 + \left[ \frac{1}{\sqrt{2}}s \right]^2$ . Now going back to the problem we get:

$$p_1 \cdot p_2 = (p_1) \cdot (2s + 1)$$

$$= 2sp_1 + p_1$$

$$= 2s \left[ \frac{1}{\sqrt{2}}(s+2) \right]^2 + 2s \left[ \frac{1}{\sqrt{2}}s \right]^2 + \left[ \frac{1}{\sqrt{2}}(s+2) \right]^2 + \left[ \frac{1}{\sqrt{2}}s \right]^2.$$

Therefore,

$$p_1 \cdot p_2 = s[(s+2)^2 + s^2] + \left[ \left( \frac{s+2}{\sqrt{2}} \right)^2 + \left( \frac{s}{\sqrt{2}} \right)^2 \right],$$

which is a representation for Theorem 3.2 ( $r(x)^2 + s(x)^2 + x(t(x)^2 + v(x)^2)$ ). Now we continue by multiplying that result by  $p_3(s) := s + 2$ ,

$$q(s) = p_1 \cdot p_2 \cdot p_3 = (p_1 \cdot p_2) \cdot (s + 2)$$

$$= s(P_1 \cdot P_2) + 2(P_1 \cdot P_2)$$

$$= s \left[ s[(s+2)^2 + s^2] + \left[ \left( \frac{s+2}{\sqrt{2}} \right)^2 + \left( \frac{s}{\sqrt{2}} \right)^2 \right] \right] + 2 \left[ s[(s+2)^2 + s^2] + \left[ \left( \frac{s+2}{\sqrt{2}} \right)^2 + \left( \frac{s}{\sqrt{2}} \right)^2 \right] \right]$$

$$= s^2[(s+2)^2 + s^2] + s \left[ \left( \frac{s+2}{\sqrt{2}} \right)^2 + \left( \frac{s}{\sqrt{2}} \right)^2 \right] + 2s[(s+2)^2 + s^2] + 2 \left[ \left( \frac{s+2}{\sqrt{2}} \right)^2 + \left( \frac{s}{\sqrt{2}} \right)^2 \right]$$

By grouping the terms, we get

$$q(s) = s \left[ \left( \frac{s+2}{\sqrt{2}} \right)^2 + \left( \frac{s}{\sqrt{2}} \right)^2 + (\sqrt{2}(s+2))^2 + (\sqrt{2}s)^2 \right] + [2s^4 + 4s^3 + 6s^2 + 4s + 4]$$

$$= [5(s^2 + 2s + 2)]s + [2(1 + s^2)(s^2 + 2s + 2)].$$

Each expression inside the brackets is non-negative, so using Theorem 2.2 we get

$$q(s) = \left[ \frac{\sqrt{5}}{\sqrt{2}}(s+2) \right]^2 + \left[ \frac{\sqrt{5}}{\sqrt{2}}s \right]^2 \Big] s + \left[ ((\sqrt{2})^2 + (\sqrt{2}s)^2) \left( \left[ \frac{1}{\sqrt{2}}(s+2) \right]^2 + \left[ \frac{1}{\sqrt{2}}s \right]^2 \right) \right]$$

$$= \underbrace{\left( \left[ \frac{\sqrt{5}}{\sqrt{2}}(s+2) \right]^2 + \left[ \frac{\sqrt{5}}{\sqrt{2}}s \right]^2 \right)}_{R(s)^2} s + \underbrace{[s^2 + s + 2]}_{U(s)^2} + \underbrace{[-s^2 - s]}_{W(s)^2}$$

which is also a representation for Theorem 3.2.

Therefore,  $q(s) = [R(s)^2 + T(s)^2]s + [U(s)^2 + W(s)^2]$ .

From now on, we will start going backwards to end up with the representations in terms of  $x$ , which is the variable that we started with. So let's replace  $s$  by  $(x - 1)$ , so that:

$$q(s) = q(x - 1) = [R(x - 1)^2 + T(x - 1)^2](x - 1) + [U(x - 1)^2 + W(x - 1)^2],$$

and we also have  $q(x - 1) = x^4 p(\frac{1}{x})$ . Now let  $v = \frac{1}{x}$  so that

$$q(\frac{1}{v} - 1) = \frac{1}{v^4} p(v) = [R(\frac{1}{v} - 1)^2 + T(\frac{1}{v} - 1)^2](\frac{1}{v} - 1) + [U(\frac{1}{v} - 1)^2 + W(\frac{1}{v} - 1)^2]$$

and

$$p(v) = v^4 \left[ [R(\frac{1-v}{v})^2 + T(\frac{1-v}{v})^2] (\frac{1-v}{v}) + [U(\frac{1-v}{v})^2 + W(\frac{1-v}{v})^2] \right].$$

Now we substitute  $\frac{1-v}{v}$  into the original  $R(s)^2$ ,  $T(s)^2$ ,  $U(s)^2$  and  $W(s)^2$ , and we get:

$$\begin{aligned} p(v) &= \left[ \left[ \sqrt{\frac{5}{2}} \left( \frac{1-v}{v} + 2 \right) \right]^2 + \left[ \sqrt{\frac{5}{2}} \left( \frac{1-v}{v} \right) \right]^2 \right] (v^3 - v^4) \\ &\quad + v^4 \left[ \left( \frac{1-v}{v} \right)^2 + \left( \frac{1-v}{v} + 2 \right)^2 + v^4 \left[ - \left( \frac{1-v}{v} \right)^2 - \left( \frac{1-v}{v} \right) \right]^2 \right] \\ &= \left[ \left[ \sqrt{\frac{5}{2}} \left( \frac{1+v}{v} \right) \right]^2 + \left[ \sqrt{\frac{5}{2}} \left( \frac{1-v}{v} \right) \right]^2 \right] v^2 (v-v^2) + [(1+v)^2 + (v-v^2) + 2v^2]^2 + [-(1-v)^2 - (v-v^2)]^2 \\ &= \left[ \left[ \sqrt{\frac{5}{2}} (1+v) \right]^2 + \left[ \sqrt{\frac{5}{2}} (1-v) \right]^2 \right] (1-v)v + [2v^2 - v + 1]^2 + [v - 1]^2 \end{aligned}$$

Now let's replace  $p(v)$  by  $p(x)$ ; then we get our final answer:

$$p(x) = \left[ \left[ \sqrt{\frac{5}{2}} (1+x) \right]^2 + \left[ \sqrt{\frac{5}{2}} (1-x) \right]^2 \right] (1-x)x + [2x^2 - x + 1]^2 + [x - 1]^2$$

Therefore,  $p(x) = [A(x)^2 + B(x)^2](1-x)x + C(x)^2 + D(x)^2$ , which is the representation of Theorem 4.1.  $\square$

Since we have proven the case of Theorem 4.1 when  $\deg p(x)$  is even, we will now prove the case when  $\deg p(x)$  is odd.

**Case II.** Assume  $p(x) \geq 0$  for  $0 \leq x \leq 1$  and suppose  $\deg p(x)$  is odd. Consider Theorem 2.3 (Factorization Theorem):

$$p(x) = Aq_1 \cdots q_m \cdot (x - x_1)^{2k_1} \cdots (x - x_s)^{2k_s} \cdot (x - y_1)^{2m_1+1} \cdots (x - y_r)^{2m_r+1},$$

where  $A$  is a constant and each  $q_i(x)$  is an irreducible quadratic such that  $q_i(x) > 0$  for all  $x$ . It is understood that any of the groups of factors  $q_i(x)$ ,  $(x - x_i)^{2k_i}$ ,  $(x - y_i)^{2m_i+1}$  may be missing. Furthermore, the  $x_i$ 's and  $y_i$ 's are all distinct.

Now,

$$p(x) = \underbrace{Aq_1 \cdots q_m [(x - x_1)^{k_1} \cdots (x - x_s)^{k_s} (x - y_1)^{m_1} \cdots (x - y_r)^{m_r}]^2}_{Q(x)} \underbrace{(x - y_1) \cdots (x - y_{r-1})(x - y_r)}_{T(x)}.$$

Next, notice that each  $y_i$  is not in the interval  $(0,1)$ . Otherwise,  $p(x)$  changes sign at  $y_i$ , so  $p(x)$  changes sign inside  $(0,1)$ , which contradicts the assumption that  $p(x) \geq 0$  for  $0 \leq x \leq 1$ . Since each  $y_i$  is not in  $(0,1)$ , either  $T(x) \geq 0$  for all  $x$  in  $[0,1]$  or  $T(x) \leq 0$  for all  $x$  in  $[0,1]$ . In the second case, replace  $T(x)$  by  $-T(x)$  and  $A$  by  $-A$ , so we may assume  $T(x) \geq 0$  for all  $x$  in  $[0,1]$ .

Let  $S(x) = Q(x)T(x)$ , so that  $p(x) = S(x)(x - y_r)$ .

**Subcase i).**  $A > 0$ . Then  $S(x) \geq 0$  for all  $x$  in  $[0,1]$ . Since the degree of  $S(x)$  is even, by Case I of the theorem,

$$S(x) = a(x)^2 + b(x)^2 + x(1 - x) \left[ c(x)^2 + d(x)^2 \right].$$

Now consider  $x - y_r$ . We have  $p(x) = S(x)(x - y_r)$ , where  $S(x) \geq 0$  on  $[0,1]$  and  $y_r$  is not in the interval  $(0,1)$ .

We claim that  $y_r \leq 0$ . Otherwise  $y_r \geq 1$ , so  $x - y_r < 0$  for some  $x$  in  $[0,1]$ . Then  $p(x) = S(x)(x - y_r) < 0$  for some  $x$  in  $[0,1]$ , which is a contradiction. Therefore, it is true that  $y_r \leq 0$ . Let  $d = -y_r \geq 0$ , so  $d = e^2$  for  $e = \sqrt{d}$ . So then  $x - y_r = x + e^2 = (x^2 + e^2) + x(1 - x)(1^2 + 0^2)$ , which shows that  $x - y_r$  is in Class II.

Now  $p(x) = S(x)(x - y_r)$  is the product of two Class II polynomials. Therefore, the proof of Subcase *i*) follows from the following:

**Lemma 4.5.** *Class II is closed under multiplication.*

*Proof.* Let  $p_1(x) = a_1^2 + b_1^2 + x(1 - x)(c_1^2 + d_1^2)$ , and let  $p_2(x) = a_2^2 + b_2^2 + x(1 - x)(c_2^2 + d_2^2)$ , where the  $a_i$ 's,  $b_i$ 's,  $c_i$ 's and  $d_i$ 's are polynomials.

Then

$$p_1 p_2 = x(1 - x) [(a_2^2 + b_2^2)(c_1^2 + d_1^2) + (a_1^2 + b_1^2)(c_2^2 + d_2^2)]$$

$$+ [(a_1^2 + b_1^2)(a_2^2 + b_2^2) + x^2(1 - x)^2(c_1^2 + d_1^2)(c_2^2 + d_2^2)].$$

Note that each term in brackets is non-negative for every  $x$ . Therefore, by Theorem 2.2,  $(a_2^2 + b_2^2)(c_1^2 + d_1^2) + (a_1^2 + b_1^2)(c_2^2 + d_2^2)$  is the sum of 2 squares and  $(a_1^2 + b_1^2)(a_2^2 + b_2^2) + x^2(1 - x)^2(c_1^2 + d_1^2)(c_2^2 + d_2^2)$  is also the sum of 2 squares. Therefore,  $p_1p_2$  is in Class II.  $\square$

**Subcase ii).**  $A < 0$ . In this case we can replace  $A$  by  $-A$  and  $x - y_r$  by  $-(x - y_r)$ . Now  $S(x) \geq 0$  for  $0 \leq x \leq 1$  and  $\deg S(x)$  is even, so by Case I,  $S(x)$  is in Class II.

$$\text{We have } p(x) = \underbrace{S(x)}_{\text{Class II}} \cdot (-(x - y_r))$$

To complete the proof, using Lemma 4.5, we want to show that  $-(x - y_r)$ , ( $= -x + y_r$ ) is in Class II. We already know that  $y_r$  is not in  $(0,1)$ . In fact,  $y_r \geq 1$ ; otherwise  $-x + y_r$  is negative for some  $x \in (0,1)$ , so  $p(x) = S(x)(-x + y_r) < 0$  for some  $x \in (0,1)$ , which is a contradiction.

Let  $k = y_r \geq 1$ . We claim that  $-x + k$  can be represented as:

$$-x + k = (Ax + B)^2 + x(1 - x)C^2, \text{ for certain constants } A, B \text{ and } C.$$

$$\begin{aligned} \text{Now } (Ax + B)^2 + x(1 - x)C^2 &= A^2x^2 + 2ABx + B^2 + C^2x - C^2x^2 \\ &= (A^2 - C^2)x^2 + (2AB + C^2)x + B^2. \end{aligned}$$

Equating this to  $-x + k$  we get the following system of equations:

$$(i) \quad A^2 = C^2$$

$$(ii) \quad 2AB + C^2 = -1$$

$$(iii) \quad B^2 = k.$$

From equation (iii), let

$$(iv) \quad B = \sqrt{k}.$$

From equations (i) and (iv), (ii) becomes

$$(4.1) \quad A^2 + 2\sqrt{k}A + 1 = 0$$

and we think of equation (4.1) as a quadratic with an unknown variable  $A$ .

The discriminant in (4.1) is:  $b^2 - 4ac = (2\sqrt{k})^2 - 4(1)(1) = 4(\sqrt{k})^2 - 4 = 4k - 4 = 4(k - 1) \geq 0$  (since  $k \geq 1$ ), which means that equation (4.1) has real roots. Now let's find them by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where, by substitution, we get:

$$\begin{aligned} A &= \frac{-(2\sqrt{k}) \pm \sqrt{(2\sqrt{k})^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-2\sqrt{k} \pm \sqrt{4k - 4}}{2} \\ &= \frac{-2\sqrt{k} \pm \sqrt{4(k - 1)}}{2} \\ &= \frac{-2\sqrt{k} \pm 2\sqrt{k - 1}}{2} \\ &= -\sqrt{k} \pm \sqrt{k - 1}. \end{aligned}$$

Using these values of  $A$ , with  $B = \sqrt{k}$  and  $C = |A|$ , we now have

$$-x + k = (Ax + B)^2 + x(1 - x)C^2.$$

So,  $-x + k$  is in Class II. Thus the proof of Subcase (ii) is completed by an application of Lemma 4.5. This completes the proof of Theorem 4.1.  $\square$

**Example 4.6.** Now let's do an example that represents Subcase (ii).

$$\text{Let } p(x) = -x^3 + 2x^2 - x + 2;$$

from the graph (See page 31.f),  $p(x) \geq 0$  for  $0 \leq x \leq 1$ .

Now  $p(x)$  factors as

$$p(x) = -(x^2 + 1)(x - 2),$$

so in Subcase *ii*),  $A = -1$ . Let's write  $p(x)$  as

$$p(x) = \underbrace{(x^2 + 1)}_{S(x)} \cdot \underbrace{(-x + 2)}_{Z(x)}.$$

Clearly,  $s(x)$  is in Class II because it is a sum of 2 squares.

To show that  $p(x)$  is in Class II we will use Lemma 4.5, so we want to show that  $Z(x)$  is also in Class II. According to Subcase *ii*),  $-x + 2$  is in Class II since we have

$$Z(x) = (Ax + B)^2 + x(1 - x)C^2,$$

$A = -\sqrt{k} - \sqrt{k-1}$ ,  $C = |A|$  and  $B = \sqrt{k}$ , where  $k = 2$ .

Therefore, by substitution:

$$\begin{aligned} Z(x) &= ((-\sqrt{k} - \sqrt{k-1})x + \sqrt{k})^2 + x(1-x)|-\sqrt{k} - \sqrt{k-1}|^2 \\ &= ((-\sqrt{2} - 1)x + \sqrt{2})^2 + x(1-x)|-\sqrt{2} - 1|^2. \end{aligned}$$

Thus, by Lemma 4.5,  $p(x)$  is in Class II.

In more detail, recall from Lemma 4.5 that Class II is closed under multiplication. We have

$$\begin{aligned} S(x) &= x^2 + 1^2 + x(1-x)(0^2 + 0^2) \\ &\equiv a_1^2 + b_1^2 + x(1-x)(c_1^2 + d_1^2), \end{aligned}$$

$$\text{and } Z(x) = ((-\sqrt{2} - 1)x + \sqrt{2})^2 + 0^2 + x(1-x)(|-\sqrt{2} - 1|^2 + 0^2)$$

$$\equiv a_2^2 + b_2^2 + x(1-x)(c_2^2 + d_2^2)$$

Then

$$p(x) = S(x) \cdot Z(x)$$

$$= \left[ (a_1^2 + b_1^2)(a_2^2 + b_2^2) + x^2(1-x)^2(c_1^2 + d_1^2)(c_2^2 + d_2^2) \right] + x(1-x) \left[ (a_2^2 + b_2^2)(c_1^2 + d_1^2) + (a_1^2 + b_1^2)(c_2^2 + d_2^2) \right].$$

Since  $c_1^2 = d_1^2 = b_2^2 = d_2^2 = 0$ , the representation becomes smaller:

$$\begin{aligned} p(x) &= (a_1^2 + b_1^2)a_2^2 + x(1-x)((a_1^2 + b_1^2)c_2^2) \\ &= (a_1a_2)^2 + (b_1a_2)^2 + x(1-x)((a_1c_2)^2 + (b_1c_2)^2) \end{aligned}$$

Therefore, by substitution we get:

$$p(x) = (\sqrt{2}x + x^2(-1 - \sqrt{2}))^2 + (x(-1 - \sqrt{2}) + \sqrt{2})^2 + x(1-x) \left[ (x|-\sqrt{2}-1|)^2 + (|-\sqrt{2}-1|)^2 \right].$$

Thus, we have shown that  $p(x)$  is in Class II.  $\square$

**Corollary 4.7.** *If  $p(x) \geq 0$  for  $\alpha \leq x \leq \beta$ , then  $p(x) = A(x)^2 + B(x)^2 + (x - \alpha)(\beta - x)(C(x)^2 + D(x)^2)$ .*

For  $\alpha = 0$ ,  $\beta = 1$  this is Theorem 4.1; for  $\alpha = -1$ ,  $\beta = 1$ , this is the result in Polya-Szego [6, Part VI, Sec. 6, 46, page 78].

*Proof.* Let  $s = \frac{x-\alpha}{\beta-\alpha}$ , so  $x = \alpha + (\beta - \alpha)s$ . Let  $q(s) = p(x) = p(x + (\beta - \alpha)s)$ .

If  $0 \leq s \leq 1$ , then  $\alpha \leq x \leq \beta$ , so  $p(x) \geq 0$ . So  $q(s) \geq 0$  if  $0 \leq s \leq 1$ , and by Theorem 4.1,  $q(s) = a(s)^2 + b(s)^2 + s(1-s)(c(s)^2 + d(s)^2)$ .

Now let's go back in terms of  $x$ :

$$\begin{aligned} p(x) &= q(s) = q\left(\frac{x-\alpha}{\beta-\alpha}\right) \\ &= a\left(\frac{x-\alpha}{\beta-\alpha}\right)^2 + b\left(\frac{x-\alpha}{\beta-\alpha}\right)^2 + \left(\frac{x-\alpha}{\beta-\alpha}\right)\left(1 - \frac{x-\alpha}{\beta-\alpha}\right) \left[ c\left(\frac{x-\alpha}{\beta-\alpha}\right)^2 + d\left(\frac{x-\alpha}{\beta-\alpha}\right)^2 \right]. \end{aligned}$$

Since all the individual terms  $a\left(\frac{x-\alpha}{\beta-\alpha}\right)$ ,  $b\left(\frac{x-\alpha}{\beta-\alpha}\right)$ ,  $c\left(\frac{x-\alpha}{\beta-\alpha}\right)$ ,  $d\left(\frac{x-\alpha}{\beta-\alpha}\right)$  are still in the form of

polynomials, we can then write the last expression as follows:

$$\begin{aligned} p(x) &= A(x)^2 + B(x)^2 + \left(\frac{x-\alpha}{\beta-\alpha}\right)\left(1 - \frac{x-\alpha}{\beta-\alpha}\right)[C(x)^2 + D(x)^2] \\ &= A(x)^2 + B(x)^2 + \left(\frac{x-\alpha}{\beta-\alpha}\right)\left(\frac{x-\beta}{\beta-\alpha}\right)[C(x)^2 + D(x)^2] \\ &= A(x)^2 + B(x)^2 + \frac{(x-\alpha)(x-\beta)}{(\beta-\alpha)^2}[C(x)^2 + D(x)^2] \\ &= A(x)^2 + B(x)^2 + (x-\alpha)(x-\beta)\left[\left(\frac{C(x)}{(\beta-\alpha)}\right)^2 + \left(\frac{D(x)}{(\beta-\alpha)}\right)^2\right], \end{aligned}$$

which is the representation we want.

□

## 5. APPENDIX

Positive polynomials play an important role in the study of Moment Theory [1]. Now we will show how, with the use of Theorem 2.2, we can test the positivity of a linear functional. Let  $P_n$  be the set of polynomials with  $\deg p \leq n$ . Since  $P_n$  is closed under addition and multiplication by a real scalar,  $P_n$  is a vector space over  $\mathbb{R}$ . Consider

$$L : P_n \longrightarrow \mathbb{R},$$

such that  $L$  is linear. To be linear,  $L$  needs to fulfill the following properties:

i) For polynomials  $p, q \in P_n$ ,  $L(p + q) = L(p) + L(q)$ .

ii) For  $a \in \mathbb{R}$ ,  $p \in P_n$ ,  $L(ap) = aL(p)$ .

Let  $P_n^+ = \{p \in P_n : p(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$ . We say that  $L$  is *positive* if  $p \in P_n^+ \implies L(p) \geq 0$ .

**Example 5.1.** Suppose  $a_1, \dots, a_m$  are non-negative numbers and  $x_1, \dots, x_m$  are real numbers. For  $p \in P_n$ , let  $L(p) = a_1p(x_1) + \dots + a_m p(x_m)$ . Then it is easy to see that  $L$  is positive linear functional. However, not every positive linear functional  $L : P_n \longrightarrow \mathbb{R}$  has this simple form.  $\square$

We want a test to see when a linear functional

$$L : P_{2n} \longrightarrow \mathbb{R}$$

is positive. We proceed by taking the monomial basis of  $P_{2n}$ :  $1, x, x^2, \dots, x^n, \dots, x^{2n}$ , and letting  $\beta_i = L(x^i)$  ( $0 \leq i \leq 2n$ ). Now consider the following Hankel matrix,

$$H(n) = \begin{pmatrix} \beta_0 & \beta_1 & \dots & \beta_n \\ \beta_1 & \beta_2 & \dots & \beta_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_n & \beta_{n+1} & \dots & \beta_{2n} \end{pmatrix}.$$

**Definition 5.2.**  $H(n)$  is positive if for every vector  $a = (a_0, a_1, \dots, a_n)$ ,  $\langle H(n) a, a \rangle \geq 0$ .

**Remark 5.3.** There are several methods for checking whether  $H(n)$  is positive, including nested determinants or checking that all of the eigenvalues of  $H(n)$  are non-negative.

**Theorem 5.4.**  $L : P_{2n} \longrightarrow \mathbb{R}$  is a positive linear functional if and only if  $H(n) \geq 0$ .

*Proof.* For polynomials  $p(x) = a_0 + a_1x + \dots + a_nx^n$  and  $q(x) = b_0 + b_1x + \dots + b_nx^n$ , let

$$\hat{p} = a = (a_0, a_1, \dots, a_n) \text{ and } \hat{q} = b = (b_0, b_1, \dots, b_n)$$

Then for  $p, q \in P_n$ ,  $pq \in P_{2n}$ , and

$$(5.1) \quad L(pq) = \langle H(n) \hat{p}, \hat{q} \rangle.$$

Suppose  $L : P_{2n} \rightarrow \mathbb{R}$  is positive. We want to show  $H(n) \geq 0$ . Let  $a = (a_0, a_1, \dots, a_n)$ . Then we must show that  $\langle H(n) a, a \rangle \geq 0$ . If

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n,$$

then  $\hat{p} = a$ . By equation (5.1)  $\langle H(n) a, a \rangle = \langle H(n) \hat{p}, \hat{p} \rangle = L(p^2)$ ; but  $p^2 \in P_{2n}^+$ , so  $L(p^2) \geq 0$ . Therefore,  $\langle H(n) a, a \rangle = L(p^2) \geq 0$ , so  $H(n) \geq 0$ . Now suppose  $H(n) \geq 0$ . We want to prove that  $L$  is positive. Let  $p \in P_{2n}^+$ ; we want to show  $L(p) \geq 0$ . By Theorem 2.2,

$$p(x) = r(x)^2 + s(x)^2$$

where  $\deg p \leq 2n \implies \deg r \leq n$  and  $\deg s \leq n$ . Now,  $L(p) = L(r(x)^2 + s(x)^2) = L(r(x)^2) + L(s(x)^2)$ . Therefore, by equation (5.1):

$$L(p) = \langle H(n) \hat{r}, \hat{r} \rangle + \langle H(n) \hat{s}, \hat{s} \rangle \geq 0.$$

This completes the proof.  $\square$

Thus, Theorem 2.2 does help determine the positivity of a linear functional.

## 6. GRAPHS

## REFERENCES

- [1] N.I. Akhiezer, The Classical Moment Problem, Hafner Publ. Co., New York, 1965.
- [2] R. Curto and L. Fialkowm Recursiveness, positivity, and truncated moment problems, Houston J. Math. 17(1991), 603-635.
- [3] R. Curto and L. Fialkow, Solution to the parabolic moment problem, Integral Equations and Operator Theory 50(2004), 169-196.
- [4] E.K. Haviland, On the momentum problem for distributions in more than one dimension II, Amer. J. Math 58(1936), 164-168.
- [5] D. Hilbert, Uber die Darstellung defniter formen als Sumen von Formenquadraten, Math. Ann. 32. 32(1888), 342-350.
- [6] G. Polya and G. Szego, Problems and Theorems in Analysis, Vol. II, SpringerVerlag, New York-Heidelberg-Berlin, 1976.
- [7] K. Schmudgen, The K-moment problem for compact semi-algebraic sets, Math, Ann. 289 (1991), 203-206.

CYNDHIA SILVERIO, ELECTRICAL ENGINEERING MAJOR (CLASS OF 2009) AND ROSEMARY HOLGUIN,  
MATH SECONDARY EDUCATION MAJOR (CLASS 2008), STATE UNIVERSITY OF NEW YORK AT NEW PALTZ,  
NEW PALTZ, NY 12561, USA