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THE TRUNCATED MOMENT PROBLEM ON PARALLEL LINES

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ABSTRACT. Let $\beta \equiv \beta^{(2d)}$ denote a bivariate real sequence of degree $2d$, and let M_d denote the associated moment matrix. We prove that β admits a representing measure supported in the union of two parallel lines (the variety of a suitable quadratic polynomial $p(x,y)$) if and only if i) M_d is posititve semidefinite, ii) M_d is recursively generated, iii) the variety $\mathcal{V}$ of M_d satisfies $\operatorname{rank} M_d \le \operatorname{card} \mathcal{V}$, and iv) there is a dependence relation $p(X,Y) = 0$ in the columns of M_d.

1. Introduction

Let $\beta \equiv \beta^{(2d)} = \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\le 2d}$ denote a real n-dimensional multisequence of degree $2d$, and let K denote a closed subset of $\mathbb{R}^n$. *The Truncated K-Moment Problem* (TKMP) for β concerns the existence of a positive Borel measure μ, supported in K, such that

$$\beta_i = \int_K x^i \, d\mu \ (i \in \mathbb{Z}_+^n, \ |i| \le 2d). \tag{1.1}$$

(Here, for $x \equiv (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $i \equiv (i_1,\ldots,i_n) \in \mathbb{Z}_+^n$, we set $|i| = i_1 + \cdots + i_n$ and $x^i = x_1^{i_1}\cdots x_n^{i_n}$.) A measure as in (1.1) is a *K-representing measure* for β; for $K = \mathbb{R}^d$, we refer to TKMP as *The Truncated Moment Problem* (TMP) and to μ simply as a *representing measure.* Two general, but abstract, solutions to the truncated K-moment problem are known; one involves flat extensions of positive moment matrices [CF8] (cf. Theorem 1.4 below for $K = \mathbb{R}^d$), the other entails extensions of K-positive linear functionals [CF10]. (For general references to moment problems, see [Akh] [AK] [KN] [ST]; for the connection between the *Full Moment Problem* and TMP, see [Sto2].)

By a *concrete solution* to TKMP we mean a set of conditions for K-representing measures that can be effectively tested in numerical examples. Concrete solutions to the truncated K-moment problem are known in only a few cases, including, for $n = 1$, $K = \mathbb{R}$, $[0,+\infty)$, or $\cup_{i=1}^m [a_i, b_i]$ (cf. [AK] [KN] [CF1]), and, for $n = 2$, the case when K is an algebraic curve $p(x,y) = 0$ with $\deg p \le 2$ (cf. [CF9, Theorem 1.2], Theorem 1.1 (below)).

Let $M_d \equiv M_d(\beta)$ denote the moment matrix associated with β (see below for terminology and notation). The rows and columns of M_d are denoted by X^i and are indexed (in degree-lexicographic order) by the monomials x^i in $\mathcal{P}_d \equiv \{p \in \mathbb{R}[x_1,\ldots,x_n] : \deg p \le d\}$. Corresponding to $p \equiv \sum_{i\in\mathbb{Z}_+^n, |i|\le d} a_i x^i \in \mathcal{P}_d$ is the element $p(X) \equiv \sum a_i X^i$ of $\operatorname{Col} M_d$, the column space of M_d; M_d is *recursively generated* if whenever p, q, $pq \in \mathcal{P}_d$ and $p(X) = 0$, then $(pq)(X) = 0$. Positive semidefiniteness and recursiveness of M_d are

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necessary conditions for representing measures [CF2] [F1]. For $p \in \mathcal{P}_d$, M_d is *p-pure* if the only dependence relations in $\operatorname{Col} M_d$ are those of the form $(pq)(X) = 0$ ($q \in \mathcal{P}_{d-\deg p}$) [F3]. Let $\mathcal{V} \equiv \mathcal{V}(M_d)$ denote the *algebraic variety* corresponding to M_d, i.e., $\mathcal{V} = \bigcap_{p\in\mathcal{P}_d, p(X)=0} \mathcal{Z}(p)$ (where $\mathcal{Z}(p) = \{x \in \mathbb{R}^n : p(x) = 0\}$). A necessary condition for representing measures is the "variety condition", $r \equiv \operatorname{rank} M_d \leq v \equiv \operatorname{card} \mathcal{V}(M_d)$ (cf. [CF8, Corollary 2.12]).

Now let $n = 2$. For a polynomial $p(x, y)$, let $K = \mathcal{Z}(p) := \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}$. For $\deg p(x, y) = 1$, it is proved in [CF3] that $\beta \equiv \beta^{(2d)}$ has a $\mathcal{Z}(p)$-representing measure if and only if M_d is positive, recursively generated, and has a column relation $p(X, Y) = 0$. Consider the following solution to TKMP for $K = \mathcal{Z}(p)$ with $\deg p(x, y) = 2$.

Theorem 1.1. *([CF9, Theorem 2.1]) Let $d \geq 2$ and suppose $\deg p(x, y) = 2$. $\beta \equiv \beta^{(2d)}$ has a representing measure supported in $\mathcal{Z}(p)$ if and only if $M_d \equiv M_d(\beta)$ is positive semidefinite, recursively generated, satisfies the variety condition, and has a column dependence relation $p(X, Y) = 0$.*

The conditions of the theorem are "concrete" in that they can be verified using elementary linear algebra, or, for $\operatorname{card} \mathcal{V}(M_d)$, estimated with the aid of computer algebra systems.

In [CF6], R.E. Curto and the author showed that the existence of representing measures in the *Truncated Complex Moment Problem* (TCMP) is stable under invertible degree one mappings, and also that TCMP is equivalent to TMP. As discussed in [F4], in TMP, invertible degree one mappings $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ are of the form $\tau(x, y) = (a + \alpha x + \gamma y, b + \delta x + \lambda y)$, with $\alpha\lambda - \gamma\delta \neq 0$. It is well-known that under such a mapping, the degree 2 curve $p(x, y) = 0$ may be transformed into one of the following nine basic varieties: $x^2 + y^2 = 1$, $y = x^2$, $xy = 1$, $xy = 0$, $x^2 = 1$, $x^2 = 0$, $x^2 = -1$, $x^2 + y^2 = 0$, and $x^2 + y^2 = -1$ (cf. [SH, p. 405]). In a series of papers, we proved Theorem 1.1 for each of the first four varieties, corresponding to a circle [CF6], parabola [CF7], hyperbola [CF9], or degenerate hyperbola (intersecting lines) [CF9].

In [F4] the author noted that the preceding analysis of Theorem 1.1 is incomplete, since it does not consider all nine of the varieties. However, in [F4] we showed that if M_d is positive, then the column relations $X^2 = -1$ and $X^2 + Y^2 = -1$ cannot occur. Further, we showed in [F4] that if $M_d \geq 0$ and either $X^2 = 0$ or $X^2 + Y^2 = 0$, then $\operatorname{rank} M_2 \leq 3$. In this case, if $\operatorname{rank} M_1 = 3$, then it follows from [CF8] that $\operatorname{rank} M_d = \operatorname{rank} M_1 = 3$ and that M_d has a unique representing measure. In the subcase when $\operatorname{rank} M_1 < 3$, it follows from [CF3] that M_d has a measure if and only if it is recursively generated. Thus, to complete the analysis of Theorem 1.1, it remains to consider the variety $x^2 = 1$, corresponding to two parallel lines. The purpose of this note is to establish Theorem 1.1 in this case, thereby validating Theorem 1.1 for any polynomial of degree 2.

Theorem 1.2. *Let $n = 2$, $d \geq 2$. Suppose $\deg p(x, y) = 2$ and $\mathcal{Z}(p)$ consists of 2 parallel lines. Then $\beta \equiv \beta^{(2d)}$ has a representing measure supported in $\mathcal{Z}(p)$ if and only if M_d is positive semidefinite, recursively generated, satisfies the variety condition, and $p(X, Y) = 0$ in the column space of M_d.*

We note that for $d = 2$, Theorem 1.2 was proved in [F4] using an approach based in part on computer algebra calculations. The proof of Theorem 1.2 presented in the sequel uses a different approach, and does not depend on computer algebra. Theorem 1.1 cannot be extended to planar curves of degree 3; this was shown in [CFM] with an example of M_3 in which $\mathcal{Z}(p)$ consists of three parallel lines. A complete solution to TKMP for $K = \mathcal{Z}(y - x^3)$ appears in [F3], but for the variety of a general degree 3 curve, TKMP is largely unsolved. In a different direction, in [Sto1], J. Stochel proved that if $\deg p(x, y) = 2$, then a full moment sequence $\beta^{(\infty)}$ has a representing measure supported in $\mathcal{Z}(p)$ if and only if the corresponding moment matrix M_∞ is positive semidefinite and satisfies $p(X, Y) = 0$ in its column space. As noted in [CF9], this result can be derived from Theorem 1.1 by applying [Sto2], which connects the full and truncated moment problems.

We conclude this section with some terminology and background results that we will employ in the sequel. Unless otherwise stated, we are in the general case, i.e., $n \geq 1$. For $p \equiv \sum_{i \in \mathbb{Z}_+^n, |i| \leq d} a_i x^i \in \mathcal{P}_d$, let $\hat{p} \equiv (a_i)$ denote the coefficient vector of p relative to the basis $\mathcal{B}_d$ of monomials in $\mathcal{P}_d$ in degree-lexicographic order. Define $\Lambda_\beta : \mathcal{P}_{2d} \mapsto \mathbb{R}$ by $\Lambda(\sum_{|i| \leq 2d} a_i x^i) = \sum a_i \beta_i$. Following [CF2] [CF8], we associate to $\beta \equiv \beta^{(2d)}$ the *moment matrix* $M_d \equiv M_d(\beta)$, with rows and columns X^i indexed by the elements of $\mathcal{B}_d$. The entry in row X^i, column X^j of M_d is β_{i+j} $(i, j \in \mathbb{Z}_+^n,\ |i|, |j| \leq d)$, so M_d is a real symmetric matrix characterized by $\langle M_d \hat{p}, \hat{q} \rangle = \Lambda_\beta(pq)$ $(p, q \in \mathcal{P}_d)$. If μ is a representing measure for β, then $\langle M_d \hat{p}, \hat{p} \rangle = \Lambda_\beta(p^2) = \int p^2 \, d\mu \geq 0$, and since M_d is real symmetric, it follows that M_d is positive semidefinite $(M_d \succeq 0)$.

For $p(x) \equiv \sum a_i x^i \in \mathcal{P}_d$, we have the column space element $p(X) \equiv \sum a_i X^i$, and a calculation shows that $p(X) = M_d \hat{p}$. In the sequel, we often write $\langle p(X), q(X) \rangle$ to denote $\langle M_d \hat{p}, \hat{q} \rangle$, even though, strictly speaking, $\langle p(X), q(X) \rangle$ means $\langle M_d \hat{p}, M_d \hat{q} \rangle$. Thus, we often denote β_{i+j} by $\langle X^i, X^j \rangle$. If β admits a representing measure μ, then

$$for\ p \in \mathcal{P}_d,\ \operatorname{supp} \mu \subseteq \mathcal{Z}(p) \Longleftrightarrow p(X) = 0\ (cf.\ [CF8,\ Prop.\ 2.10]). \tag{1.2}$$

It follows from (1.2) that $\operatorname{supp} \mu \subseteq \mathcal{V}(M_d)$, whence

$$r \equiv \operatorname{rank} M_d \leq \operatorname{card} \operatorname{supp} \mu \leq v \equiv \operatorname{card} \mathcal{V}(M_d)\ (cf.\ [CF8,\ Cor.\ 2.12]). \tag{1.3}$$

In the sequel we will frequently cite the following basic result of [CF2] [CF8] concerning the existence of a "minimal" representing measure, a representing measure μ satisfying $\operatorname{card} \operatorname{supp} \mu = \operatorname{rank} M_d$.

Theorem 1.3. *(Flat Extension Theorem, cf. [CF8, Thm. 1.1-1.2]) $\beta \equiv \beta^{(2d)}$ has a $\operatorname{rank} M_d$-atomic representing measure if and only if $M_d \succeq 0$ and M_d admits a flat moment matrix extension, i.e., a moment matrix extension M_{d+1} satisfying $\operatorname{rank} M_{d+1} = \operatorname{rank} M_d$. In this case, $\beta^{(2d+2)}$ admits a unique representing measure, $\mu \equiv \mu_{M_{d+1}}$, and μ satisfies $\operatorname{supp} \mu = \mathcal{V}(M_{d+1})$ and $\operatorname{card} \operatorname{supp} \mu = \operatorname{rank} M_d$. Further, M_{d+1} admits unique successive positive extensions $M_{d+2}, M_{d+3}, \ldots$, and these are flat extensions.*

Note that for the case of flat data ($M_d \succeq 0$ and $\operatorname{rank} M_d = \operatorname{rank} M(d-1)$), Theorem 1.3 (applied to $M(d-1)$) implies the existence of a unique ($\operatorname{rank} M_d$)-atomic representing measure for $\beta^{(2d)}$.

Suppose M_d is positive and admits a flat extension M_{d+1}. The unique representing measure for M_{d+1} referred to in Theorem 1.3 may be explicitly computed as follows (cf.

[CF8, Theorem 1.2]). Let $r = \operatorname{rank} M_d$, so that $\operatorname{card} \mathcal{V}(M_{d+1}) = r$ and $\mathcal{V}(M_{d+1}) \equiv \{w_i\}_{i=1}^r$. Let $\mathcal{B} \equiv \{X^{i_1}, \dots, X^{i_r}\}$ denote a basis for $\operatorname{Col} M_d$, and consider the Vandermonde-type matrix

$$V \equiv V_{\mathcal{B}} := \begin{pmatrix} w_1^{i_1} & \dots & w_r^{i_1} \\ \vdots & & \vdots \\ w_1^{i_r} & \dots & w_r^{i_r} \end{pmatrix}. \tag{1.4}$$

Then V is invertible, and [CF8] shows that $\beta^{(2d+2)}$ has the unique representing measure $\mu \equiv \mu_{M_{d+1}}$, of the form $\mu = \sum_{i=1}^r \rho_i \delta_{w_i}$, where δ_{w_i} is the unit-mass atomic measure with support $\{w_i\}$, and $\rho \equiv (\rho_1, \dots, \rho_r)$ is determined by $\rho^t = V^{-1}(\beta_{i_1}, \dots, \beta_{i_r})^t$ (where $\cdot^t$) denotes transpose). ([CF8] shows that μ is independent of $\mathcal{B}$.)

We next recall some properties of positive moment matrix extensions that we will refer to in the sequel. A key ingredient in our proofs is a result of Bayer and Teichmann [BT], which generalizes the classical theorem of Tchakaloff [Tch] concerning multivariable cubature. The result of Bayer and Teichmann implies that if $\beta \equiv \beta^{(2d)}$ admits a K-representing measure, then β admits a *finitely atomic* K-representing measure ν. Since ν has convergent power moments of all orders, it follows that M_d $(= M_d[\nu])$ admits successive positive, recursively generated extensions, namely, $M_{d+1}[\nu], M_{d+2}[\nu], \dots$.

By combining Theorem 1.3 with [BT], we have the following solution of the truncated K-moment problem for $K = \mathbb{R}^d$, expressed in terms of moment matrix extensions. A generalization to the case when K is a closed basic semialgebraic subset of $\mathbb{R}^d$ appears in [CF8, Cor. 1.4].

Theorem 1.4. *(Moment Matrix Extension Theorem, cf. [CF2] [CF8]) $\beta^{(2d)}$ has a representing measure if and only if there is an integer $k \ge 0$ such that M_d admits a positive moment matrix extension M_{d+k}, which in turn admits a flat extension M_{d+k+1}, i.e., $\operatorname{rank} M_{d+k+1} = \operatorname{rank} M_{d+k}$. In this case, we may take $k \le dim\, \mathcal{P}_{2n} - \operatorname{rank} M_d$.*

Theorem 1.4 is not, by itself, a concrete solution to TMP, but it does provide a framework for obtaining concrete solutions such as Theorem 1.1. Theorem 1.4 was originally proved in [CF2] for representing measures having convergent moments of all orders, but [BT] shows that this restriction in the hypothesis is unnecessary.

Consider a real symmetric block matrix $\widetilde{M} \equiv \begin{pmatrix} M & B \\ B^t & C \end{pmatrix}$. A result of Smul'jan [Smu] implies that $\widetilde{M} \succeq 0$ if and only if $M \succeq 0$, there exists a matrix W such that $B = MW$ (equivalently, $\operatorname{Ran} B \subseteq \operatorname{Ran} M$ [D]), and $C \succeq C^\flat \equiv W^t M W$ (note that $C^\flat$ is independent of W satisfying $B = MW$). In this case, the matrix $M^\flat \equiv [M; B] := \begin{pmatrix} M & B \\ B^t & C^\flat \end{pmatrix}$ is a positive *flat extension* of M, i.e., $\operatorname{rank} M^\flat = \operatorname{rank} M$. Consider a moment matrix extension

$$M_{d+1} \equiv \begin{pmatrix} M_d & B_{d+1} \\ B_{d+1}^t & C_{d+1} \end{pmatrix}.$$

If $M_d \succeq 0$, then M_{d+1} is a flat (hence positive) extension of M_d if and only if $B_{d+1} = M_d W$ (for some W) and $C_{d+1} = C^\flat \equiv W^t M_d W$; equivalently, $M_{d+1} = [M_d; B_{d+1}]$. Suppose $M_{d+1} \succeq 0$ and let $p \in \mathcal{P}_d$; the Extension Principle [F1] shows that if $p(X) = 0$ in

$\operatorname{Col} M_d$, then $p(X) = 0$ in $\operatorname{Col} M_{d+1}$, i.e., column dependence relations in M_d extend to M_{d+1}. It follows that

$$M_{d+1} \geq 0 \Longrightarrow \mathcal{V}(M_{d+1}) \subseteq \mathcal{V}(M_d). \tag{1.5}$$

Finally, for the planar case ($d = 2$), we consider the block matrix decomposition $M_d \equiv (M[i,j])_{0\leq i,j\leq d}$, where $M[i,j]$ is the matrix with $i+1$ rows and $j+1$ columns of the form

$$M[i,j] \equiv \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & & \cdots & \beta_{i-1,j+1} \\ \beta_{i+j-2,2} & & & \cdots & \beta_{i-2,j+2} \\ \vdots & & & \cdots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \beta_{j-2,i+2} & \cdots & \beta_{0,i+j} \end{pmatrix}. \tag{1.6}$$

Note that $M[i,j]$ has all of the moments in $\beta^{(2d)}$ of degree $i+j$ and has the Hankel-like property of being constant on cross-diagonals; in particular, in the extension M_{d+1}, block $C_{d+1} \equiv M[n+1,n+1]$ is a Hankel matrix. Note that $B_{d+1} = (M[i,d+1])_{0\leq i\leq d}$. For $0 \leq k \leq d$, we set $[B_{d+1}]_k = (M[i,d+1])_{0\leq i\leq k}$. Similarly, for $0 \leq k \leq d$ and $i,j \geq 0$, $i+j \leq d+1$, we define $[X^iY^j]_k := (\beta_{i,j},\ldots,\beta_{k-j,j},\ldots,\beta_{i,k-i})^t$, the truncation of column X^iY^j to rows X^rY^s with $r+s \leq k$.

2. The (x^2-1)-pure truncated moment problem

In this section we establish the existence of representing measures for a positive, recursively generated (x^2-1)-pure moment matrix M_d, i.e., a moment matrix $M_d \geq 0$ whose column dependence relations are precisely those that can be determined from $X^2 = 1$ by recursiveness, including all relations of the form

$$X^{i+2}Y^j = X^iY^j \ (i,j \geq 0, i+j \leq d-2). \tag{2.1}$$

Our hypothesis implies that $\mathcal{V}(M_d)$ coincides with the union of parallel lines given by $\Gamma \equiv \{(x,y) : x^2 = 1\}$, and that $\operatorname{rank} M_d = 2d+1$, with a basis for $\operatorname{Col} M_d$ of the form

$$\mathcal{B} = \{1, X, Y, XY, Y^2, \ldots, \ldots, XY^i, Y^{i+1}, \ldots, XY^{d-1}, Y^d\}.$$

Theorem 2.1. *If M_d is positive, recursively generated, and (x^2-1)-pure, then M_d admits infinitely many distinct flat extensions and corresponding $(2d+1)$-atomic representing measures.*

The proof that we present is motivated by the analysis of the (x^3-y)-pure case in [F3], but the conclusion that measures always exist in the (x^2-1)-pure case contrasts with our results in the (x^3-1)-pure case. We begin with the following computational formula for the moments of M_d; this result is valid for any moment matrix satisfying (2.1).

Lemma 2.2. $\beta_{i+2,j} = \beta_{i,j}$ $(i,j \geq 0,\ i+j+2 \leq 2d)$.

Proof. Suppose $\beta_{i+2,j} = \langle X^kY^l, X^pY^q\rangle$ with $k,l,p,q \geq 0$, $k+l$, $p+q \leq d$, $k+p = i+2$, $l+q = j$. If $k \geq 2$, then (2.1) implies $X^kY^l = X^{k-2}Y^l$, so $\beta_{i+2,j} = \langle X^kY^l, X^pY^q\rangle = \langle X^{k-2}Y^l, X^pY^q\rangle = \beta_{p+k-2,l+q} = \beta_{i,j}$. The case when $p \geq 2$ is similar, using $\langle X^kY^l, X^pY^q\rangle = \langle X^pY^q, X^kY^l\rangle$. Since $k+p = i+2 \geq 2$, we may now assume $k =$

$p = 1$, whence $i = 0$. If $q \geq 1$, then $\beta_{i+2,j} = \beta_{2,j} = \langle XY^l, XY^q \rangle = \langle Y^{l+1}, X^2Y^{q-1} \rangle = \langle X^2Y^{q-1}, Y^{l+1} \rangle = \langle Y^{q-1}, Y^{l+1} \rangle = \beta_{0,j}$. Finally, if $k = p = 1$ and $q = 0$, then $i = 0$, $l = j \leq 2d - 2$, so $\beta_{i+2,j} = \beta_{2,j} = \langle XY^j, X \rangle = \langle X^2Y^j, 1 \rangle = \langle Y^j, 1 \rangle = \beta_{0,j} = \beta_{i,j}$. □

As discussed in Section 1, the existence of a representing measure for $\beta^{(2d)}$ implies that M_d admits a positive, recursively generated moment matrix extension

$$M_{d+1} \equiv \begin{pmatrix} M_d & B_{d+1} \\ B_{d+1}^t & C_{d+1} \end{pmatrix}.$$

We next describe concretely the structure of block B_{d+1} for such an extension. Any such block $B_{d+1} \equiv ([M[j, d+1])_{0 \leq j \leq d}$ must satisfy the following requirement:

Property 2.3. *For $0 \leq j \leq d-1$, the data in $M[j, d+1]$ agrees with the data in M_d. In particular, for $0 \leq i \leq d+1$ and $k, l \geq 0$ with $k + l \leq d-1$, the element of B_{d+1} in row X^kY^l and column X^iY^{d+1-i}, $\langle X^iY^{d+1-i}, X^kY^l \rangle$, coincides with the "old" moment $\beta_{i+k,d+1-i+l}$ of M_d.*

More generally, throughout block B_{d+1}, the Hankel property of each block $M[j, d+1]$ must hold (cf. (1.9)):

Property 2.4. $\langle X^iY^{d+1-i}, X^kY^l \rangle = \langle X^{i-1}Y^{d+2-i}, X^{k+1}Y^{l-1} \rangle$ *($1 \leq i \leq d+1$, $k \geq 0$, $l \geq 1$, $k + l \leq d$).*

Note that within a block $M[j, d+1]$, in any band of consecutive columns, Property 2.3 implies Property 2.4.

Positivity of M_{d+1} also entails $\operatorname{Ran} B_{d+1} \subseteq \operatorname{Ran} M_d$, so we must show that the block that we construct satisfies this range inclusion. Positivity of M_{d+1} and the Extension Principle imply that the column relations (2.1) must hold in $\operatorname{Col} M_{d+1}$. The desired recursiveness of M_{d+1} then implies that in $(\, M_d \; B_{d+1} \,)$ we must have column dependence relations of the form

$$X^{d+1} = X^{d-1},\ X^dY = X^{d-2}Y, \ldots, X^2Y^{d-1} = Y^{d-1}; \tag{2.2}$$

thus, columns X^{d+1}, $X^dY, \ldots, X^2Y^{d-1}$ in B_{d+1} inherit the required Hankel property (Property 2.4) from the band of columns X^{d-1}, $X^{d-2}Y, \ldots$, Y^{d-1} in M_d. Moreover, these columns satisfy Property 2.3: for $d + 1 \geq i \geq 2$, $k, l \geq 0$, $k + l \leq d-1$, we have $\langle X^iY^{d+1-i}, X^kY^l \rangle = \langle X^{i-2}Y^{d+1-i}, X^kY^l \rangle = \beta_{i-2+k,d+1-i+l} = \beta_{i+k,d+1-i+l}$ (from Lemma 2.2). Next, we use "old" moments to define block $M[i, d+1]$ ($0 \leq i \leq d-1$) in columns XY^d and Y^{d+1}, i.e., in B_{d+1},

$$\langle X^iY^{d+1-i}, X^kY^l \rangle) := \beta_{i+k,d+1-i+l} \quad (0 \leq i \leq 1,\ l, k \geq 0, l + k \leq d-1). \tag{2.3}$$

Thus, all of the blocks $M[j, d+1]$ ($0 \leq j \leq d-1$) satisfy Properties 2.3 and 2.4.

To complete the definition of B_{d+1}, we next define the elements of columns XY^d and Y^{d+1} in block $M[d, d+1]$ (cf. (2.5) below). To insure moment matrix structure in this block, we propagate the elements of the previously defined column X^2Y^{d-1} along the cross diagonals of $B[d, d+1]$, as follows. For $k = 0$, 1 and $i, j \geq 0$ with $i + j = d$ and $0 \leq j \leq d - 2 + k$, we define

$$\langle X^kY^{d+1-k}, X^iY^j \rangle := \langle X^2Y^{d-1}, X^{k+i-2}Y^{j-k+2} \rangle \tag{2.4}$$

$(= \langle Y^{d-1}, X^{k+i-2}Y^{j-k+2}\rangle = \beta_{k+i-2,d+j-k+1}$ (from (2.2))). To complete the definition of $B[d,d+1]$, we choose $r,s \in \mathbb{R}$ and set $\langle XY^d, Y^d\rangle = \langle Y^{d+1}, XY^{d-1}\rangle := r$, and $\langle Y^{d+1}, Y^d\rangle := s$. Thus, $B[d,d+1]$ is of the form

$$\begin{pmatrix} \beta_{2d+1,0} & \beta_{2d,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{d,d+1} \\ \beta_{2d,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2,2d-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2,2d-1} & r \\ \beta_{d+1,d} & \beta_{d,d+1} & \cdot & \cdot & \cdot & \cdot & \beta_{2,2d-1} & r & s \end{pmatrix}, \tag{2.5}$$

with $\beta_{i,2d+1-i} = \beta_{i-2,2d+1-i}$ $(2 \le i \le 2d+1)$.

Having defined $B_{d+1} \equiv B_{d+1}[r,s]$ to be consistent with recursiveness for M_{d+1}, we next establish the range inclusion $\operatorname{Ran} B_{d+1} \subseteq \operatorname{Ran} M_d$. It is clear from (2.2) that columns $X^{d+1}, \dots, X^2Y^{d-1}$ of B_{d+1} belong to the column space of M_d, which coincides with $\operatorname{Ran} M_d$, so it suffices to consider columns XY^d, and Y^{d+1}. Let J denote the compression of M_d to rows and columns indexed by the elements of basis $\mathscr{B}$; thus, J is positive definite $(J \succ 0)$. For $k,l \ge 0$ with $k+l \le d+1$, let $[X^kY^l]_{\mathscr{B}}$ denote the compression of column X^kY^l of $\begin{pmatrix} M_d & B_{d+1} \end{pmatrix}$ to the rows indexed by the elements of $\mathscr{B}$. Note that the columns of J are of the form $[X^pY^q]_{\mathscr{B}}$ (in degree-lexicographic order) with $p,q \ge 0$, $p+q \le d$, $p \le 1$. Since J is invertible, for $0 \le i \le 1$ and $j = d+1-i$, we may express $[X^iY^j]_{\mathscr{B}}$ as a linear combination of the columns of J, i.e.,

$$[X^iY^j]_{\mathscr{B}} = \sum_{X^pY^q \in \mathscr{B}} c_{pq}^{(ij)} [X^pY^q]_{\mathscr{B}} \; (c_{pq}^{(ij)} \in \mathbb{R}). \tag{2.6}$$

We claim that the same relation holds in the full columns of $\begin{pmatrix} M_d & B_{d+1} \end{pmatrix}$.

Lemma 2.5. *In* $\operatorname{Col}\begin{pmatrix} M_d & B_{d+1} \end{pmatrix}$, $X^iY^j = \sum_{X^pY^q \in \mathscr{B}} c_{pq}^{(ij)} X^pY^q$ $(0 \le i \le 1,\ j = d+1-i)$.

To prove Lemma 2.5 it suffices to show that for $k,l \ge 0$, $k+l \le d$, in columns X^iY^j of B_{d+1} $(0 \le i \le 1,\ j = d+1-i)$,

$$\langle X^iY^j, X^kY^l\rangle = \sum_{X^pY^q \in \mathscr{B}} c_{pq}^{(ij)} \langle X^pY^q, X^kY^l\rangle. \tag{2.7}$$

We note that (2.6) already shows that (2.7) holds for the rows indexed by $\mathscr{B}$:

$$\text{(2.7) holds whenever } k,l \ge 0,\ k+l \le d,\ k \le 1. \tag{2.8}$$

To establish (2.7), we begin with the following reductions, which we will also use in Section 3.

Lemma 2.6. *If* row X^kY^l *of* $(M_d\ B_{d+1})$ *is a linear combination of rows* $X^{k'}Y^{l'}$ *for which (2.7) holds, then (2.7) holds for* row X^kY^l.

Proof. Suppose row $X^kY^l = \sum a_{k'l'}$ row $X^{k'}Y^{l'}$, i.e.,

$$\langle X^uY^v, X^kY^l\rangle = \sum a_{k'l'} \langle X^uY^v, X^{k'}Y^{l'}\rangle \; (u,v \ge 0, u+v \le d+1).$$

Now $\langle X^iY^j, X^kY^l\rangle = \sum a_{k'l'}\ \langle X^iY^j, X^{k'}Y^{l'}\rangle = \sum a_{k'l'}\ \sum c_{pq}^{ij}\langle X^pY^q, X^{k'}Y^{l'}\rangle$

$= \sum c_{pq}^{ij}\sum a_{k'l'}\ \langle X^pY^q, X^{k'}Y^{l'}\rangle = \sum c_{pq}^{ij}\langle X^pY^q, X^kY^l\rangle$, so (2.7) holds for row X^kY^l. □

Lemma 2.7. *In* $(\ M_d\ B_{d+1}\ )$, row$X^{k+2}Y^l =$ rowX^kY^l *($k, l \geq 0$, $k+l+2 \leq d$).*

Proof. Since M_d is real symmetric, it follows from (2.1) that the result holds in the rows of M_d. It then follows from (2.2) that the result holds for the rows of $(\ M_d\ B_{d+1}\ )$ in columns $X^{d+1},\dots,X^2Y^{d-1}$ of B_{d+1}. It thus remains to prove that in columns XY^d and Y^{d+1},

$$\langle X^iY^{d+1-i}, X^{2+k}Y^l\rangle = \langle X^iY^{d+1-i}, X^kY^l\rangle\ (i = 0, 1,\ k, l \geq 0,\ k+l+2 \leq d). \qquad (2.9)$$

We consider first the case when $2+k+l \leq d-1$. From (2.3),

$\langle X^iY^{d+1-i}, X^{2+k}Y^l\rangle = \beta_{2+k+i,d+1-i+l}$

$= \langle X^{2+k+i}Y^{l+1-i}, Y^d\rangle$ (in M_d, since $2+k+i,\ l+1-i \geq 0$, $(2+k+i)+(l+1-i) \leq d$)

$= \langle X^{k+i}Y^{l+1-i}, Y^d\rangle$ (from (2.1))

$= \beta_{k+i,l+1-i+d} \equiv \langle X^iY^{d+1-i}, X^kY^l\rangle$ (from (2.3), since $k+l \leq d-3$).

In the remaining case we have $2+k+l = d$. From (2.4),

$\langle X^iY^{d+1-i}, X^{2+k}Y^l\rangle := \langle X^2Y^{d-1}, X^{k+i}Y^{l+2-i}\rangle$

$= \langle Y^{d-1}, X^{k+i}Y^{l+2-i}\rangle$ (from (2.2))

$= \beta_{k+i,d+l+1-i}$ (in M_d)

$= \langle X^iY^{d+1-i}, X^kY^l\rangle$ (from (2.3), since $k+l = d-2$). □

Proof. (Proof of Lemma 2.5). It remains to prove (2.7) for $k, l \geq 0$, $k+l \leq d$, and for this we use induction on $\rho \equiv k+l$. For $\rho = 0,\ 1$, the result follows from (2.8), since in these cases, $k \leq 1$. Assume (2.7) holds whenever $\rho < k'+l'$ ($k', l' \geq 0$, $k'+l' \leq d$). Consider X^kY^l with $k, l \geq 0$, $k+l = k'+l'$. If $k \leq 1$, then the result follows from (2.8). If $k \geq 2$, then Lemma 2.7 shows that row$X^kY^l =$ row$X^{k-2}Y^l$. By induction, (2.7) holds for row $X^{k-2}Y^l$, so the result follows from Lemma 2.6. □

The preceding discussion yields the following result.

Proposition 2.8. *If M_d is positive and recursively generated, with column relations determined entirely from $X^2 = 1$ by recursiveness, then M_d admits a moment matrix block $B_{d+1} \equiv B_{d+1}[r,s]$ compatible with a recursively generated extension M_{d+1}, and any such block B_{d+1} satisfies* Ran $B_{d+1} \subseteq$ Ran M_d.

For the (x^2-1)-pure case, having just described the structure of block B_{d+1} for a positive, recursively generated moment matrix extension $M_{d+1} \equiv$

$\begin{pmatrix} M_d & B_{d+1} \\ B_{d+1}^t & C_{n+1} \end{pmatrix}$, we next consider conditions for the existence of block $C_{d+1} \equiv M[d+1, d+1]$ for such an extension. Since $\operatorname{Ran} B_{d+1} \subseteq \operatorname{Ran} M_d$ (Proposition 2.8), there exists a matrix W such that $B_{d+1} = M_d W$. As discussed in Section 1, $M_{d+1} \succeq 0$ if and only if

$$C_{d+1} \succeq C^\flat \equiv B_{d+1}^t W \ (= W^t M_d W). \tag{2.10}$$

Further, M_{d+1} is a flat extension of M_d if and only if $C^\flat$ has the form of a moment matrix block C_{d+1} and $C_{d+1} = C^\flat$.

Since M_{d+1} is to be positive, the Extension Principle implies that each of the column relations in (2.1) persists in M_{d+1}. From this and the required recursiveness in M_{d+1}, it follows that each of the column relations in (2.2) must hold in M_{d+1}; in particular, these relations must hold in the columns of $\begin{pmatrix} B_{d+1}^t & C_{d+1} \end{pmatrix}$. The construction of $[M_d; B_{d+1}]$ shows that relations (2.2) also hold in $\begin{pmatrix} B_{d+1}^t & C^\flat \end{pmatrix}$, so C_{d+1} agrees with $C^\flat$ in columns $X^{d+1}, \dots, X^2 Y^{d-1}$. From (2.2), these columns agree with columns $X^{d-1}, \dots, Y^{d-1}$ of the moment matrix block B_{d+1}^t, so these columns form a Hankel band in $C^\flat$. Since $C^\flat \equiv (C_{ij}^\flat)_{1 \le i,j \le d+2}$ is positive, hence real symmetric, it follows that $C^\flat$ has the form of a moment matrix block C_{d+1}, i.e., $C^\flat$ is Hankel, if and only if

$$C_{d+2,d}^\flat = C_{d+1,d+1}^\flat. \tag{2.11}$$

We next compute $C^\flat$ explicitly so as to analyze (2.11). As above, let $J \equiv [M_d]_{\mathcal{B}}$ denote the compression of M_d to the rows and columns indexed by basis $\mathcal{B}$, so that $J > 0$. Let us write

$$J \equiv \begin{pmatrix} M & x \\ x^t & \Delta \end{pmatrix},$$

where M is the compression of M_d to the rows and columns indexed by the elements of basis $\mathcal{B}$ except for Y^d (the final basis element in the degree-lexicographic ordering) and

$$\begin{pmatrix} x \\ \Delta \end{pmatrix}$$

is the compression of column Y^d in M_d to rows indexed by $\mathcal{B}$; thus, $\Delta = \langle M_d \widehat{Y^d}, \widehat{Y^d} \rangle = \beta_{0,2d}$. Since $J > 0$, then $M > 0$ and $\Delta > x^t M^{-1} x$. Thus we have

$$\beta_{0,2d} > \beta_{0,2d}^\flat \equiv x^t M^{-1} x.$$

A calculation shows that

$$J^{-1} = \begin{pmatrix} P & v \\ v^t & \epsilon \end{pmatrix},$$

where

$$P = M^{-1}(1 + \epsilon x x^t M^{-1}), \ v = -\epsilon M^{-1} x, \ \epsilon = \frac{1}{\Delta - x^t M^{-1} x} (> 0).$$

To compute $C^\flat$, let $\widetilde{W} = J^{-1} [B_{d+1}]_{\mathcal{B}}$, where $[B_{d+1}]_{\mathcal{B}}$ is the compression of B_{d+1} to the rows indexed by $\mathcal{B}$; thus, $[M_d]_{\mathcal{B}} \widetilde{W} = [B_{d+1}]_{\mathcal{B}}$. We next define a matrix W, with the same number of rows as M_d and the same number of columns as $\widetilde{W}$. If $X^i Y^j$ is in basis $\mathcal{B}$, then row $X^i Y^j$ of W coincides with row $X^i Y^j$ of $\widetilde{W}$. If $X^i Y^j$ is not in $\mathcal{B}$, then row $X^i Y^j$ of W is a row of zeros. Lemma 2.5 implies that $B_{d+1} = M_d W$, so (2.10) implies $C^\flat = B_{d+1}^t W$. A straightforward calculation shows that $B_{d+1}^t W = [B_{d+1}]_{\mathscr{B}}^t \widetilde{W}$, so we have

$$C^\flat = [B_{d+1}]_{\mathscr{B}}^t \widetilde{W}. \tag{2.12}$$

Turning to (2.11), note that

$$C^\flat_{d+2,d} = \langle C_{d+1}\widehat{x^2y^{d-1}}, \widehat{y^{d+1}}\rangle = \langle M_{d+1}\widehat{y^{d-1}}, \widehat{y^{d+1}}\rangle = \langle B_{d+1}\widehat{y^{d+1}}, \widehat{y^{d-1}}\rangle = \beta_{0,2d},$$

i.e.,

$$C^\flat_{d+2,d} = \beta_{0,2d}. \tag{2.13}$$

Let $\widetilde{u} := [XY^d]_{\mathscr{B}} \equiv \begin{pmatrix} u_1 \\ \vdots \\ u_m \\ r \end{pmatrix}$ (where $m := (\mathrm{card}\,\mathscr{B}) - 1$). From (2.12),

$$C^\flat_{d+1,d+1} = \langle C^\flat \widehat{xy^d}, \widehat{xy^d}\rangle = \widetilde{u}^t J^{-1}\widetilde{u}. \tag{2.14}$$

Recall that in column XY^d, elements in rows of degree at most $d-1$ are "old moments", while those from rows of degree d come from (2.4) (except in row Y^d, where r appears). We thus have $u \equiv \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} \beta_{1,d} \\ \beta_{0,d} \\ \vdots \\ \beta_{1,2d-1} \\ \beta_{0,2d-1} \end{pmatrix}$ (where all but the final element are "old" moments and the final element is computed using (2.4)).

To compute $C^\flat_{d+1,d+1}$ explicitly, let $v := \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$, and let p_i denote the $i-th$ row of P $(1 \le i \le m)$. Then

$$\begin{aligned}
C^\flat_{d+1,d+1} &= \langle J^{-1}\widetilde{u}, \widetilde{u}\rangle \\
&= \left\langle \begin{pmatrix} p_1 & v_1 \\ \vdots & \vdots \\ p_m & v_m \\ v^t & \epsilon \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \\ r \end{pmatrix}, \begin{pmatrix} u_1 \\ \vdots \\ u_m \\ r \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} \langle p_1, u\rangle + v_1 r \\ \vdots \\ \langle p_m, u\rangle + v_m r \\ v^t u + \epsilon r \end{pmatrix}, \begin{pmatrix} u_1 \\ \vdots \\ u_m \\ r \end{pmatrix} \right\rangle \\
&= (\langle p_1, u\rangle + v_1 r)u_1 + \cdots + (\langle p_m, u\rangle + v_m r)u_m + (v^t u + \epsilon r)r \\
&= \epsilon r^2 + 2v^t u r + \langle P u, u\rangle \equiv \psi(r),
\end{aligned}$$

i.e.,

$$C^\flat_{d+1,d+1} = \psi(r). \tag{2.15}$$

Thus, from (2.13) and (2.15), the flatness condition (2.11) is equivalent to the existence of r such that

$$\beta_{0,2d} = \psi(r). \tag{2.16}$$

Viewing $\psi(r)$ as a quadratic in r, whose lead coefficient is positive, the minimum value of $\psi(r)$ occurs at $r_{min} := \frac{-v^t u}{\epsilon}$, and

$$\psi(r_{min}) = \langle Pu, u\rangle - \frac{(v^t u)^2}{\epsilon}.$$

We claim that

$$\psi(r_{min}) = \beta^{\flat}_{0,2d} (= x^t M^{-1} x). \tag{2.17}$$

Since $P = M^{-1}(1 + \epsilon x x^t M^{-1})$, then $\langle Pu, u\rangle = \langle (1 + \epsilon x x^t M^{-1})u, M^{-1}u\rangle$

$$= \langle u, M^{-1}u\rangle + \epsilon\langle x^t M^{-1}u, x^t M^{-1}u\rangle = \langle u, M^{-1}u\rangle + \epsilon(x^t M^{-1}u)^2.$$

Thus, (2.17) is equivalent to

$$\langle u, M^{-1}u\rangle + \epsilon(x^t M^{-1}u)^2 = \frac{(v^t u)^2}{\epsilon} + x^t M^{-1}x. \tag{2.18}$$

Since $v^t u = -\epsilon x^t M^{-1}u$, then (2.18) is equivalent to

$$\langle u, M^{-1}u\rangle + \epsilon(x^t M^{-1}u)^2 = \frac{\epsilon^2 (x^t M^{-1}u)^2}{\epsilon} + x^t M^{-1}x,$$

or

$$\langle u, M^{-1}u\rangle = \langle x, M^{-1}x\rangle. \tag{2.19}$$

Consider the compression of M to consecutive rows Y^j, XY^j $(0 \le j \le d-1)$ and consecutive columns Y^i, XY^i $(0 \le i \le d-1)$:

$$M_{ij} := \begin{pmatrix} \beta_{0,i+j} & \beta_{1,i+j} \\ \beta_{1,i+j} & \beta_{2,i+j} \end{pmatrix} = \begin{pmatrix} \beta_{0,i+j} & \beta_{1,i+j} \\ \beta_{1,i+j} & \beta_{0,i+j} \end{pmatrix}.$$

Let $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and set $\mathcal{U} = U^{(d)}$ (the direct sum of d copies of U). Then $\mathcal{U} = \mathcal{U}^t = \mathcal{U}^{-1}$ and $\mathcal{U}^{-1} M \mathcal{U} = M$, whence $\mathcal{U}^{-1} M^{-1} \mathcal{U} = M^{-1}$. Note that

$$x = \begin{pmatrix} \beta_{0,d} \\ \beta_{1,d} \\ \vdots \\ \beta_{0,2d-1} \\ \beta_{1,2d-1} \end{pmatrix},$$

so that $\mathcal{U}x = u$. We have

$$\langle u, M^{-1}u\rangle = \langle \mathcal{U}x, M^{-1}\mathcal{U}x\rangle = \langle x, \mathcal{U}M^{-1}\mathcal{U}x\rangle = \langle x, \mathcal{U}^{-1}M^{-1}\mathcal{U}x\rangle = \langle x, M^{-1}x\rangle.$$

This establishes (2.19) and thus proves (2.17). Now, $\beta_{0,2d} > \beta^{\flat}_{0,2d} = \psi(r_{min})$. Thus, there exists r such that $\psi(r) = \beta_{0,2d}$. With this choice of r (and any s), (2.16) holds, so $C^{\flat}$ is Hankel, and thus

$$\begin{pmatrix} M_d & B_{d+1}[r,s] \\ B_{d+1}[r,s]^t & C^{\flat} \end{pmatrix}$$

is a flat moment matrix extension of M_d. This completes the proof of Theorem 2.1.

We conclude this section with an example illustrating Theorem 2.1.

Example 2.9. Consider M_3 defined by

$$M_3 := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & a \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & b \\ 1 & 0 & 0 & 1 & 0 & a & 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & b & 1 & 0 & a & 0 \\ 0 & 0 & a & 0 & b & 0 & 0 & a & 0 & e \end{pmatrix}, \tag{2.20}$$

with $a-1>b>0$ and $e=a^2+b^2+1$. A calculation shows that M_3 is (x^2-1)-pure, with $\operatorname{rank} M_3 = 7$ and column basis $\mathscr{B} = \{1, X, Y, XY, Y^2, XY^2, Y^3\}$. Following the proof of Theorem 2.1, we compute $\widetilde{u} = (0, 0, b, a, 0, 0, r)$ $(\beta_{1,6} := r)$ and

$$\psi(r) = r^2 - 4abr + a^2 + b^2 + 4a^2b^2.$$

Now $\beta_{0,6} = e = a^2+b^2+1$, so one solution to $\psi(r) = e$ is $r = 1+2ab$. With this value for r, we see that in a flat extension M_4 we must have the column relation

$$XY^3 = (-a+b)Y + (a-b)XY + Y^3,$$

and we define $Q(x,y) := xy^3 - ((-a+b)y + (a-b)xy + y^3)$. Further, for fixed $s \in \mathbb{R}$, if we define $\beta_{07} := s$, then we find that in the corresponding flat extension M_4 we have the column relation

$$Y^4 = \tfrac{-1}{a+b-1}1 + \tfrac{-1}{a+b-1}X - asY - bsXY \\ + \frac{1-a+a^2+ab}{a+b-1}Y^2 + \frac{1-b+b^2+ab}{a+b-1}XY2 + sY^3,$$

and we define

$$S(x,y) := y^4 - (\tfrac{-1}{a+b-1} + \tfrac{-1}{a+b-1}x - asy - bsxy \\ + \frac{1-a+a^2+ab}{a+b-1}y^2 + \frac{1-b+b^2+ab}{a+b-1}xy^2 + sy^3).$$

The support of a 7-atomic measure for M_3 thus consists of the common zeros of x^2-1, $Q(x,y)$, and $S(x,y)$. With $x=1$, we see that every y satisfies $Q(1,y)=0$, and [CF8, Theorem 1.2] insures that $S(1,y)=0$ has 4 distinct real roots. (For example, with $b=1$, $a=3$, $e=11$, $s=0$, these 4 roots are $\pm\frac{7-\sqrt{43}}{3}$ and $\pm\frac{7+\sqrt{43}}{3}$.) With $x=-1$, we see that the 3 roots of $Q(-1,y)=0$ are $y=0$, $y=\pm\sqrt{a-b}$, while $S(-1,y)=0$ has an additional root, $y=s$. Thus there are exactly 7 common zeros for p, Q, and S (4 on the line $x=1$, 3 on the line $x=-1$), and there is a unique representing measure for M_3 supported on these points; the densities for the measure can be computed as described in Section 1.

3. THE TRUNCATED MOMENT PROBLEM ON PARALLEL LINES

In this section we prove Theorem 1.2, which we restate in an equivalent form (as discussed in Section 1). Let $p(x,y) := x^2 - 1$, so that $\Gamma \equiv \mathcal{Z}(p)$ consists of the parallel lines $x = 1$ and $x = -1$.

Theorem 3.1. *Let $d \geq 2$. $\beta \equiv \beta^{(2d)}$ has a representing measure supported in Γ if and only if M_d is positive semidefinite, recursively generated, satisfies the variety condition $v \leq r$, and has a column dependence relation $X^2 = 1$.*

The necessity of the conditions is clear from the discussion in Section 1, so we focus on sufficiency. We assume that M_d is positive and recursively generated, with the column relation $X^2 = 1$. If M_1 is singular, the existence of a representing measure follows from [CF3], so we assume in the sequel that $M_1 > 0$. If M_d is p-pure, then $\mathcal{V}(M_d) = \Gamma$, so $v > r$, and the existence of representing measures follows from Theorem 2.1. We may thus assume that M_d is not p-pure. Suppose next that M_d has a nontrivial column dependence relation of the form $Y^j = q(X,Y)$, where $0 \leq j \leq d$, $\deg q \leq j$, and q has no y^j term. Then since we also have $X^i = 1$ with $i = 2$, M_d is *recursively determinate* in the sense of [F2]. Since $i + j - 2 = j \leq d$, it follows from [CF11, Corollary 2.4] that M_d has a unique flat extension (and a corresponding representing measure).

We may thus assume that M_d is *not* recursively determinate. In the remaining case, there is a minimal i, $2 \leq i \leq d$ such that M_d has a column relation of the form

$$XY^i = a_0 1 + b_1 X + a_1 Y + \cdots + b_j XY^{j-1} + a_j Y^j + \cdots + b_i XY^{i-1} + a_i Y^i. \tag{3.1}$$

Setting $S(x,y) := a_0 + b_1 x + a_1 y + \cdots + b_1 xy^{i-1} + a_i y^i$, and $q(x,y) = xy^i - S(x,y)$, we have $q(X,Y) = 0$ in $\operatorname{Col} M_d$, and thus

$$\mathcal{V}(M_d) \subseteq \mathcal{V}(M_{i+1}) \subseteq \mathcal{Z}(p) \cap \mathcal{Z}(q) \equiv V_1 \cup V_2, \tag{3.2}$$

where $V_1 := \{(1,y) : q(1,y) = 0\}$ and $V_2 := \{(-1,y) : q(-1,y) = 0\}$. In this case, we have a basis for $\operatorname{Col} M_d$ of the form

$$\mathcal{B} := \{1,\ X,\ Y, \ldots, XY^j, Y^{j+1}, \ldots, XY^{i-1},\ Y^i,\ Y^{i+1}, \ldots, Y^k, \ldots, Y^d\}. \tag{3.3}$$

Let

$$Q_1(y) := q(1,y) = (1 - a_i)y^i - ((a_0 + b_1) + (a_1 + b_2)y + \cdots + (a_{i-1} + b_i)y^{i-1})$$

and

$$Q_2(y) := q(-1,y) = -(1 + a_i)y^i + (b_i - a_{i-1})y^{i-1} + \cdots + (b_1 - a_0).$$

For $j = 1,2$, if $Q_j \not\equiv 0$, then $\operatorname{card} V_j = \operatorname{card} \mathcal{Z}(Q_j) \leq i$. Thus, if $Q_1 \not\equiv 0$ and $Q_2 \not\equiv 0$, then $v = \operatorname{card} \mathcal{V}(M_d) \leq \operatorname{card} \mathcal{V}(M_{i+1}) \leq 2i < 2i + 2 = \operatorname{rank} M_{i+1} \leq \operatorname{rank} M_d$, so M_d fails to satisfy the variety condition (and thus has no measure). It follows that if M_d satisfies the variety condition, i.e., $r \leq v$, then $Q_1 \equiv 0$ or $Q_2 \equiv 0$. We will show that in this case M_d admits a flat extension M_{d+1}, and thus has a representing measure (necessarily contained in Γ). The proofs of the cases when $Q_1 \equiv 0$ and when $Q_2 \equiv 0$ are very similar, so we will consider only the former case.

In the sequel we assume $Q_1 \equiv 0$, or, equivalently,

$$a_0 = -b_1,\ a_1 = -b_2,\ \ldots, a_{i-1} = -b_i,\ a_i = 1. \tag{3.4}$$

To construct block B_{d+1} for a flat extension, we begin by propagating the relation $X^2 = 1$ to determine columns $X^{d+1}, \ldots, X^2Y^{d-1}$ as in (2.2). Thus, as discussed in Section 2, this band of columns satisfies Properties 2.3 and 2.4. Further, we propagate relation (3.1) to define

$$XY^d := (Sy^{d-i})(X,Y) = a_0Y^{d-i} + b_1XY^{d-i} + a_1Y^{d-i+1} + \cdots + b_iXY^{d-1} + a_iY^d. \quad (3.5)$$

At this point, we must verify that XY^d is Hankel with respect to column X^2Y^{d-1}, which coincides with Y^{d-1} by (2.2). Thus, it is sufficient to show that

$$Y^{d-1} = (Sxy^{d-i-1})(X,Y), \quad (3.6)$$

since $(Sxy^{d-i-1})(X,Y)$ is readily seen to be Hankel with respect to $(Sy^{d-i})(X,Y)$ (= XY^d). Indeed, for this last assertion, denoting $S(X,Y) \equiv \sum_{r+s\leq i} a_{rs}X^rY^s$, for $k \geq 1$, $l \geq 0$, $k+l \leq d$, we have

$$\begin{aligned}\langle (Sy^{d-i})(X,Y), X^kY^l\rangle &= \langle \sum a_{rs}X^rY^{s+d-i}, X^kY^l\rangle \\ &= \sum a_{rs}\langle X^{r+1}Y^{s+d-i-1}, X^{k-1}Y^{l+1}\rangle \\ &= \langle (Sxy^{d-i-1})(X,Y), X^{k-1}Y^{l+1}\rangle. \end{aligned} \quad (3.7)$$

To establish (3.6), we begin with

$$\begin{aligned}(Sxy^{d-i-1})(X,Y) &= a_0XY^{d-i-1} + b_1Y^{d-i-1} + a_1XY^{d-i} + b_2Y^{d-i} + a_2XY^{d-i+1} \\ &+ \cdots + a_{i-1}XY^{d-2} + b_iY^{d-2} + a_iXY^{d-1}. \end{aligned} \quad (3.8)$$

Further, by recursiveness in M_d,

$$\begin{aligned}XY^{d-1} &= (Sy^{d-i-1})(X,Y) = a_0Y^{d-i-1} + b_1XY^{d-i-1} + a_1Y^{d-i} + b_2XY^{d-i} + a_2Y^{d-i+1} \\ &+ \cdots + a_{i-1}Y^{d-2} + b_iXY^{d-2} + a_iY^{d-1}. \end{aligned} \quad (3.9)$$

Since $a_i = 1$, (3.8) and (3.9) imply

$$\begin{aligned}(Sxy^{d-i-1})(X,Y) &= (a_0+b_1)(Y^{d-i-1} + XY^{d-i-1}) \quad +(a_1+b_2)(Y^{d-i} + XY^{d-i}) + \\ &\cdots + (a_{i-1}+b_i)(Y^{d-2} + XY^{d-2}) + Y^{d-1}. \end{aligned} \quad (3.10)$$

It now follows from (3.4) that $(Sxy^{d-i-1})(X,Y) = Y^{d-1}$. Thus, X^2Y^{d-1} is well-defined and B_{d+1} satisfies Property 2.4 in columns $X^{d+1}, \ldots, X^2Y^{d-1}, XY^d$.

We next verify that Property 2.3 holds in column XY^d. We must show that

$$For\ k,l \geq 0,\ k+l \leq d-1,\ \langle XY^d, X^kY^l\rangle = \beta_{k+1,d+l}. \quad (3.11)$$

By definition of XY^d, we have

$$\begin{aligned}\langle XY^d, X^kY^l\rangle &= a_0\langle Y^{d-i}, X^kY^l\rangle + b_1\langle XY^{d-i}, X^kY^l\rangle + a_1\langle Y^{d-i+1}, X^kY^l\rangle \quad (3.12) \\ &+ b_2\langle XY^{d-i+1}, X^kY^l\rangle + a_2\langle Y^{d-i+2}, X^kY^l\rangle + \cdots + b_i\langle XY^{d-1}, X^kY^l\rangle + a_i\langle Y^d, X^kY^l\rangle. \end{aligned}$$

Now, in M_d we have

$$\begin{aligned}\beta_{k+1,d+l} &= \langle XY^{d-1}, X^kY^{l+1}\rangle = a_0\langle Y^{d-i-1}, X^kY^{l+1}\rangle + b_1\langle XY^{d-i-1}, X^kY^{l+1}\rangle \\ &+ a_1\langle Y^{d-i}, X^kY^{l+1}\rangle + b_2\langle XY^{d-i}, X^kY^{l+1}\rangle + a_2\langle Y^{d-i+1}, X^kY^{l+1}\rangle \\ &+ \cdots + b_i\langle XY^{d-2}, X^kY^{l+1}\rangle + a_i\langle Y^{d-1}, X^kY^{l+1}\rangle. \end{aligned} \quad (3.13)$$

Note that the degree of each column X^rY^s on the right side of the last equation is at most $d-1$, so in M_d, $\langle X^rY^s, X^kY^{l+1}\rangle = \langle X^rY^{s+1}, X^kY^l\rangle$, and thus the expressions in (3.12) and (3.13) are equal. This completes the proof that columns $X^{d+1},\dots,XY^d$ satisfy Properties 2.3 and 2.4.

To define column Y^{d+1}, we use old moments in blocks $M[0,d+1],\dots,M[d-1,d+1]$, which insures that Properties 2.3 and 2.4 hold in the portions of Y^{d+1} in these blocks. To define Y^{d+1} in block $M[d,d+1]$, we use the Hankel requirement to define

$$\langle Y^{d+1}, X^kY^l\rangle := \langle XY^d, X^{k-1}Y^{l+1}\rangle \; (k\geq 1,\ l\geq 0, k+l=d). \tag{3.14}$$

To complete Y^{d+1} in B_{d+1}, for $\alpha\in\mathbb{R}$, we define $\langle Y^{d+1}, Y^d\rangle := \alpha$.

The preceding discussion shows that B_{d+1} satisfies Properties 2.3 and 2.4. We will show below that $\operatorname{Ran} B_{d+1}\subseteq \operatorname{Ran} M_d$, so that $B_{d+1}=M_dW$ for some W. Assuming this, observe that $C^\flat := W^tM_dW$ is necessarily Hankel (i.e., $C^\flat$ is a moment matrix block of the form C_{d+1}). Indeed, the flat extension construction requires that relations (2.1) and (3.5) hold in $C^\flat$. Essentially the same argument that was used to show that X^2Y^{d-1} is well-defined in B_{d+1} also shows that X^2Y^{d-1} is well-defined in $C^\flat$. It thus follows as in the discussion of B_{d+1} that the band of columns $X^{d+1},\dots,XY^d$ is constant on each cross-diagonal segment within the band. Since $C^\flat$ is real symmetric, we thus see that $C^\flat$ is Hankel.

To complete the proof of Theorem 3.1, it remains to prove that $\operatorname{Ran} B_{d+1}\subseteq \operatorname{Ran} M_d$. The argument is similar to that given in Section 2, but now using for basis $\mathcal{B}$ the basis in (3.3). Note that by definition, columns $X^{d+1},\dots,XY^d$ of B_{d+1} belong to $\operatorname{Col} M_d$(= $\operatorname{Ran} M_d$), so it suffices to show that $Y^{d+1}\in \operatorname{Col} M_d$. Let $J := [M_d]_{\mathcal{B}}$ (>0) and denote

$$[Y^{d+1}]_{\mathcal{B}} := \sum_{X^rY^s\in\mathcal{B}} a_{rs}[X^rY^s]_{\mathcal{B}}. \tag{3.15}$$

We claim that in B_{d+1}, $Y^{d+1} = \sum_{X^rY^s\in\mathcal{B}} a_{rs}X^rY^s$. Equivalently, we seek to show that for $k,l\geq 0$, $k+l\leq d$,

$$\langle Y^{d+1}, X^kY^l\rangle = \langle \sum_{X^rY^s\in\mathcal{B}} a_{rs}X^rY^s, X^kY^l\rangle. \tag{3.16}$$

From (3.15), (3.16) holds if X^kY^l is in $\mathcal{B}$. If X^kY^l satisfies $k\geq 2$, then (3.16) holds by exactly the same argument as in Section 2 (using Lemmas 2.6-2.7, but with (2.6)-(2.7) replaced by (3.15)-3.16)). Thus, to establish (3.16) it remains to consider rows X^kY^l of the form XY^{i+u} ($0\leq u\leq d-i-1$). To show that (3.16) holds for these rows, Lemma 2.6 (or, rather, an exact analogue with (2.7) replaced by (3.16)) implies that it suffices to verify that each such row is a linear combination of rows satisfying (3.16). To this end, we require a preliminary result. Let us write the column relation in (3.1) as

$$XY^i = \sum_{j<i} c_jXY^j + \sum_{k<i+1} Y^k. \tag{3.17}$$

Lemma 3.2. *In* $(\,M_d\ B_{d+1}\,)$,

$$\operatorname{row} XY^{i+u} = \sum_{j<i} c_j \operatorname{row} XY^{j+u} + \sum_{k<i+1} \operatorname{row} Y^{k+u}\ (u\geq 0,\ u+i+1\leq d). \tag{3.18}$$

Proof. In M_d, from (3.17) and recursiveness, we have the column relation

$$XY^{i+u} = \sum_{j<i} c_j XY^{j+u} + \sum_{k<i+1} d_k Y^{k+u} \ (u \geq 0,\ u+i+1 \leq d). \tag{3.19}$$

Since M_d is real symmetric, (3.19) implies that (3.18) holds in the rows of M_d. (2.2) and (3.5) now imply that (3.18) holds for the rows of the band of columns $X^{d+1}, \dots, XY^d$. It remains to prove that (3.18) holds in Y^{d+1}; we need to prove that

$$\langle Y^{d+1}, XY^{i+u}\rangle = \sum_{j<i} c_j \langle Y^{d+1}, XY^{j+u}\rangle + \sum_{k<i+1} d_k \langle Y^{d+1}, Y^{k+u}\rangle \ (u \geq 0,\ u+i+1 \leq d). \tag{3.20}$$

We consider first the case $i+u+1 \leq d-1$. Then $\langle Y^{d+1}, XY^{i+u}\rangle := \beta_{1,d+1+i+u}$

$= \langle Y^d, XY^{i+u+1}\rangle = \langle XY^{i+u+1}, Y^d\rangle$ (in M_d)

$= \langle \sum_{j<i} c_j XY^{j+1+u} + \sum_{k<i+1} d_k Y^{k+1+u}, Y^d\rangle$ (from (3.17) and recursiveness in M_d)

$= \sum_{j<i} c_j \beta_{1,j+1+u+d} + \sum_{k<i+1} d_k \beta_{0,k+1+u+d}$ (in M_d, since $j+1+u < i+1+u \leq d-1$ and $k+1+u < i+2+u \leq d$)

$= \sum_{j<i} c_j \langle Y^{d+1}, XY^{j+u}\rangle + \sum_{k<i+1} d_k \langle Y^{d+1}, Y^{k+u}\rangle$ (using "old" moments, since $1+j+u < 1+i+u \leq d-1$ and $k+u \leq< i+1+u \leq d-1$).

We next consider the case $i+u+1 = d$. From (3.14), we have $\langle Y^{d+1}, XY^{i+u}\rangle := \langle XY^d, Y^{i+u+1}\rangle$

$= \sum_{j<i} c_j \langle XY^{j+d-i}, Y^{i+u+1}\rangle + \sum_{k<i+1} d_k \langle Y^{k+d-i}, Y^{i+u+1}\rangle$ (from (3.17) and recursiveness in $(\,M_d\ B_{d+1}\,)$)

$= \sum_{j<i} c_j \beta_{1,j+1+u+d} + \sum_{k<i+1} d_k \beta_{0,k+1+u+d}$ (in M_d, since $1+j+d-i < 1+d$, $k+d-i < i+1+d-i = d+1$, and $i+u+1 = d$)

$= \sum_{j<i} c_j \langle Y^{d+1}, XY^{j+u}\rangle + \sum_{k<i+1} d_k \langle Y^{d+1}, Y^{k+u}\rangle$ (using "old" moments, since $1+j+u < 1+i+u = d$ and $k+u < i+1+u = d$). □

Corollary 3.3. *In* $(\,M_d\ B_{d+1}\,)$, *each row* XY^{i+u} $(0 \leq u \leq d-i-1)$ *is a linear combination of rows which satisfy (3.16).*

Proof. The proof is by induction on u, $0 \leq u \leq d-i-1$. For $u = 0$, Lemma 3.2 shows that row XY^i is a linear combination of rows corresponding to elements of basis $\mathscr{B}$, and each of these rows clearly satisfies (3.16). Assume the conclusion holds for rows XY^{i+w} $(0 \leq w \leq u-1)$, so that each such row satisfies (3.16). Since each row Y^k also satisfies (3.16), Lemma 3.2 implies that XY^{i+u} is a linear combination or rows satisfying (3.16). □

This completes the proof of the range inclusion and so also completes the proof of Theorem 3.1.

We conclude two examples illustrating Theorem 3.1.

Example 3.4. To illustrate the recursively determinate case, we modify Example 2.9 by changing the value of β_{06} from a^2+b^2+1 to a^2+b^2. Then M_3 is positive, with $\operatorname{rank} M_3 = 6$, and the column relations $X^2 = 1$ and $Y^3 = aY + bXY$ show that M_3 is recursively determinate. A unique flat extension is determined by defining $X^4 = X^2$ and $Y^4 = aY^2 + bXY^2$. The support of the corresponding 6-atomic measure for M_3 consists of the common zeros of x^2-1 and $y^3 - ay - bxy$. With $x = 1$, the roots of $y^3 = ay + by$ are $y = 0$ and $y = \pm\sqrt{a+b}$. With $x = -1$, the roots of $y^3 = ay - by$ are $y = 0$ and $y = \pm\sqrt{a-b}$. These 6 points provide the support of the unique representing measure for M_3.

The final example concerns a case where M_3 is neither (x^2-1)-pure nor recursively determinate.

Example 3.5. Consider M_3 defined by

$$M_3 := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 2 & 0 & 6 \end{pmatrix}.$$

Then M_3 is positive and recursively generated, with column relations generated by $X^2 = 1$ and $XY^2 = 1 + X - Y^2$. Setting $q(x,y) = xy^2 - (1 + x - y^2)$, we see that $q(-1,y) \equiv 0$, so the proof of Theorem 3.1 shows that M_3 has infinitely many flat extensions (and 6-atomic measures), parameterized by a choice for $\alpha \equiv \beta_{07}$. A calculation based on the proof of Theorem 3.1 shows that in M_4 we must have $Y^4 = -2\cdot 1 - X - 2\alpha Y + \alpha XY + 4Y^2 + \alpha Y^3$, so the support of a measure consists of the common zeros of x^2-1, $q(x,y)$ and $r(x,y) := y^4 - (-2 - x - 2\alpha y + \alpha xy + 4y^2 + \alpha y^3)$. For a numerical example, let $\alpha = -2$. The roots of $r(1,y) = 0$ are $y = 1$, $y = -1$, $y = -3$, but the latter value is not a root of $q(1,y) = 0$, so it must be excluded. The roots of $r(-1,y) = 0$ satisfy $y \approx -2.59$, ≈ -1.39, $y \approx 0.15$, $y \approx 1.83$, and these are also roots of $q(-1,y) = 0$. The resulting 6 points (2 on $x = 1$, 4 on $x = -1$) form the support of the unique 6-atomic measure for M_3 corresponding to $\alpha = -2$.

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Page line

100 14^ $X^2 + Y^2 = -1$

101 5^ case of *flat data* italics

109 9^ Let U = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that $UM_{ij} = M_{ij}U$.
Set $\mathcal{U} = U^{(d)} \ldots$

110 12^ $XY^2 + sY^3$,

111 19 there is a minimal i, $1 \leq i \leq d-1$,

111 21 $+ \; b_i \, x \, y^{i-1} + a_i \, y^i$

115 3 We conclude with two examples .....

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