NON-NEGATIVE POLYNOMIALS

CARLOS ANGLAS AND DWAYNE WOOLWARD

Contents

Research was conducted during the Spring, 2010 and Summer I, 2010 semesters with Professor Lawrence A. Fialkow as the mentor. During Spring, 2010 the project was sponsored by SUNY/NSF Alliance for Minority Participation at SUNY New Paltz. The New Paltz AMP Director is Professor Stacie Nunes from the Department of Physics. During Summer I, 2010 the project was sponsored under National Science Foundation Grant DMS0758378. Special thanks to the sponsors for their generous help and support of this project.

Introduction

The use of positive polynomials in one or more variables is essential in several areas of mathematics, including optimization and algebraic geometry [4][3]. As for polynomials that are non-negative on the real line, half-line, or on an interval, this paper derives a "weighted sums of squares" description. These results are known from Polya and Szego [5], where the proofs are based on complex analysis. In this paper, starting from the *Factorization Theorem* for real polynomials, we prove these theorems using only basic algebra. These proofs explain how to construct the weighted sums of squares expressions, and we illustrate these constructions with examples in the text and in the Mathematica appendices. In section 1 (Theorem 1.1), we discuss how every polynomial which is non-negative on the whole real line is a sum of two squares of polynomials. In section 2 (Theorem 2.1), we use Theorem 1.1 to demonstrate that every non-negative polynomial on the half line $[0, +\infty)$ is of the form $f^2+g^2+x(h^2+k^2)$ for certain polynomials f, g, h, and k. In section 3 (Theorem 3.1), we use Theorems 1.1 and 2.1 to demonstrate that every non-negative polynomial on the interval $[0, 1]$ can be represented in the form $f^2 + g^2 + x(1 - x)(h^2 + k^2)$ for certain polynomials f, g, h, and k. In Section 4, we look at non-negative polynomials in two variables. First, by studying the example of Motzkin, we prove that not every non-negative polynomial of *deg* 6 can be represented as a sum of squares. We also study a method for identifying when a polynomial is a sum of squares.

1. Non-Negative Polynomials on the Whole Real Line

1.1. **Section A.** In this section we look at non-negative polynomials on the real line and introduce our first and most important theorem. For polynomials $f(x)$ and $g(x)$, the expression $f(x)^2+g(x)^2$ is known as a sum of two squares. We know that if $p(x)$ is a sum of two squares, then $p(x) \geq 0$ for all x. But if $p(x) \geq 0$ for all x (a non-negative polynomial), then is $p(x)$ necessarily a sum of two squares? The following theorem, from Polya and Szego, answers this question.

Theorem 1.1. *(See Polya and Szego [5, Part VI, Sec. 6, 44, page 77].) If we have a polynomial $p(x) \geq 0$ for all x, then $p(x)$ admits polynomials $f(x)$ and $g(x)$ such that $p(x) = f(x)^2 + g(x)^2$ for all x.*

In an attempt to prove Theorem 1.1, we first turn to the quadratic $q(x) = ax^2 + bx + c$, assuming $a \neq 0$ and $q(x) \geq 0$ for all x . Consider the three possible cases:

(i) $q(x)$ has no real roots
(ii) $q(x)$ has one repeated, real root
(iii) $q(x)$ has two distinct, real roots

1.1.1. *Case 1: No Real Roots.* In order to compute the roots of $q(x) = 0$, we use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.1}$$

From this we see there are no real roots when the discriminant of the quadratic, $b^2 - 4ac$, is negative. Suppose $q(x) > 0$ for all x. We write $q(x)$ as

$$q(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right). \tag{1.2}$$

Since $\lim_{x\to+\infty} x^2 + \frac{b}{a}x + \frac{c}{a} = +\infty$, it follows that $a > 0$. We can now express $q(x)$ as a sum of two squares by completing the squares, as follows:

$$q(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) =$$

$$= a\left[x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= a\left(x + \frac{b}{2a}\right)^2 + a\left[-\left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= \left[\sqrt{a}\left(x + \frac{b}{2a}\right)\right]^2 + \left[-a\left(\frac{b}{2a}\right)^2 + c)\right]$$

$$= \left[(\sqrt{a}) \left(x + \frac{b}{2a} \right) \right]^2 + \frac{4ac - b^2}{4a}.$$

From the final expression, we let

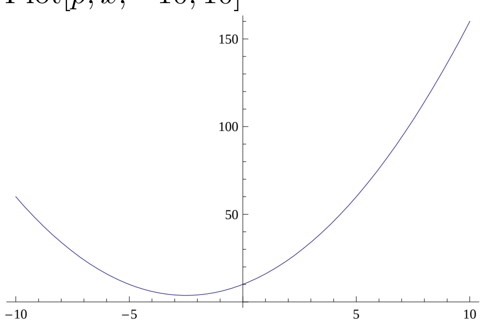
$$f(x) = \sqrt{a} \left(x + \frac{b}{2a} \right)$$

and let

$$\alpha = \sqrt{\frac{4ac - b^2}{4a}}.$$

So now we have $q(x) = f(x)^2 + (\alpha)^2$, a sum of two squares.
We illustrate this case with the following example.

Example 1.2. We start with a polynomial, not in its sum of squares form.
$p = x^2 + 5x + 10$
$10 + 5x + x^2$
Factor$[p]10 + 5x + x^2$
Now we check to see if p has any roots using the discriminant, $b^2 - 4ac < 0$
$a = 1$
$b = 5$
$c = 10$
$b^2 - 4ac = -15$
-15 is less than zero so p has no real roots.
Plot$[p, x, -10, 10]$

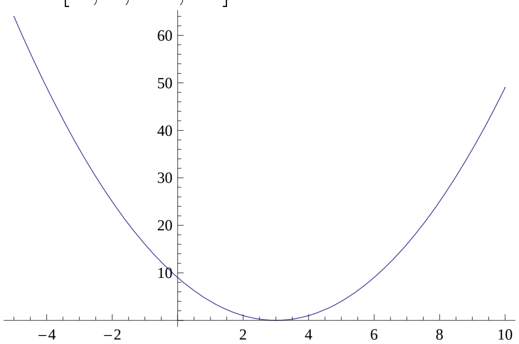


The plot shows us that p is positive for all x. Now we can convert p into its sum of two squares form by completing the square.
$\left(\sqrt{a} \left(x + \frac{b}{2a} \right) \right)^2 + \left(\sqrt{\frac{4ac - b^2}{4a}} \right)^2 = \frac{15}{4} + \left(\frac{5}{2} + x \right)^2$
$f = \frac{5}{2} + x$
$\alpha = \sqrt{\frac{15}{4}}$
$f^2 + \alpha^2 = \frac{15}{4} + \left(\frac{5}{2} + x \right)^2$

This our new sum of squares form.
Expand[%] $= 10 + 5x + x^2$
Expanded, we get the polynomial we started off with.

1.1.2. *Case 2: One Repeated, Real Root.* Suppose $q(x) = 0$ has one root, x_o, repeated twice, so that $q(x) = a(x - x_0)^2$, with $a \neq 0$. Here a must be positive since $q(x) \geq 0$ for all x. To make $q(x)$ a sum of two squares we simply take $f(x) = \sqrt{a}(x - x_o)$ and $g(x) = 0$, i.e., $q(x) = f(x)^2 + 0^2$. Here is an example of a polynomial with repeated roots.

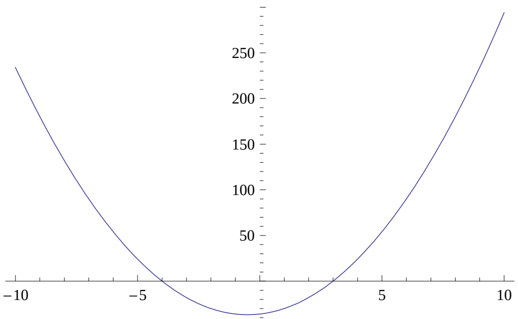
Example 1.3. We have a polynomial with a repeated root, $a(x - x_0)^2$
$P = (x - 3)^2$
Expand[P] $= 9 - 6x + x^2$
Factor[P] $= (-3 + x)^2$
Plot[$P, x, -5, 10$]



Plot shows p is non-negative for all x
$f = (x - 3)$
$g = 0$
Expand[$f^2 + g^2$] $= 9 - 6x + x^2$
Expand[$P - (f^2 + g^2)$] $= 0$
So our sum of two squares is simply $(x - 3)^2 + 0^2$

1.1.3. *Case 3: Two Distinct, Real Roots.* For there to be two distinct real roots, then $q(x) = a(x - x_o)(x - x_1)$, with $a \neq 0$ and $x_o \neq x_1$ But then $q(x)$ changes sign at $x = x_o$ and $x = x_1$, so this case cannot occur in Theorem 1.1. Here is an example of a polynomial with two distinct roots.

Example 1.4. A polynomial of the form $a(x - x_o)(x - x_1)$ with $a = 3, x_o = -3, x_1 = 4$
$P = -36 + 3x + 3x^2$
Factor[P] $= 3(-3 + x)(4 + x)$
Plot[$P, x, -10, 10$]



The plot clearly shows that the function is negative as it goes below the x - axis, it changes sign at the roots $+3, -4$. This is a contradiction of the rule $p \geq 0$ for all x, so this case cannot occur.

1.2. **Section B.** In the proof of Theorem 1.1, we use the following theorem.

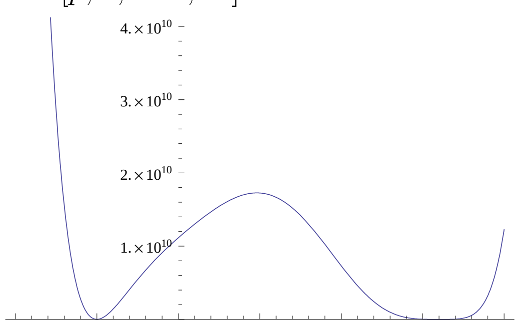
Theorem 1.5. *(Factorization Theorem) If $p(x)$ is a polynomial, then*

$$p(x) = Aq_1(x)\ldots q_k(x)(x-x_1)^{n_1}\ldots(x-x_r)^{n_r}(x-y_1)^{m_1}\ldots(x-y_s)^{m_s},$$

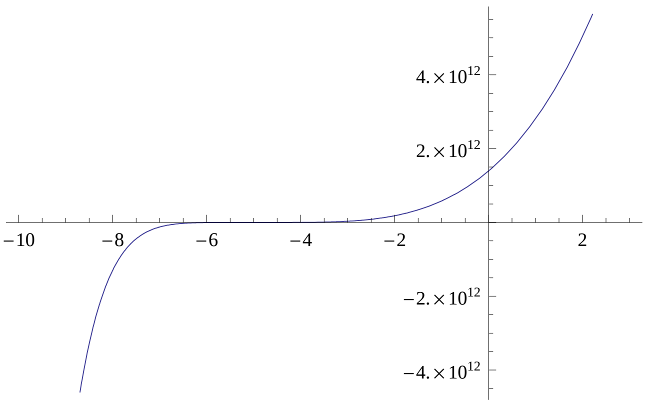
where A is a constant, each of $q_1(x)\ldots q_k(x)$ are strictly non-zero quadratics, $x_1\ldots x_r$, $y_1\ldots y_s$ are all distinct, $n_1\ldots n_r$ are even exponents, and $m_1\ldots m_s$ are odd exponents, (Note that any of these groups may be missing in the expression).

For the proof, see Appendix A. Here is an example of the factorization theorem.

Example 1.6. We start with our polynomial p
$p = 11150950400 + 1944616960x - 94936064x^2 + 12993152x^3 - 4111322x^4 - 650222x^5 + 150728x^6 - 8798x^7 + 166x^8$
$\text{Factor}[p] = 166(-16+x)^4(5+x)^2(41+x+x^2)$
p is now in its factored form given by the Factorization Theorem. Notice $A = 166, q_1 = x^2+x+41$ with no real roots, and then there are some more linear terms with even exponents.
$\text{Plot}[p, x, -10, 20]$



Plot tells us p is non-negative for all x. But what happens when we add a term with an odd exponent? Observe r.
$r = 166(x^2+x+41)(x-16)^4(x+5)^2(x+5)^3$
$\text{Factor}[r] = 166(-16+x)^4(5+x)^5(41+x+x^2)$
$\text{Plot}[r, x, -10, 3]$



This plot tells us that the linear term with the odd exponent made p negative for $x < -5$.

1.3. **Section C.** In this section we introduce a lemma which shows that a product of two terms, each a sum of two squares, can also be expressed as a sum of two squares. The lemma can be applied as many times as needed, depending on the number of terms in a product. For example, if you have three terms, each sums of two squares, you will use the lemma twice, once to combine the first two terms into a sum of two squares, and once more to combine with the last term. In the end, this will result in a polynomial of the form $f(x)^2 + g(x)^2$.

Lemma 1.7. *If $A(x) = f(x)^2 + g(x)^2$ (Type I) and $B(x) = h(x)^2 + k(x)^2$ (Type I), then $A(x) \times B(x)$ can be expressed as $R(x)^2 + S(x)^2$ (Type I), where $R(x)$ and $S(x)$ are polynomials; in brief, Type I$\times$ Type I = Type I.*

Proof. We can expand the product $A(x)B(x)$:

$$A(x)B(x) = f^2h^2 + f^2k^2 + g^2h^2 + g^2k^2$$

$$= \left(f^2h^2 + 2fhgk + g^2k^2\right) + \left(f^2k^2 - 2fhgk + g^2h^2\right).$$

Factoring the sub-expressions further will give us two sums of squares:

$$f^2h^2 + 2fghk + g^2k^2 = (fh + gk)^2$$

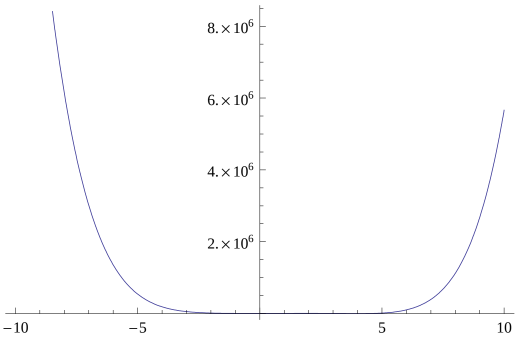
$$g^2h^2 - 2fghk + f^2k^2 = (fk - gh)^2.$$

So $A(x)B(x) = R(x)^2 + S(x)^2$ with $R(x) = fh + gk$ and $S(x) = fk - gh$.

□

The following example shows how Lemma 1.7 is applied.

Example 1.8. We start with a polynomial of several terms
$p = 3360 + 224x + 1002x^2 - 353x^3 + 101x^4 - 71x^5 + 12x^6$
$\text{Factor}[p] = (-4 + x)^2(15 + x + 3x^2)(14 + 7x + 4x^2)$
$\text{Plot}[p, x, -10, 10]$



Plot shows polynomial is non-negative for all x.
Expand$[(15 + x + 3x^2)] = 15 + x + 3x^2$
$a = 3$
$b = 1$
$c = 15$

$$\left((\sqrt{a})\left(x + \frac{b}{2a}\right)\right)^2 + \left(\sqrt{\frac{4ac - b^2}{4a}}\right)^2 = \frac{179}{12} + 3\left(\frac{1}{6} + x\right)^2$$

Prove that the completed square expression and the original are equal. Expand $[(15 + x + 3x^2) - \%] = 0$
Now combine $(-4 + x)^2(15 + x + 3x^2)$ using the Lemma.
$X =$ Expand$[(-4 + x)^2(15 + x + 3x^2)] = 240 - 104x + 55x^2 - 23x^3 + 3x^4$
X represents the first two terms.
$f = (-4 + x)$
$g = 0$

$$h = \sqrt{\frac{179}{12}}$$

$$k = \sqrt{3}\left(\frac{1}{6} + x\right)$$

We'll let $A = (fh + gk)$ and $B = (fk - gh)$

$$A = fh + gk = \frac{1}{2}\sqrt{\frac{179}{3}}(-4 + x)$$

$$B = fk - gh = \sqrt{3}(-4 + x)(\frac{1}{6} + x)$$

Expand$[X - (A^2 + B^2)] = 0$
There is the product of our two sum of squares. Double-checking, we get a zero to show we haven' t changed anything.
Convert polynomial to sum of squares form.
Expand$[14 + 7x + 4x^2] = 14 + 7x + 4x^2$
$a = 4$
$b = 7$
$c = 14$

$$\left((\sqrt{a})\left(x + \frac{b}{2a}\right)\right)^2 + \left(\sqrt{\frac{4ac - b^2}{4a}}\right)^2 = \frac{175}{16} + 4\left(\frac{7}{8} + x\right)^2$$

Now lets combine the third term in sums of two squares form using the Lemma again.
$f = A = \frac{1}{2}\sqrt{\frac{179}{3}}(-4+x)$
$g = B = \sqrt{3}(-4+x)\left(\frac{1}{6}+x\right)$
$h = \sqrt{\frac{175}{16}}$
$k = 2\left(\frac{7}{8}+x\right)$
$AA = fh + gk = \frac{5}{8}\sqrt{\frac{1253}{3}}(-4+x) + 2\sqrt{3}(-4+x)\left(\frac{1}{6}+x\right)\left(\frac{7}{8}+x\right)$
$AA = \text{Factor}[AA] = \frac{((-4+x)(7+5\sqrt{1253}+50x+48x^2))}{(8\sqrt{3})}$
$\text{Expand}[AA] = -\frac{7}{2\sqrt{3}} - \frac{5\sqrt{\frac{1253}{3}}}{2} - \frac{193x}{8\sqrt{3}} + \frac{5}{8}\sqrt{\frac{1253}{3}}x - \frac{71x^2}{4\sqrt{3}} + 2\sqrt{3}x^3$
$BB = fk - gh = -\frac{5}{4}\sqrt{21}(-4+x)\left(\frac{1}{6}+x\right) + \sqrt{\frac{179}{3}}(-4+x)\left(\frac{7}{8}+x\right)$
$BB = \text{Factor}[BB] = -\frac{(((-4+x)(5\sqrt{7}-7\sqrt{179}+30\sqrt{7}x-8\sqrt{179}x)))}{(8\sqrt{3})}$
$\text{Expand}[BB] = \frac{5\sqrt{\frac{7}{3}}}{2} - \frac{7\sqrt{\frac{179}{3}}}{2} + \frac{115}{8}\sqrt{\frac{7}{3}}x - \frac{25}{8}\sqrt{\frac{179}{3}}x - \frac{5\sqrt{21}x^2}{4} + \sqrt{\frac{179}{3}}x^2$
$\text{Expand}[p - (AA^2 + BB^2)] = 0$
Now we have one big sum of squares that equals our original polynomial p.

Proof of Theorem 1.1

Proof. Suppose $p(x) \geq 0$ for every real number x. By the *Factorization Theorem*, we have $p(x) = Aq_1(x) \ldots q_k(x)\,(x-x_1)^{n_1} \ldots (x-x_r)^{n_r}\,(x-y_1)^{m_1} \ldots (x-y_s)^{m_s}$, where the q_i are quadratics with no real roots, $x_1, \ldots, x_r,\ y_1 \ldots, y_s$ are distinct, $n_1, \ldots, n_r$ are even, say $n_i = 2d_i$, and $m_1, \ldots, m_s$ are odd. We can also assume $A \neq 0$. Since $p(x) \geq 0$ for all x, and m_i is odd, we cannot have any factors $(x-y_i)^{m_i}$, since $p(x)$ will change sign at the root y_i. So $p(x) = Aq_1(x) \ldots q_k(x)(x-x_1)^{n_1} \ldots (x-x_r)^{n_r}$. If some $q_i(x) < 0$ for all x, we can write

$$p(x) = (\mathbf{-1})Aq_1(x) \ldots (\mathbf{-}q_i(x)) \ldots (q_k(x))(x-x_1)^{n_1} \ldots (x-x_r)^{n_r}$$

(notice the -1's cancel, leaving the value of $p(x)$ unchanged). We now have $p(x) = \widetilde{A}\widetilde{q_i}(x) \ldots \widetilde{q_k}(x)(x-x_1)^{n_1} \ldots (x-x_r)^{n_r}$, where each $\widetilde{q_i}(x)$ is a strictly positive quadratic. Since $\widetilde{q_i}(x) > 0$ for all x, each factor $(x-x_i)^{n_i} \geq 0$ for all x, and $p(x) \geq 0$ for all x, we must have $\widetilde{A} > 0$. So $\widetilde{A}$ is a square of the form $\sqrt{\widetilde{A}}^2$, each $\widetilde{q_i}(x)$ is a sum of 2 squares (by section A, case 1), and each $(x-x_i)^{n_i}$ is a square, $(x-x_i)^{n_i} = \left[(x-x_i)^{d_i}\right]^2$. The proof is now completed by applying Lemma 1.7 to the factors in $p(x)$, one after another. ☐

The following corollary generalizes Lemma 1.7.

Corollary 1.9. *If* $A(x) = \sum_{i=1}^{n} {f_i}^2(x)$ *and* $B(x) = \sum_{i=1}^{n} {g_i}^2(x)$*, then* $A(x) \times B(x)$ *can be expressed as* $R(x)^2 + S(x)^2$*, a sum of two squares.*

Proof. Since $A(x) \times B(x)$ is non-negative for every x, the Type I representation follows from Theorem 1.1. $\square$

2. Non-Negative Polynomials on the Half-Line

In this section we look at non-negative polynomials on the half-line and introduce our second theorem.

Theorem 2.1. *(See Polya and Szego [5, Part VI, Sec. 6, 45, page 78].) If we have a polynomial $p(x)$ that satisfies $p(x) \geq 0$ for every $x \geq 0$, then there are polynomials $f(x)$, $g(x)$, $h(x)$, and $k(x)$ such that $p(x) = f(x)^2 + g(x)^2 + x[h(x)^2 + k(x)^2]$ for all x.*

We can use Theorem 1.1 to prove Theorem 2.1. But before we begin we must define a couple of things to make the process easier.

- A *Type I polynomial* is a polynomial $p(x)$ with a representation as in Theorem 1.1, a sum of two squares, i.e. $f(x)^2 + g(x)^2$.
- A *Type II polynomial* is a polynomial $p(x)$ with a representation as in the form of Theorem 2.1, ([Type I] + x[Type I]), i.e. $f(x)^2 + g(x)^2 + x[h(x)^2 + k(x)^2]$.

Now that we have defined the two different types, we can continue. In Theorem 1.1, we used Lemma 1.7, which shows that Type I $\times$ Type I = Type I. Similarly, to prove Theorem 2.1, we need the following result:

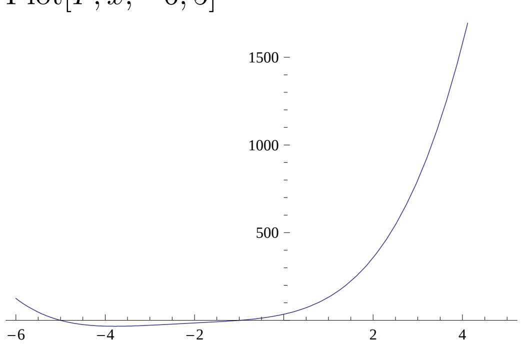
Lemma 2.2. *If $x_i \leq 0$, then Type II $\times$ $(x - x_i)$ is also of the form Type II.*

Proof. Suppose $p(x)$ is Type II: $p = f^2 + g^2 + x(h^2 + k^2)$, where f, g, h, and k are all polynomials. Let $q(x) := p(x - x_i) = x(f^2 + g^2) - x_i(f^2 + g^2) + x^2(h^2 + k^2) - (x)(x_i)(h^2 + k^2)$

$$= \underbrace{\left[x^2(h^2 + k^2) - x_i(f^2 + g^2)\right]}_{P(x)} + x \underbrace{\left[(f^2 + g^2) - x_i(h^2 + k^2)\right]}_{Q(x)}.$$

Since $-x_i \geq 0$, we can see that $P(x) \geq 0$ and $Q(x) \geq 0$ for each x, so by Theorem 1.1, $P(x)$ and $Q(x)$ have Type I representations. At the same time, we see that $q(x) \equiv P(x) + xQ(x)$ has a Type II representation. $\square$

Example 2.3. Suppose P is a non-negative polynomial for $x \geq 0$.
$P = 7 + 10x + 4x^2 + x^3$
Factor$[P] = (1 + x)(7 + 3x + x^2)$
$x_o = -5$
$Q = \text{Expand}[P(x - x_o)] = 35 + 57x + 30x^2 + 9x^3 + x^4$
Plot$[P, x, -6, 5]$



From the plot, we can clearly see that $P > 0$ when $x > 0$.
Factor$[Q] = (1 + x)(5 + x)\left(7 + 3x + x^2\right)$
Convert to sum of squares form using Lemma 1.7 from Theorem 1.1.
Z=Expand$[7 + 3x + x^2] = 7 + 3x + x^2$
$a = 1$
$b = 3$
$c = 7$
$\left((\sqrt{a})\left(x + \frac{b}{2a}\right)\right)^2 + \left(\sqrt{\frac{4ac-b\hat{}2}{4a}}\right)^2 = \frac{19}{4} + \left(\frac{3}{2} + x\right)^2$
Expand$[Z - \%] = 0$
Prove P has Type II.
$f = \dfrac{\sqrt{19}}{2}$
$g = \left(\dfrac{3}{2} + x\right)$
$h = \dfrac{\sqrt{19}}{2}$
$k = \left(\dfrac{3}{2} + x\right)$
P has a Type II representation.
Expand$[P - (f^2 + g^2 + x(h^2 + k^2))] = 0$
Apply Lemma 2.2 from Theorem 2.1 with $x_o = -5$: $TypeII \times (x - x_o) = TypeII$.
We show that Q is Type II.
$R = x^2(h^2 + k^2) - x_o(f^2 + g^2) = 5\left(\frac{19}{4} + \left(\frac{3}{2} + x\right)^2\right) + x^2\left(\frac{19}{4} + \left(\frac{3}{2} + x\right)^2\right)$
Factor$[R] = (5 + x^2)(7 + 3x + x^2)$
$\left((\sqrt{a})\left(x + \frac{b}{2a}\right)\right)^2 + \left(\sqrt{\frac{4ac-b^2}{4a}}\right)^2 = \frac{19}{4} + \left(\frac{3}{2} + x\right)^2$
$f = x$
$g = \sqrt{5}$
$h = \frac{\sqrt{19}}{2}$
$k = \frac{3}{2} + x$
$A = fk - gh = -\frac{\sqrt{95}}{2} + x\left(\frac{3}{2} + x\right)$
$B = fh + gk = \frac{\sqrt{19}x}{2} + \sqrt{5}\left(\frac{3}{2} + x\right)$
Expand$[R - (A^2 + B^2)] = 0$
R has Type I representation
$S = (f^2 + g^2) - x_o(h^2 + k^2) = \frac{19}{4} + \left(\frac{3}{2} + x\right)^2 + 5\left(\frac{19}{4} + \left(\frac{3}{2} + x\right)^2\right)$
Factor$[S] = 6(7 + 3x + x^2)$
$f = 0$
$g = \sqrt{6}$
$h = \frac{\sqrt{19}}{2}$
$k = \left(\frac{3}{2} + x\right)$

$J = fk - gh = -\sqrt{\frac{57}{2}}$
$K = fh + gk = \sqrt{6}\left(\frac{3}{2} + x\right)$
Expand$[S - (J^2 + K^2)] = 0$
S has a Type I representation.
Expand$[Q - (A^2 + B^2 + x(J^2 + K^2))] = 0$
Polynomial Q has a Type II representation.

Proof of Theorem 2.1

Proof. We use the Factorization Theorem to write p from Theorem 2.1 as:

$$p(x) = A\left[q_1 \ldots q_k (x - y_1)^{n_1} \ldots (x - y_m)^{n_m} (x - x_1)^{s_1} \ldots (x - x_p)^{s_p}\right],$$

where A is a constant, the q_i's are quadratics that cannot be factored (i.e., they have no real roots), the exponents n_i are even, the exponents s_i are odd, and all of $x_1, \ldots x_p, y_1, \ldots y_m$ are distinct. (Note some of these factors may be missing in any expression p.) Since $p(x) \geq 0$ for all $x \geq 0$, we can assume that each q_i is positive for all x, each $x_i \leq 0$ (since $p(x)$ changes sign at x_i), and $A > 0$. Let's say $s_i = 2t_i + 1$, with $t_i \geq 0$, so that $(x - x_i)^{s_i} = (x - x_i)^{2t_i}(x - x_i)$. Let

$$r(x) := Aq_1 \ldots q_k (x - y_1)^{n_1} \ldots (x - y_m)^{n_m} (x - x_1)^{2t_1} \ldots (x - x_p)^{2t_p},$$

so that $p(x) = r(x)(x - x_i) \ldots (x - x_p)$.
Clearly, $r(x) \geq 0$ for every x, so $r(x)$ has a Type I representation by Theorem 1.1, and therefore a Type II representation:

$$r(x) = f(x)^2 + g(x)^2 = f(x)^2 + g(x)^2 + x(0^2 + 0^2).$$

If there are no x_i's, then we just have $p(x) = r(x)$, so $p(x)$ has a Type II representation. If there are some x_i's, they must be ≤ 0, since $p(x)$ changes sign at x_i.
Now, we can apply Lemma 2.2 to

$$p(x) = \underbrace{r(x)(x - x_1)(x - x_2) \ldots (x - x_p)}_{R(x)}$$

Since each $x_i \leq 0$, Lemma 2.2 implies that $r(x)(x - x_1)$, and each partial product $r(x)(x - x_1) \ldots (r - x_i)$ is Type II, so $p(x) \equiv R(x)$ is Type II. □

Before continuing to the next section we will introduce three facts concerning degrees of polynomials. We will use these facts in Theorem 3.1.

2.1. **Fact 1.** First we compute the degree of a sum of squares of two polynomials.
Let $q(x) = a_0 + a_i x + \ldots + a_n x^n$, where $a_n \neq 0$ (so the *deg* $q = n$).
Let $r(x) = b_0 + b_i x + \ldots + b_m x^m$, where $b_m \neq 0$ (so the *deg* $r = m$).
Then $s(x) := q(x)^2 + r(x)^2$

$$= {a_o}^2 + \ldots + 2a_{n-1}a_n x^{2n-1} + {a_n}^2 x^{2n} + {b_o}^2 + \ldots + 2b_{m-1}b_m x^{2m-1} + b_m^2 x^{2m}.$$

But what is the degree of $s(x)$? Let's consider the degrees n and m.

$$deg\ s(x) = \begin{cases} 2n & n > m \\ 2m & m > n \\ 2n = 2m & n = m. \end{cases}$$

According to this, the degree of the sum is twice n if $n > m$, twice m if $m > n$, or twice n or m if $n = m$. So we can say the degree of the sum is twice the highest degree,

$$deg\ (q(x)^2 + r(x)^2) = 2\max(deg\ q(x), deg\ r(x)) \tag{2.1}$$

2.2. **Fact 2.** For any polynomials $a(x)$, $b(x)$, the degree of their sum, $a(x) + b(x)$, satisfies

$$deg\ (a(x) + b(x)) \leq \max\left(deg\ a(x), deg\ b(x)\right). \tag{2.2}$$

Also, if the highest degree coefficients in $a(x)$ and $b(x)$ are both positive, then

$$deg\ (a(x) + b(x)) = max(deg\ a(x), deg\ b(x)). \tag{2.3}$$

2.3. **Fact 3.** Consider a Type II polynomial:

$$p(x) = f(x)^2 + g(x)^2 + x\left[h(x)^2 + g(x)^2\right]$$

Then by Fact 1, $deg\ \ (f(x)^2 + g(x)^2) = 2max(deg\ f(x), deg\ g(x))$. Also by Fact 1, $deg\ (h(x)^2 + k(x)^2) = 2max(deg\ h(x), deg\ k(x))$. Then $deg\ \ (x(h(x)^2 + k(x)^2)) =$

$$1 + 2max(deg\ h(x), deg\ k(x)) = max(1 + 2deg\ h(x), 1 + 2deg\ k(x)).$$

So, by Fact 2,

$$deg\ \left(f(x)^2 + g(x)^2 + x\left[h(x)^2 + g(x)^2\right]\right) =$$
$$max(deg\ (f(x)^2 + g(x)^2), deg\ x(h(x)^2 + k(x)^2)) =$$
$$max(2max(deg\ f, deg\ g)), 1 + 2max(deg\ h(x), deg\ k(x)).$$

So we discover that

$$deg\ (f^2 + g^2 + x(h^2 + k^2)) = max\left(2deg\ f, 2deg\ g, 1 + 2deg\ h, 1 + 2deg\ k\right). \tag{2.4}$$

We will use Fact 3, mainly equation 2.4, in the proof of the next theorem.

3. Non-Negative Polynomials on $[0,1]$

In this section, we look at polynomials that are positive in the interval $[0,1]$. The next theorem is due to M. Fekete (1935) (see [3], Theorem 2.7 and page 49).

Theorem 3.1. *The polynomial $p(x)$ satisfies $p(x) \geq 0$ for all x on the interval $[0,1]$ if and only if $p(x)$ is a Type III polynomial, i.e. there are polynomials $f(x)$, $g(x)$, $h(x)$, and $k(x)$ such that $p(x) = f(x)^2 + g(x)^2 + x(1-x)[h(x)^2 + k(x)^2]$.*

In the proof of this theorem we will utilize more resources, which we will discuss now. The following corollaries concern the degree of a Type II polynomial.

Corollary 3.2. *Consider the Type II polynomial,*
$p(x) = f(x)^2 + g(x)^2 + x(h(x)^2 + k(x)^2)$ where $f, g, h,$ and k are polynomials. If $deg\ p(x) = 2m$, then $deg\ f(x), deg\ g(x) \leq m$ and $deg\ h(x), deg\ k(x) \leq m-1$.

Proof. By equation 2.4, we can say the following:
$2m = deg\ p(x) \geq deg\ f(x)^2 = 2 \times deg\ f(x)$, or more simply $deg\ f(x) \leq m$. Similarly, $deg\ g(x) \leq m$. Also, $2m = deg\ p(x) \geq deg\ xh(x)^2 = 1 + 2 \times deg\ h(x)$, so $m \geq \frac{1}{2} + deg\ h(x)$. Finally, we get $deg\ h(x) \leq m - \frac{1}{2}$, but since $deg\ h(x)$ is an integer, we must have $deg\ h(x) \leq m-1$. Similarly, $deg\ k(x) \leq m-1$. □

Corollary 3.3. *Consider the Type II polynomial,*
$p(x) = f(x)^2 + g(x)^2 + x(h(x)^2 + k(x)^2)$ where $f, g, h,$ and k are polynomials. If $deg\ p(x) = 2m+1$, then $deg\ f(x), deg\ g(x), deg\ h(x), deg\ k(x) \leq m$.

Proof. By equation 2.4, we can say the following:
$2m+1 = deg\ p(x) \geq deg\ f(x)^2 \geq 2 \times deg\ f(x)$, so $m + \frac{1}{2} \geq deg\ f(x)$, or more simply, $deg\ f(x) \leq m$. Similarly, $deg\ g(x) \leq m$. Also $2m+1 = deg\ p(x) \geq deg\ xh(x)^2 = 1 + 2\times$ deg $h(x)$, or more simply, $deg\ h(x) \leq m$. Similarly, $deg\ k(x) \leq m$. □

The following Lemma will be utilized to help us create Type III polynomials.

Lemma 3.4. *If $r(x)$ and $s(x)$ are both Type III polynomials, then the product of $r(x)$ and $s(x)$ is also a Type III polynomial.*

Proof. Suppose $r(x) = f(x)^2 + g(x)^2 + x(1-x)[h(x)^2 + k(x)^2]$
and $s(x) = a(x)^2 + b(x)^2 + x(1-x)[c(x)^2 + d(x)^2]$ where $f, g, h, k, a, b, c,$ and d are polynomials.
Then $r(x) \times s(x) =$

$$\underbrace{[f(x)^2 + g(x)^2][a(x)^2 + b(x)^2] + x^2(1-x)^2[h(x)^2 + k(x)^2][c(x)^2 + d(x)^2]}_{P(x)}$$

$$+x(1-x)\underbrace{\left[f(x)^2 + g(x)^2\right]\left[c(x)^2 + d(x)^2\right] + [a(x)^2 + b(x)^2][h(x)^2 + k(x)^2]}_{Q(x)}$$

So $r(x) \times s(x) = P(x) + x(1-x)Q(x)$
Clearly, both $P(x)$ and $Q(x)$ are non-negative for all x, so, by Theorem 1.1, each polynomial is a Type I. So $r(x) \times s(x)$ is a Type III polynomial. □

Proof of Theorem 3.1

To prove Theorem 3.1, we now consider the two possible cases:

(i) polynomials of even degree

(ii) polynomials of odd degree

3.1. **Case 1: Polynomials of Even Degree.** Suppose $p(x) \geq 0$ for $0 \leq x \leq 1$ and suppose $p(x)$ has even degree, $deg\ p(x) = 2k$, so that

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2k} x^{2k}.$$

For $x \neq 0$,

$$p\left(\frac{1}{x}\right) = a_0 + a_1 \frac{1}{x} + a_2 \frac{1}{x} + \ldots + a_{2k} \frac{1}{x^{2k}},$$

so we can define

$$q(x) := x^{2k} p\left(\frac{1}{x}\right) = a_0 x^{2k} + a_1 x^{2k-1} + \ldots + a_{2k}.$$

If $x \geq 1$, then $0 < \frac{1}{x} \leq 1$, so $p\left(\frac{1}{x}\right) \geq 0$, and so $q(x) = x^{2k} p\left(\frac{1}{x}\right) \geq 0$.

Let $r(x) := q(x+1)$. If $x \geq 0$, then $x + 1 \geq 1$, so $q(x+1) \geq 0$, and we find that $r(x) \geq 0$ for $x \geq 0$. By Theorem 2.1, there exists polynomials $A(x)$, $B(x)$, $C(x)$, and $D(x)$ such that

$$r(x) = A(x)^2 + B(x)^2 + x[C(x)^2 + D(x)^2].$$

We have $r(x) = q(x+1)$, so $deg\ r = deg\ q = 2k$. Corollary 3.2 shows that $deg\ A$, $deg\ B \leq k$ and $deg\ C$, $deg\ D \leq k - 1$. Now,

$$r(x-1) = A(x-1)^2 + B(x-1)^2 + (x-1)\left[(C(x-1)^2 + D(x-1)^2\right]$$

and

$$r(x-1) = q(x-1+1) = q(x) = x^{2k} p\left(\frac{1}{x}\right).$$

For $x \neq 0$, let $u = \frac{1}{x}$ and in turn $x = \frac{1}{u}$. Then, $\frac{1}{u^{2k}} \times p(u) = r\left(\frac{1}{u} - 1\right)$

$$= A\left(\frac{1}{u} - 1\right)^2 + B\left(\frac{1}{u} - 1\right)^2 + \left(\frac{1}{u} - 1\right)\left[C\left(\frac{1}{u} - 1\right)^2 + D\left(\frac{1}{u} - 1\right)^2\right],$$

so, $p(u) =$

$$u^{2k} \times \left[A\left(\frac{1}{u} - 1\right)^2 + B\left(\frac{1}{u} - 1\right)^2 + \left(\frac{1}{u} - 1\right)\left[C\left(\frac{1}{u} - 1\right)^2 + D\left(\frac{1}{u} - 1\right)^2\right]\right].$$

Next, distribute the u^{2k} term to each polynomial A, B, C, and D. For example, since $deg\ A \leq k$, we can write $A(x) = c_o + c_1 x + \ldots + c_k x^k$. Now,

$$A\left(\frac{1-u}{u}\right) = c_o + c_1\left(\frac{1-u}{u}\right) + \ldots + c_k\left(\frac{(1-u)^k}{u^k}\right),$$

so

$$u^{2k} \times \left[A \left(\frac{1}{u} - 1 \right)^2 \right] = u^{2k} \times A \left(\frac{1-u}{u} \right)^2 .$$

Consider

$$\tilde{A} := u^k A \left(\frac{1-u}{u} \right). \tag{3.1}$$

Since, $deg\ A \leq k$, then $\tilde{A}(u)$

$$= u^k c_o + c_1 \left(1-u\right) u^{k-1} + \ldots + c_k (1-u)^k.$$

Now we see that $\tilde{A}(u)$ is a polynomial in u. Similarly, $\tilde{B}(u) := u^k B \left(\frac{1}{u} - 1 \right)$ is a polynomial, where $B(x) = d_o + d_1 x + \ldots d_k x^k$ (from Corollary 3.2, $deg\ B \leq k$).
Next,

$$u^{2k} \left(\frac{1-u}{u} \right) \left[C \left(\frac{1-u}{u} \right)^2 + D \left(\frac{1-u}{u} \right)^2 \right]$$

$$= u^2 \left(\frac{1-u}{u} \right) \left[u^{2k-2} C \left(\frac{1-u}{u} \right)^2 + u^{2k-2} D \left(\frac{1-u}{u} \right)^2 \right]$$

$$= u(1-u)[\tilde{C}(u)^2 + \tilde{D}(u)^2],$$

where $\tilde{C}(u) := u^{k-1} C \left(\frac{1-u}{u} \right)$ and $\tilde{D}(u) := u^{k-1} D \left(\frac{1-u}{u} \right)$. Since, from Corollary 3.2, $deg\ C$, $deg\ D \leq k-1$, we see that $\tilde{C}(u)$ and $\tilde{D}(u)$ are polynomials. So,

$$p(u) = \tilde{A}(u)^2 + \tilde{B}(u)^2 + u(1-u)[\tilde{C}(u)^2 + \tilde{D}(u)^2]$$

and therefore p is Type III. This concludes the proof of Case 1.

Here is an example illustrating equation 3.1.

Example 3.5. Let $A(x) = 1014x^7 + 366x^6 + 99x^5 + 32x^4 + 678x^3 + 44x^2 + 4x + 14$.
Then $\tilde{A}(x) := x^7 A \left(\frac{1-x}{x} \right)$.
$\tilde{A}(x) = \left(\frac{1014(1-x)^7}{x^7} + \frac{366(1-x)^6}{x^6} + \frac{99(1-x)^5}{x^5} + \frac{32(1-x)^4}{x^4} + \frac{678(1-x)^3}{x^3} + \frac{44(1-x)^2}{x^2} + \frac{4(1-x)}{x} + 14 \right) x^7$
Expand$[\tilde{A}(x)] = 1014 - 6732x + 19197x^2 - 30463x^3 + 29710x^4 - 18592x^5 + 7219x^6 - 1339x^7$
So we see that $\tilde{A}(x)$ is a polynomial in x as we claimed in Equation 3.1.

3.2. **Case 2: Polynomials of Odd Degree.** Suppose $p(x) \geq 0$ for $0 \leq x \leq 1$ and suppose that $p(x)$ has odd degree, $deg\ p(x) = 2m+1$. By the Factorization Theorem,

$$p(x) = A q_1 \ldots q_r (x - y_1)^{s_1} \ldots (x - y_t)^{s_t} (x - x_1)^{t_1} \ldots (x - x_u)^{t_u},$$

where $q_1 \ldots q_r$ are quadratics with no real roots, $x_1 \ldots x_u, y_1 \ldots y_t$ are distinct, s_i is even, and t_i is odd. We let $t_i = 2v_i + 1$, so

$$(x - x_i)^{t_i} = (x - x_i)^{2v_i} (x - x_i).$$

Now we have $p(x) =$

$$A\underbrace{q_1 \dots q_r (x-y_1)^{s_1} \dots (x-y_t)^{s_t} (x-x_1)^{2v_1} \dots (x-x_u)^{2v_u}}_{Q(x)} (x-x_i) \dots (x-x_u).$$

We can assume that each $q_i > 0$ for all x, so $Q(x) \geq 0$ for all x. Since $p(x) \geq 0$ for $0 \leq x \leq 1$, it is clear that the x_i's are not in the interval $[0,1]$ (otherwise, $p(x)$ would change sign in $[0,1]$.)
To continue further we need two Lemmas.

Lemma 3.6. *If $x_0 \leq 0$, then $x - x_0$ is of Type III.*

Proof. We want a Type III representation for $x - x_0$ of the form

$$p(x) := (x - x_0) = (Ax + B)^2 + x(1-x)C^2, \tag{3.2}$$

where A, B, and C are constants.
Let $\alpha = -x_0 \geq 0$, so we can now say
$x + \alpha \equiv x - x_0 = A^2x^2 + 2ABx + B^2 + xC^2 - x^2C^2$.
Now we compare coefficients on each side of the equation:
$B^2 = \alpha \geq 0$, so $B = \sqrt{\alpha}$.
Next,

$$2AB + C^2 = 1. \tag{3.3}$$

Also, $A^2 - C^2 = 0$, so $A^2 = C^2$, or $\mid A \mid = \mid C \mid$.
So we can substitute A for C in equation 3.3 to get
$2AB + A^2 = 1$, so finally we have

$$A^2 + 2\sqrt{\alpha}A - 1 = 0. \tag{3.4}$$

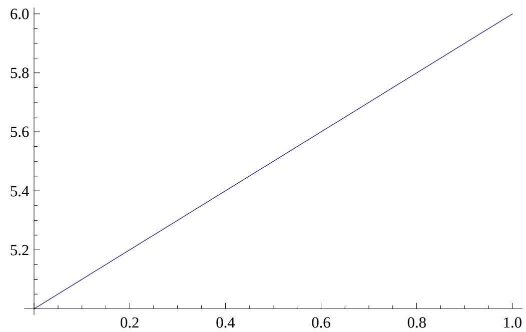
For the discriminant of this quadratic in A we get

$$b^2 - 4ac = 4\alpha - 4(1)(-1) = 4(\alpha + 1) > 0.$$

This tells us that we can solve equation 3.4 for A, and get the Type III representation such as in equation 3.2. □

The following is an example of Lemma 3.6

Example 3.7. We start with P, a $(x - x_o)$ term where $x_o \leq 0$.
$P = (x + 5)$
Plot$[P, x, 0, 1]$



Think of the number 5 as a square, $(\sqrt{\alpha})^2$ and since we know h and k cannot be 0 or else there will be no x term.
$x_0 = -5$
$\alpha = -x_0 = 5$
$B = \sqrt{\alpha} = \sqrt{5}$
$H = A$
Use equation 3.4 from Lemma 3.6
$0 == 2AB + A^2 - 1 = -1 + 2\sqrt{5}A + A^2$
$a = 1$
$b = 2B = 2\sqrt{5}$
$c = -1$
Apply the quadratic formula
$\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) = \frac{1}{2}\left(-2\sqrt{5} + 2\sqrt{6}\right)$
Expand[%] $= -\sqrt{5} + \sqrt{6}$
$\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{1}{2}\left(-2\sqrt{5} - 2\sqrt{6}\right)$
Expand[%] $= -\sqrt{5} - \sqrt{6}$
$A = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) = \frac{1}{2}\left(-2\sqrt{5} + 2\sqrt{6}\right)$
$H = A = \frac{1}{2}\left(-2\sqrt{5} + 2\sqrt{6}\right)$
$B = \sqrt{\alpha} = \sqrt{5}$
Verify that the expression is equal to 1
$A^2 + 2BA = \sqrt{5}\left(-2\sqrt{5} + 2\sqrt{6}\right) + \frac{1}{4}\left(-2\sqrt{5} + 2\sqrt{6}\right)^2$
FullSimplify[%] $= 1$
$A^2 + 2BA = 1$ so the above expression should be zero.
Expand$[((x - x_o)) - ((A^2 + 2BA)\, x - x_o)] = 0$
Type III representation below.
Expand$[(x - x_o) - ((Ax + B)^2 + x(1-x)H^2)] = 0$
It is concluded that original linear term $x + 5$ has a Type III representation.

Lemma 3.8. *If $x_o \geq 1$, then $-(x - x_o)$ is of Type III.*

Proof. We try for a Type III representation of the form

$$p(x) = -(x - x_0) = (Ax + B)^2 + x(1-x)C^2. \tag{3.5}$$

Now, $-x + x_0 = A^2x^2 + 2ABx + B^2 + xC^2 - x^2C^2$. To solve 3.5 we need $x_0 = B^2$, so $B = \sqrt{x_0}$. We also need

$$2AB + C^2 = -1 \tag{3.6}$$

and
$A^2 - C^2 = 0$, so $A^2 = C^2$, or $\mid A \mid = \mid C \mid$.
So we can substitute A for C in equation 3.6 and get

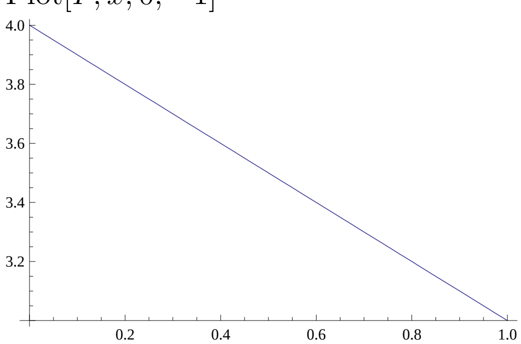
$$2AB + A^2 = -1. \tag{3.7}$$

So finally we need $A^2 + 2\sqrt{x_0}A + 1 = 0$.
From the discriminant, we get
$b^2 - 4ac = 4(x_0 - 1) \geq 0$. Therefore we can find A, B, and C that satisfy equation 3.5, so $p(x)$ is Type III.

□

The following example illustrates Lemma 3.8.

Example 3.9. Here is an example of why $-(x - x_o)$, where $x_o \geq 1$, is of Type III.
$P = -(x - 4)$
$\text{Plot}[P, x, 0, -1]$



Plot shows P is non-negative for $[0, 1]$.
$x_o = 4$
$B = \sqrt{x_o} = 2$
$H = A$
Solve using equation 3.4
$0 == 2AB + A^2 + 1 = 1 + 4A + A^2$
$a = 1$
$b = 2B = 4$
$c = 1$
Apply the quadratic formula.
$\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) = \frac{1}{2}\left(-4 + 2\sqrt{3}\right)$
$\text{Expand}[\%] = -2 + \sqrt{3}$
$\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{1}{2}\left(-4 - 2\sqrt{3}\right)$

Expand[%] $= -2 - \sqrt{3}$
$A = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) = \frac{1}{2}\left(-4 + 2\sqrt{3}\right)$
$H = A = \frac{1}{2}\left(-4 + 2\sqrt{3}\right)$
$B = \sqrt{x_o} = 2$
Verify the expression is - 1
$A^2 + 2BA = 2\left(-4 + 2\sqrt{3}\right) + \frac{1}{4}\left(-4 + 2\sqrt{3}\right)^2$
FullSimplify[%] $= -1$
Expand$[(-(x - x_o)) - ((A^2 + 2BA)x + x_o)] = 0$
Expand$[-(x - x_o) - ((Ax + B)^2 + x(1 - x)H^2)] = 0$
$-(x - 4)$ has a Type III representation.

For the proof of Theorem 3.1 with *deg p* odd, we will now look at two different cases depending on the sign of A from the factorization theorem.

3.2.1. *Case i.* When $A > 0$, $p(x) =$

$$Aq_1 \dots q_r(x - y_1)^{s_1} \dots (x - y_t)^{s_t}(x - x_1)^{2v_1}$$
$$\dots (x - x_m)^{2v_m}(x - x_1) \dots (x - x_i)(x - x_{i+1}) \dots (x - x_m),$$

where $x_1 \dots x_i \leq 0$ and $x_{i+1} \dots x_m \geq 1$. Let

$$Q(x) := Aq_1 \dots q_r(x - y_1)^{s_1} \dots (x - y_t)^{s_t}(x - x_1)^{2v_1} \dots (x - x_m)^{2v_m}.$$

$Q(x) \geq 0$ because $A > 0$ and the q_i's are strictly positive quadratics. These parameters make $Q(x)$ a Type I polynomial, which can be expressed as a Type III polynomial. So now we have

$$p(x) = Q(x)\underbrace{(x - x_1) \dots (x - x_i)}_{O(x)}\underbrace{(x - x_{i+1}) \dots (x - x_m)}_{E(x)},$$

where $x_1 \dots x_i$ are ≤ 0 and $x_{i+1}, \dots, x_m$ are ≥ 1. By Lemma 3.6, each of the factors in $(x - x_1) \dots (x - x_i)$ are Type III so by Lemma 3.4, $O(x)$ is Type III. For $0 < x < 1$, each factor in $E(x)$ is negative, so there must be an even number of these factors in $E(x)$. We can rewrite $E(x)$ as $(-(x - x_{i+1})) \dots (-(x - x_m))$(since the number of factors is even, the value does not change). Since each $-(x - x_{i+j})$ is Type III by Lemma 3.6, then $E(x)$ is Type III by Lemma 3.4. Now, $p(x) = Q(x) \times O(x) \times E(x)$, so by Lemma 3.4, $p(x)$ is Type III.

3.2.2. *Case ii.* : When $A < 0$, $p(x) =$

$$-\tilde{A}q_1q_1 \dots q_r(x - y_1)^{s_1} \dots (x - y_t)^{s_t}(x - x_1)^{2v_1}$$
$$\dots (x - x_m)^{2v_m}(x - x_1) \dots (x - x_i)(x - x_{i+1}) \dots (x - x_m),$$

where $\tilde{A} > 0$, each q_i is strictly positive, $x_1 \dots x_i \leq 0$, and $x_{i+1} \dots x_m \geq 1$. Let

$Q(x) := \widetilde{A}q_1 \dots q_r(x-y_1)^{s_1} \dots (x-y_t)^{s_t}(x-x_1)^{2v_1} \dots (x-x_m)^{2v_m}$. Then $Q(x)$ is non-negative for all x and we can write

$$p(x) = (-1)Q(x) \underbrace{(x-x_1)\dots(x-x_i)}_{E(x)} \underbrace{(x-x_{i+1})\dots(x-x_m)}_{O(x)}$$

For $0 < x < 1$, each factor $(x - x_j)$ with $x_j \geq 1$ is negative, so there must be an odd number of factors in $O(x)$ (since otherwise $p(x) < 0$). So we can write

$$p(x) = (-1)Q(x)E(x) \underbrace{((-1)(x-x_{i+1})\dots(-1)(x-x_m))}_{R(x)}(-1).$$

(The number of negative signs inserted in $O(x)$ is odd, so in order for them to cancel each other out we compensate with an extra negative at the end). Let $F(x) = Q(x) \times E(x)$. Since each factor in $E(x)$ is Type III by Lemma 3.6, then $E(x)$ is Type III by Lemma 3.4. So $F(x)$ is also Type III by Lemma 3.4. Since each factor $-(x - x_j)$ in $R(x)$ is Type III by Lemma 3.8, then $R(x)$ is Type III by Lemma 3.4, so now we have

$$p(x) = F(x)R(x).$$

(The leading coefficient(-1) and the very last (-1) at the end cancel out.) Since $F(x)$ and $R(x)$ are Type III, by Lemma 3.4, $p(x) = F(x) \times R(x)$ is also Type III. This completes the proof of case 3.2, polynomials with odd degree.

The following corollary tells us that Theorem 3.1 works for any interval $[a, b]$.

Corollary 3.10. *Suppose $p(x)$ is non-negative on the interval $[a, b]$, where $a < b$. Then there are polynomials $R(x), S(x), T(x)$, and $U(x)$ such that*

$$p(x) = R(x)^2 + S(x)^2 + (x-a)(b-x)[T(x)^2 + U(x)^2].$$

Proof. Consider the function, $q(x) := p\,(u(x))$ where $u \equiv u(x) := a + x(b-a)$. $q(x)$ is a polynomial with $deg\ q = deg\ p$, and if $0 \leq x \leq 1$, then $a \leq u(x) \leq b$, so $q(x) \geq 0$. According to Theorem 3.1, there are polynomials $A(x)$, $B(x)$, $C(x)$, $D(x)$ such that

$$q(x) = A(x)^2 + B(x)^2 + x(1-x)\left[C(x)^2 + D(x)^2\right].$$

We know that $u = a + x(b-a)$; if we solve for x, we get the expression:

$$x = \frac{u-a}{b-a}.$$

Now replace x in the function $q(x)$ with $\frac{u-a}{b-a}$,

$$p\,(u(x)) = A(x)^2 + B(x)^2 + x(1-x)\left[C(x)^2 + D(x)^2\right]$$

$$= A\left(\frac{u-a}{b-a}\right)^2 + B\left(\frac{u-a}{b-a}\right)^2 + \left(\frac{u-a}{b-a}\right)\left(1 - \frac{u-a}{b-a}\right)\left[C\left(\frac{u-a}{b-a}\right)^2 + D\left(\frac{u-a}{b-a}\right)^2\right].$$

Now simplify $1 - \dfrac{u-a}{b-a}$ to get

$$\frac{b-a}{b-a} - \frac{u-a}{b-a} = \frac{b-u}{b-a},$$

then replace in the expression for $p(u(x))$:

$$p\left(u(x)\right) = \widetilde{A}(u)^2 + \widetilde{B}(u)^2 + \left(\frac{(u-a)}{(b-a)^2}(b-u)\right)\left[\widetilde{C}(u)^2 + \widetilde{D}(u)^2\right],$$

where

$$\widetilde{A}(u) := A\left(\frac{u-a}{b-a}\right),\ \widetilde{B}(u) := B\left(\frac{u-a}{b-a}\right),\ \widetilde{C}(u) := C\left(\frac{u-a}{b-a}\right),\ \widetilde{D}(u) := D\left(\frac{u-a}{b-a}\right).$$

Note that $\widetilde{A}, \widetilde{B}, \widetilde{C}$, and $\widetilde{D}$ are polynomials in u.

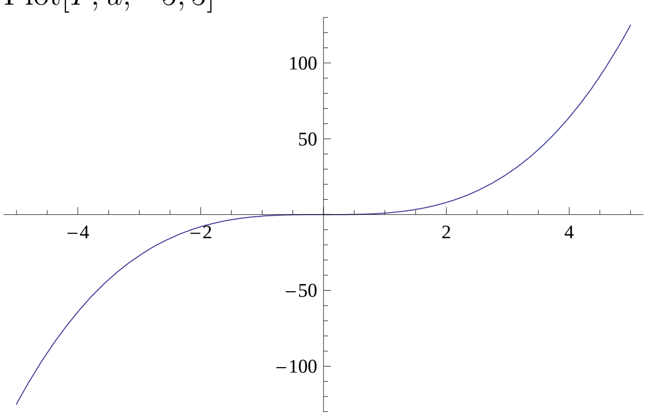
The final expression is derived by distributing $\dfrac{1}{b-a^2}$,

$$p\left(u\right) = \widetilde{A}(u)^2 + \widetilde{B}(u)^2 + (u-a)(b-u)\left[C'(u)^2 + D'(u)^2\right],$$

where $C'(u) := \dfrac{\widetilde{C}(u)}{b-a}$, $D'(u) := \dfrac{\widetilde{D}(u)}{b-a}$ are polynomials in u. □

The following example illustrates Corollary 3.10.

Example 3.11. Take $P(u) = u^3$, positive on $[2, 4]$ and give a Type III representation.
$P = u^3$
$a = 2$
$b = 4$
Plot$[P, u, -5, 5]$



$u = a + x(b-a) = 2 + 2x$
$P = (2 + 2x)^3$
Factoring the polynomial shows an odd degree, so we use Theorem 3.1 Case 2.
Factor$[P] = 8(1 + x)^3$
Expand$[P - (8(1 + x)^2(1 + x))] = 0$
Next we get the Type III representation of the polynomial P.

$(x+1) = \left(1+\frac{1}{2}\left(-2+2\sqrt{2}\right)x\right)^2 + x(1-x)\left(\frac{1}{2}\left(-2+2\sqrt{2}\right)\right)^2$

$8\left((1+x)^2\right) = \left(\sqrt{8}(1+x)\right)^2 + 0^2 + x(1-x)\left((0)^2+(0)^2\right)$

Now use Lemma 3.4 regarding multiplication of Type III polynomials.

Using Lemma 3.4, we can express P as $F + x(1-x)G$, where F and G are the following sums of two squares:

$F = X^2 + W^2 = 0^2 + \left(2\sqrt{2}(1+x)(1-x+\sqrt{2}x)\right)^2$

$G = V^2 + U^2 = 0^2 + \left(\sqrt{2}(-2+2\sqrt{2})(1+x)\right)^2.$

Now solve for x in the equation $u = a + x(b-a)$

$x = \frac{u-a}{b-a} = \frac{1}{2}(-2+u)$

$(X^2 + W^2 + x(1-x)(U^2+V^2)) =$

$(8\left(1+\frac{1}{2}(-2+u)\right)^2\left(1+\frac{2-u}{2}+\frac{-2+u}{\sqrt{2}}\right)^2 + \left(-2+2\sqrt{2}\right)^2\left(1+\frac{2-u}{2}\right)\left(1+\frac{1}{2}(-2+u)\right)^2(-2+$
$u)$

RR $=\sqrt{8}\left(1+\frac{1}{2}(-2+u)\right)\left(1+\frac{2-u}{2}+\frac{-2+u}{\sqrt{2}}\right)$

$SS = X = 0$

$TT = V = 0$

$UU = \frac{1}{(b-a)}U = \frac{\left(-2+2\sqrt{2}\right)\left(1+\frac{1}{2}(-2+u)\right)}{\sqrt{2}}$

ExpandAll$[P - (RR^2 + SS^2 + (u-a)(b-u)(TT^2+UU^2))] = 0$

So the original $P = u^3$, has a Type III representation on $[2,4]$.

(Refer to Appendix D for the full problem implementation.)

4. Non-Negative Polynomials in Two-Variables

In this section we will discuss non-negative polynomials in two variables. In 1888, David Hilbert proved that there are polynomials $p(x, y)$ that are non-negative on $\mathbb{R}^2$ but cannot be expressed as a sum of squares. The first specific example of such a polynomial was discovered by T. Motzkin in 1967. In this section we study Motzkin's example. We then illustrate a general method for indicating whether or not a polynomial $p(x, y)$ is a sum of squares.

Theorem 4.1. *Given the Motzkin polynomial*

$$M(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1, \tag{4.1}$$

(i) *$M(x, y) \geq 0$ for every x, y and*
(ii) *M cannot be expressed as a sum of squares.*

In an attempt to prove this two-part theorem we will use two methods. The first will utilize the following theorem.

Theorem 4.2. *(Inequality of Arithmetic and Geometric Means) When we have the numbers $a_1 \ldots a_n > 0$, we can say that*

$$\frac{a_1 + \ldots + a_n}{n} \geq (a_1 \ldots a_n)^{\frac{1}{n}}. \tag{4.2}$$

Proof of Theorem 4.1

Proof. Here we try to prove $M(x, y) \geq 0$ for every x, y with the Motzkin polynomial

$$M(x, y) = x^2 + y^2(x^2 + y^2 - 3) + 1.$$

If $x^2 + y^2 - 3 \geq 0$, then clearly $M(x, y) \geq 0$.
Suppose

$$x^2 + y^2 < 3,$$

let

$$w := 3 - x^2 - y^2 > 0,$$

and let

$$z := \sqrt{w}.$$

Then

$$z^2 = 3 - x^2 - y^2, \tag{4.3}$$

or

$$3 = x^2 + y^2 + z^2. \tag{4.4}$$

By equation 4.2, we can say the following:

$$\frac{x^2 + y^2 + z^2}{3} \geq (x^2y^2z^2)^{\frac{1}{3}}.$$

If we substitute in equation 4.4 we get,

$$1 = \frac{3}{3} = \frac{x^2 + y^2 + z^2}{3} \geq (x^2y^2z^2)^{\frac{1}{3}}.$$

Since

$$1 \geq (x^2y^2z^2)^{\frac{1}{3}},$$

then

$$1 \geq x^2y^2z^2.$$

Substitute z^2 in from equation 4.3:

$$1 \geq x^2y^2(3 - x^2 - y^2).$$

Distribute the -1

$$-1 \leq x^2y^2(x^2 + y^2 - 3).$$

Bring the 1 to the other side,

$$0 \leq x^2y^2(x^2 + y^2 - 3) + 1 = M(x, y),$$

and we discover that $M(x, y) \geq 0$ for every x, y.

Now we will try to prove the second part of Theorem 4.1 by attempting to express $M(x, y)$ as a sum of squares. Suppose $M(x, y)$ is a sum of squares of polynomials of degree 3:

$$M(x, y) = s(x, y) :=$$

$$\sum_{i=1}^{p}(a_i + b_ix + c_iy + d_ix^2 + e_ixy + f_iy^2 + g_ix^3 + h_ix^2y + j_ixy^2 + k_iy^3)^2,$$

where $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, j_i$, and k_i are constants. Comparing to equation 4.1, the Motzkin polynomial does not contain most of the terms in $s(x, y)$ so some of these constants may equal 0. But let's look at certain terms to see if we can really express $M(x, y)$ as a sum of squares. There is no x^2 term in equation 4.1, so the coefficient of x^2 in $s(x, y)$ must be 0. Therefore, $\sum_{i=1}^{p} b_i^2 = 0$, so each $b_i = 0$. Similarly, there is no y^2 term in Equation 4.1, so $\sum_{i=1}^{p} c_i^2 = 0$, and therefore $c_i = 0$. Most noticeably we see that the coefficient of the x^2y^2 term in equation 4.1 is -3, so we must have

$$\sum_{i=1}^{p} 2f_id_i + e_i^2 + 2b_ij_i + 2c_ih_i = \sum_{i=1}^{p} 2f_id_i + e_i^2 = -3. \tag{4.5}$$

But let's look at d_i. Since there is no x^4 term in equation 4.1, then $\sum_{i=1}^{p} d_i^2 + 2b_ig_i = 0$. But we discovered above that $b_i = 0$, so $\sum_{i=1}^{p} d_i^2 = 0$, and therefore each $d_i = 0$. When we go back to equation 4.5 we see that we now have $\sum_{i=1}^{p} e_i^2 = -3$. This is a contradiction and is not possible. (The sum of all the e_i^2 terms cannot equal a negative number.) Therefore M cannot be expressed as a sum of squares. $\square$

We now continue by studying some consequences of the following theorem of Hilbert:

Theorem 4.3. *Every polynomial $p(x_1, x_2, \cdots x_n)$ of degree $2k$ that is non-negative is a sum of squares if and only if*

(i) $n = 1, \quad k \geq 0$ *(Theorem 1.1)*
(ii) $n > 1, \quad k = 1$
(iii) $n = 2, \quad k = 2$

Hilbert's theorem shows for $n = 2$ and $deg\ p = 6$ there are polynomials that are non-negative but cannot be expressed as sums of squares. Next we study a method for determining if a polynomial $p(x, y)$ is a sum of squares.

Proposition 4.4. *A polynomial $p(x, y)$ with $deg\ p = 2k$ can be expressed as a sum of squares if and only if there is a matrix A such that*

$$p(x, y) = (A\,v)^T A\,v,$$

where

$$v \equiv v_k := \left(1, x, y, x^2, x\ y, y^2, \cdots, x^k, x^{k-1}\ y, \cdots, y^k\right).$$

Proof. The entries of Av are polynomials, so $(A\ v)^T (A\ v)$ is a sum of squares of polynomials. So if

$$p(x, y) = (A\,v)^T (A\,v),$$

then p is a sum of squares of polynomials. Conversely, if $p = \sum_{i=1}^{k} {f_i}^2$ (f_i is a polynomial), define a matrix A so that row i of A has the coefficients of f_i. Then $p(x, y) = (A\,v)^T (A\,v)$. □

To decide if a given $p(x, y)$ is a sum of squares, we use the following steps.

(i) Find a matrix M with $M = M^T$ such that $p(x, y) = v^T M v$.
(ii) Find a matrix M that satisfies step(i), and $M \succeq 0$ (M is *positive*, i.e., where all eigenvalues of M are non-negative). Then let $P = M$ so, $p(x, y) = v^T\, P\, v$.
(iii) Use Cholesky Decomposition to find a matrix A such that

$$P = A^T A.$$

(iv) Then $p(x, y) = v^T\ P\ v = v^T A^T A v = (A\,v)^T\ (A\,v)$, giving a sum of squares representation for p.

Example 4.5. $p = 7 + 2x + 14x^2 + 14y + 28xy + 49y^2$
The vector v, must count up to half of the total degree.
$v = (1, x, y)$
The matrix is self-adjoint for simplicity so there is easy variable comparison.
$P = \{\{a, b, c\}, \{b, d, e\}, \{c, e, f\}\}$
Solve p = v.P.v for P.

$$MatrixForm[P] = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Together$[p-v.P.v] = 7-a+2x-2bx+14x^2-dx^2+14y-2cy+28xy-2exy+49y^2-fy^2$
$a = 7$
$b = 1$
$c = 7$
$d = 14$
$e = 14$
$f = 49$
$p - v.P.v = x + 14x^2 + 7y + 28xy + 49y^2 - x(1 + 14x + 14y) - y(7 + 14x + 49y)$
Factor[%] = 0
P= v.M.v
$P = \{7, 1, 7\}, \{1, 14, 14\}, \{7, 14, 49\} = \begin{pmatrix} 7 & 1 & 7 \\ 1 & 14 & 14 \\ 7 & 14 & 49 \end{pmatrix}$
Check for positivity, determinant, cholesky decomposition, eigenvalues.
Det$[P] = 2891$
CholeskyDecomposition$[P] = \left\{\left\{\sqrt{7}, \frac{1}{\sqrt{7}}, \sqrt{7}\right\}, \left\{0, \sqrt{\frac{97}{7}}, 13\sqrt{\frac{7}{97}}\right\}, \left\{0, 0, 7\sqrt{\frac{59}{97}}\right\}\right\}$
A=$\left\{\left\{\sqrt{7}, \frac{1}{\sqrt{7}}, \sqrt{7}\right\}, \left\{0, \sqrt{\frac{97}{7}}, 13\sqrt{\frac{7}{97}}\right\}, \left\{0, 0, 7\sqrt{\frac{59}{97}}\right\}\right\}$
MatrixForm[A]=$\begin{pmatrix} \sqrt{7} & \frac{1}{\sqrt{7}} & \sqrt{7} \\ 0 & \sqrt{\frac{97}{7}} & 13\sqrt{\frac{7}{97}} \\ 0 & 0 & 7\sqrt{\frac{59}{97}} \end{pmatrix}$
tA=Transpose[A]=$\left\{\left\{\sqrt{7}, 0, 0\right\}, \left\{\frac{1}{\sqrt{7}}, \sqrt{\frac{97}{7}}, 0\right\}, \left\{\sqrt{7}, 13\sqrt{\frac{7}{97}}, 7\sqrt{\frac{59}{97}}\right\}\right\}$
MatrixForm[tA]=$\begin{pmatrix} \sqrt{7} & 0 & 0 \\ \frac{1}{\sqrt{7}} & \sqrt{\frac{97}{7}} & 0 \\ \sqrt{7} & 13\sqrt{\frac{7}{97}} & 7\sqrt{\frac{59}{97}} \end{pmatrix}$
Double check that P = Transpose[A].A:
$P - tA.A = \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}$
To express p as a sum of squares, we now use $(v.tA).(A.v)$
$(v.tA).(A.v) = \frac{2891y^2}{97} + \left(\sqrt{\frac{97}{7}}x + 13\sqrt{\frac{7}{97}}y\right)^2 + \left(\sqrt{7} + \frac{x}{\sqrt{7}} + \sqrt{7}y\right)^2$
This is a sum of squares expression; We double check that it equals p
$q = (v.tA).(A.v)$
Factor$[p - q] = 0$
Another way to check that a real symmetric matrix is positive is to verify that all of its eigenvalues are nonnegative :
Eigenvalues$[P] = \text{Root}\left[-2891 + 881\#1 - 70\#1^2 + \#1^3\&, 3\right]$,
$\text{Root}\left[-2891 + 881\#1 - 70\#1^2 + \#1^3\&, 2\right]$,
$\text{Root}\left[-2891 + 881\#1 - 70\#1^2 + \#1^3\&, 1\right]$
$N[\%] = 54.9159, 9.60073, 5.48334$
This means that there are vectors a, b, c such that $P.a = 54.91592601638876`a$, $P.b =$

9.600729026932235`$b$, $Pc = 5.483344956679007`c$. Further, any vector w can be expressed as $w = ra+sb+tc$, so $Pw = 54.91592601638876`(ra)+9.600729026932235`(sb)+ 5.483344956679007`(tc)$. This shows that P is like stretching by positive multipliers in three directions.

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Appendix A. Factorization Theorem

In the previous sections we used the *Factorization Theorem*:
If $p(x)$ is a polynomial, then $p(x) = Aq_1(x)\dots q_k(x)(x-x_1)^{n_1}\dots(x-x_r)^{n_r}(x-y_1)^{m_1}\dots(x-y_s)^{m_s}$, where A is a constant, $q_1(x)\dots q_k(x)$ are strictly non-zero quadratics, $x_1\dots x_r$, $y_1\dots y_s$ are all distinct, $n_1\dots n_r$ are even exponents, and $m_1\dots m_s$ are odd exponents, (Note that any of these groups may be missing in the expression).
Here we explain the theorem's origin.
Suppose z is a complex number,

$$z = x + iy,$$

where x, y are real numbers and i is imaginary, with $i^2 = -1$. Let $\overline{z}$ be the complex conjugate of z,

$$\overline{z} := x - iy.$$

Consider a complex polynomial

$$p(z) = a_0 + a_1 z + \dots + a_k z^k$$

(where the a_i's are complex).
We state without proof the following theorem (see [1]).

Theorem A.1. *(Fundamental Theorem of Algebra) There is a complex number z_0 such that $p(z_0) = 0$.*

Next, we recall the Division Theorem.

Theorem A.2. *(Division Theorem) If $f(z)$ and $g(z)$ are polynomials, then there are polynomials, $q(z)$ and $r(z)$, with deg $r <$ deg g, such that $f(z) = g(z)q(x) + r(x)$.*

Now if we have

$$f(z) = p(z)$$

and

$$g(z) = z - z_0,$$

where $p(z_0) = 0$, then

$$p(z) = (z - z_0)q(z) + r(z),$$

where

$$deg\ r < deg\ z - z_0 = 1.$$

So

$$deg\ r = 0,$$

and therefore r is a constant. Then

$$0 = p(z_0) = 0 + r,$$

so

$$r = 0.$$

So we see that if z_0 is a root of $p(z) = 0$, then $z - z_0$ is a factor of $p(z)$, i.e.,

$$p(z) = (z - z_0)q(z),$$

and

$$deg\ p = 1 + deg\ q.$$

Corollary A.3. *A non-constant polynomial* $p(z)$ *factors completely as* $A(z-z_0)\dots(z-z_{k-1})$*. Also, if* $p(z)$ *has real coefficients and if* z_i *is a root, then so is* $\overline{z_i}$*.*

Proof. The Corollary is true if $deg\ p = 1$:

$$az + b = a(z + \frac{b}{a}) \text{ where } a \neq 0.$$

Assume by induction that the Corollary is true for all complex polynomials up to $deg\ k-1$. Suppose $deg\ p = k$. From above,

$$p = (z - z_o)q,\ deg\ q = k - 1.$$

By induction, $q = A(z - z_1)\dots(z - z_k)$. So

$$p = (z - z_o)q = A(z - z_o)(z - z_1)\dots(z - z_k).$$

Suppose p has real coefficients and w is a root.

$$p(z) = \sum_{i=o}^{k} a_i z^i.$$

So

$$0 = p(w) = \sum_{i=o}^{k} a_i w^i.$$

Then

$$0 = \overline{p(w)} = \overline{\sum_{i=o}^{k} a_i w^i} = \sum_{i=o}^{k} \overline{a_i w^i} = \sum_{i=o}^{k} \overline{a_i}\,\overline{w^i} = \sum_{i=o}^{k} a_i \overline{w}^i = p(\overline{w})$$

(because $\overline{w^i} = \overline{w}^i$). □

Suppose $p(z)$ is a real polynomial. So $p(z)$ factors completely as a complex polynomial and if z_o is a non-real root, then so is $\overline{z_o}$. So now we have

$$p(z) = (x - z_o)(x - \overline{z_o})q(x).$$

We can express $f(x)$ as a real quadratic with no real roots. Let $z_o = a + ib$ where $b \neq 0$ and $\overline{z_o} = a - ib$. So

$$f(x) := (x - z_o)(x - \overline{z_o}) = x^2 - 2ax + a^2 + b^2.$$

Checking the discriminant,

$$4a^2 - 4(a^2 + b^2) = 4a^2 - 4a^2 - 4b^2,$$

we end up with

$$-4b^2 < 0$$

(since $b \neq 0$). Therefore $f(x)$ has no real roots. If we apply this argument to the factorization in Corollary A.3, we get:

(A.1) $$p(z) = (x - z_0)\dots(x - z_m)q_1(x)\dots q_k(x),$$

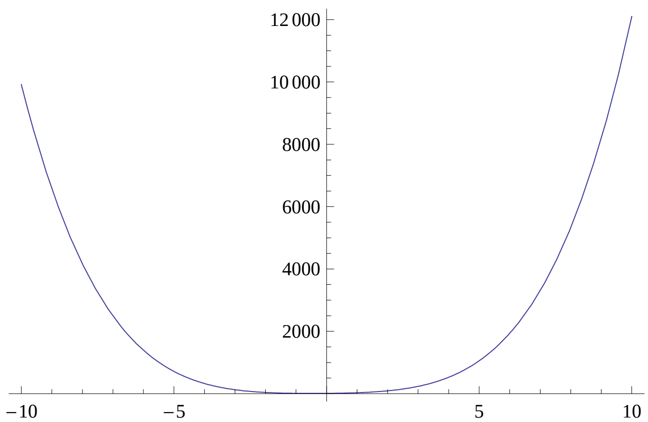
where $z_0, \ldots, z_m$ are real roots and $q_1(x) \ldots q_k(x)$ are real quadratics with no real roots. Notice that the $z_0, \ldots, z_m$ are not necessarily distinct, but the repeated factors in equation A.1 can be grouped together, and thus factors with powers of even and odd degrees are created. So finally we have our *Factorization Theorem*.

Appendix B. Section 1 Examples

```
(* Example 1.1-Find a Type I representation for the
  product of two polynomials that are non-negative for all x *)
P = (x^2 + 9) (x^2 + x + 1)
```

$(9 + x^2)\ (1 + x + x^2)$

```
Plot[P, {x, -10, 10}]
```




```
(* make (1+x+x^2) a sum of squares by using the formula derived from a quadratic*)
(* we find h and k so that we can use the product formula*)

h = x + 1 / 2
```

$\frac{1}{2} + x$

```
k = Sqrt[3 / 4]
```

$\frac{\sqrt{3}}{2}$

```
(*from the first question we find f and g*)
f = x
```

x

```
g = 3
```

3

```
(*now the product formula*)
(f^2 + g^2) * (h^2 + k^2)
```

$(9 + x^2)\ \left(\frac{3}{4} + \left(\frac{1}{2} + x\right)^2\right)$

```
(f * h + g * k) ^2 + (f * k - g * h) ^2
```

$$\left(\frac{\sqrt{3}\ x}{2}-3\left(\frac{1}{2}+x\right)\right)^2+\left(\frac{3\sqrt{3}}{2}+x\left(\frac{1}{2}+x\right)\right)^2$$

$$p=\left(\frac{\sqrt{3}\ x}{2}-3\left(\frac{1}{2}+x\right)\right)^2+\left(\frac{3\sqrt{3}}{2}+x\left(\frac{1}{2}+x\right)\right)^2$$

$$\left(\frac{\sqrt{3}\ x}{2}-3\left(\frac{1}{2}+x\right)\right)^2+\left(\frac{3\sqrt{3}}{2}+x\left(\frac{1}{2}+x\right)\right)^2$$

```
Expand[p]
```

$$9+9\,x+10\,x^2+x^3+x^4$$

```
Factor[p]
```

$$\left(9+x^2\right)\left(1+x+x^2\right)$$

```
(*notice p is equivalent to P at the beginning of the page*)
```

```
P - p
```

$$\left(9+x^2\right)\left(1+x+x^2\right)-\left(\frac{\sqrt{3}\ x}{2}-3\left(\frac{1}{2}+x\right)\right)^2-\left(\frac{3\sqrt{3}}{2}+x\left(\frac{1}{2}+x\right)\right)^2$$

```
Expand[%]
```

0

(*Example 1.2~ Find a sum of two squares representation for a polynomial of degree 12 *)

$$P = 7\,5^{17/52}\,113^{1/13}x^4 + \frac{7\,5^{1/4}x^5}{2\sqrt{2}} + 5^{43/52}\,113^{1/13}x^5 + \frac{49}{12}5^{1/4}x^6 + \frac{5^{3/4}x^6}{2\sqrt{2}} + 4\,5^{17/52}\,113^{1/13}x^6 + \sqrt{2}\,5^{1/4}x^7 + \frac{7}{12}5^{3/4}x^7 + \frac{7}{3}5^{1/4}x^8 + 7\,5^{43/52}\,113^{1/13}x^8 + \frac{7\,5^{3/4}x^9}{2\sqrt{2}} + 5\,5^{17/52}\,113^{1/13}x^9 + \frac{5\,5^{1/4}x^{10}}{2\sqrt{2}} + \frac{49}{12}5^{3/4}x^{10} + 4\,5^{43/52}\,113^{1/13}x^{10} + \frac{35}{12}5^{1/4}x^{11} + \sqrt{2}\,5^{3/4}x^{11} + \frac{7}{3}5^{3/4}x^{12}$$

$$7\,5^{17/52}\,113^{1/13}x^4 + \frac{7\,5^{1/4}x^5}{2\sqrt{2}} + 5^{43/52}\,113^{1/13}x^5 + \frac{49}{12}5^{1/4}x^6 + \frac{5^{3/4}x^6}{2\sqrt{2}} + 4\,5^{17/52}\,113^{1/13}x^6 + \sqrt{2}\,5^{1/4}x^7 + \frac{7}{12}5^{3/4}x^7 + \frac{7}{3}5^{1/4}x^8 + 7\,5^{43/52}\,113^{1/13}x^8 + \frac{7\,5^{3/4}x^9}{2\sqrt{2}} + 5\,5^{17/52}\,113^{1/13}x^9 + \frac{5\,5^{1/4}x^{10}}{2\sqrt{2}} + \frac{49}{12}5^{3/4}x^{10} + 4\,5^{43/52}\,113^{1/13}x^{10} + \frac{35}{12}5^{1/4}x^{11} + \sqrt{2}\,5^{3/4}x^{11} + \frac{7}{3}5^{3/4}x^{12}$$

Factor[P]

$$\frac{1}{12}5^{1/4}x^4\left(12\,565^{1/13} + 3\sqrt{2}\,x + 7x^2\right)\left(7 + \sqrt{5}\,x + 4x^2 + 7\sqrt{5}\,x^4 + 5x^5 + 4\sqrt{5}\,x^6\right)$$

(*Factor P(x) to see its roots where the polynomials can be condensed, combined, or easily differeniatable

q = Factor[P]

$$\frac{1}{12}5^{1/4}x^4\left(7 + \sqrt{5}\,x + 4x^2\right)\left(12\,565^{1/13} + 3\sqrt{2}\,x + 7x^2\right)\left(1 + \sqrt{5}\,x^4\right)$$

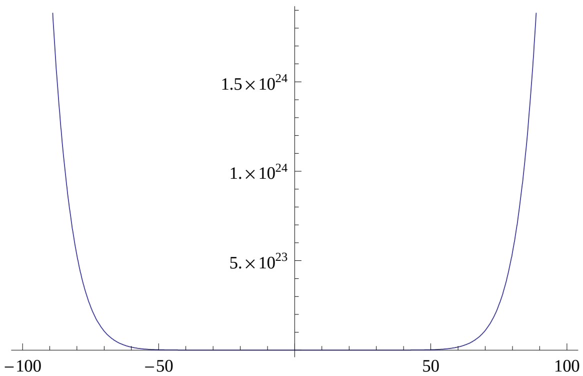
(*Expand the polynomial into its expanded form.

Expand[P]

$$7\,5^{17/52}\,113^{1/13}x^4 + \frac{7\,5^{1/4}x^5}{2\sqrt{2}} + 5^{43/52}\,113^{1/13}x^5 + \frac{49}{12}5^{1/4}x^6 + \frac{5^{3/4}x^6}{2\sqrt{2}} + 4\,5^{17/52}\,113^{1/13}x^6 + \sqrt{2}\,5^{1/4}x^7 + \frac{7}{12}5^{3/4}x^7 + \frac{7}{3}5^{1/4}x^8 + 7\,5^{43/52}\,113^{1/13}x^8 + \frac{7\,5^{3/4}x^9}{2\sqrt{2}} + 5\,5^{17/52}\,113^{1/13}x^9 + \frac{5\,5^{1/4}x^{10}}{2\sqrt{2}} + \frac{49}{12}5^{3/4}x^{10} + 4\,5^{43/52}\,113^{1/13}x^{10} + \frac{35}{12}5^{1/4}x^{11} + \sqrt{2}\,5^{3/4}x^{11} + \frac{7}{3}5^{3/4}x^{12}$$

(*Plot P(x) to make sure it is > 0 for all x.

```
Plot[P, {x, -100, 100}]
```



(*Break down the first polynomial into its SOS form, by completing the square.

A = Expand$\left[\left(\sqrt{5}\ x + 4\, x\verb|^|2 + 7\right)\right]$

$7 + \sqrt{5}\ x + 4\, x^2$

a = 4

4

b = $\sqrt{5}$

$\sqrt{5}$

c = 7

7

$\left(\sqrt{a}\ \left(x + \frac{b}{2\,a}\right)\right)\verb|^|2 + \left(\sqrt{\frac{4\,a\,c - b\verb|^|2}{4\,a}}\right)^2$

$\frac{107}{16} + 4\left(\frac{\sqrt{5}}{8} + x\right)^2$

(*Break down into SOS form, by completing the square,
first considering it's a square of the fourth degree we
take the square root of the both terms then complete the square,
then when we apply the Lemma to the quadractic form of the polynomial. Note that $1^2 = 1^4$.

1_i=Expand$\left[\sqrt{\sqrt{5}\,(x^4)} + \sqrt{1}\right]$

$1+5^{1/4}\ \sqrt{x^4}$

1_{ii}=Expand$\left[\sqrt{1} + 5^{\frac{1}{4}} x^2\right]$

$1+5^{1/4}\ x^2$

Expand$[1_i - 1_{ii}]$

$-5^{1/4}\ x^2+5^{1/4}\ \sqrt{x^4}\ = 0$*)

B = Expand[$\left(\sqrt{5}\ x\hat{}4+1\right)$]

$1+\sqrt{5}\ x^4$

a = $\sqrt{5}$

$\sqrt{5}$

b = 0

0

c = 1

1

$$\left(\left(\sqrt{a}\right)\left(x+\frac{b}{2a}\right)\right)\hat{}2+\left(\sqrt{\frac{4ac-b\hat{}2}{4a}}\right)^2$$

$1+\sqrt{5}\ x^2$

Z = Expand[A (B)]

$7+\sqrt{5}\ x+4x^2+7\sqrt{5}\ x^4+5x^5+4\sqrt{5}\ x^6$

(*Apply the Lemma*)

$$f=\frac{\sqrt{107}}{4}$$

$$\frac{\sqrt{107}}{4}$$

$$g=\sqrt{4}\left(\frac{\sqrt{5}}{8}+x\right)$$

$$2\left(\frac{\sqrt{5}}{8}+x\right)$$

h = 1

k = 5^(1/4) x^2

$5^{1/4}\, x^2$

T = f h - g k

$$\frac{\sqrt{107}}{4} - 2\, 5^{1/4}\, x^2 \left(\frac{\sqrt{5}}{8} + x\right)$$

U = f k + g h

$$\frac{1}{4}\, 5^{1/4} \sqrt{107}\, x^2 + 2 \left(\frac{\sqrt{5}}{8} + x\right)$$

(*Z = T^2 + U^2= A(B)*)

Expand[Z - (T^2 + U^2)]

0

(*Note we group the first set of polynomials which are in SOS form f^2+g^2 and combine them together with the text term which is a square to get something like this: $x^2\left(f^2+g^2\right)$*)

ZZ = T^2 + U^2

$$\left(\frac{1}{4}\, 5^{1/4} \sqrt{107}\, x^2 + 2 \left(\frac{\sqrt{5}}{8} + x\right)\right)^2 + \left(\frac{\sqrt{107}}{4} - 2\, 5^{1/4}\, x^2 \left(\frac{\sqrt{5}}{8} + x\right)\right)^2$$

Expand[ZZ]

$$7 + \sqrt{5}\, x + 4 x^2 + 7 \sqrt{5}\, x^4 + 5 x^5 + 4 \sqrt{5}\, x^6$$

(*CC is a square of the fourth degree $\left(5^{1/8}\, x^2\right)^2$= $\left(5^{1/8}\right)^2\, \left(x^2\right)^2$ so we use the terms without the squares when we apply the lemma *)

CC = Expand[(5^(1/4) x^4)]

$5^{1/4}\, x^4$

Expand[$\sqrt{CC}$]

$5^{1/8} \sqrt{x^4}$

(*Apply the Lemma, where f = T, g =U, h = $\left(5^{1/8}\right)$, k= 0 *)

f = T

$$\frac{\sqrt{107}}{4} - 2\, 5^{1/4}\, x^2 \left(\frac{\sqrt{5}}{8} + x\right)$$

g = U

$$\frac{1}{4} 5^{1/4} \sqrt{107}\ x^2 + 2 \left(\frac{\sqrt{5}}{8} + x\right)$$

h = 5^(1/8) x^2

$$5^{1/8}\ x^2$$

k = 0

0

TT = f h - g k

$$5^{1/8}\ x^2 \left(\frac{\sqrt{107}}{4} - 2\ 5^{1/4}\ x^2 \left(\frac{\sqrt{5}}{8} + x\right)\right)$$

UU = f k + g h

$$5^{1/8}\ x^2 \left(\frac{1}{4} 5^{1/4} \sqrt{107}\ x^2 + 2 \left(\frac{\sqrt{5}}{8} + x\right)\right)$$

(*$x^2(f^2+g^2)$ = $L(x)^2$+ $K(x)^2$ = CC*ZZ = TT^2+UU^2, where x is a square, f^2+g^2 are SOS. *)

Expand[CC (ZZ) - (TT^2 + UU^2)]

0

(*Now the first three sets of polynomials are in SOS form (f^2+g^2) and combine them together with the next term which is a mixture of fractions and radicals.*)

ZZZ = TT^2 + UU^2

$$5^{1/4}\ x^4 \left(\frac{1}{4} 5^{1/4} \sqrt{107}\ x^2 + 2 \left(\frac{\sqrt{5}}{8} + x\right)\right)^2 + 5^{1/4}\ x^4 \left(\frac{\sqrt{107}}{4} - 2\ 5^{1/4}\ x^2 \left(\frac{\sqrt{5}}{8} + x\right)\right)^2$$

Expand[ZZZ]

$$7\ 5^{1/4}\ x^4 + 5^{3/4}\ x^5 + 4\ 5^{1/4}\ x^6 + 7\ 5^{3/4}\ x^8 + 5\ 5^{1/4}\ x^9 + 4\ 5^{3/4}\ x^{10}$$

(*Expand the equation and complete the square*)

EE = Expand[((7/12) x^2 + (1/$\sqrt{8}$) x + (565^(1/13)))]

$$565^{1/13} + \frac{x}{2\sqrt{2}} + \frac{7 x^2}{12}$$

a = 7 / 12

$$\frac{7}{12}$$

b = $\frac{1}{2\sqrt{2}}$

$$\frac{1}{2\sqrt{2}}$$

c = $565^{1/13}$

$$565^{1/13}$$

$$\left(\left(\sqrt{a}\right)\left(x+\frac{b}{2\,a}\right)\right)\text{^}2+\left(\sqrt{\frac{4\,a\,c-b\text{^}2}{4\,a}}\right)^2$$

$$\frac{3}{7}\left(-\frac{1}{8}+\frac{7\,565^{1/13}}{3}\right)+\frac{7}{12}\left(\frac{3}{7\sqrt{2}}+x\right)^2$$

(*Apply the Lemma once again, taking the previous set of SOS and multiplying them by the final polynomial that is now in SOS form *)

f = TT

$$5^{1/8}\,x^2\left(\frac{\sqrt{107}}{4}-2\,5^{1/4}\,x^2\left(\frac{\sqrt{5}}{8}+x\right)\right)$$

g = UU

$$5^{1/8}\,x^2\left(\frac{1}{4}\,5^{1/4}\sqrt{107}\,x^2+2\left(\frac{\sqrt{5}}{8}+x\right)\right)$$

h = $\sqrt{\frac{3}{7}\left(-\frac{1}{8}+\frac{7\,565^{1/13}}{3}\right)}$

$$\sqrt{\frac{3}{7}\left(-\frac{1}{8}+\frac{7\,565^{1/13}}{3}\right)}$$

k = $\sqrt{\frac{7}{12}}\left(\frac{3}{7\sqrt{2}}+x\right)$

$$\frac{1}{2}\sqrt{\frac{7}{3}}\left(\frac{3}{7\sqrt{2}}+x\right)$$

TTT = f h − g k

$$-\frac{5^{1/8}\sqrt{7}\,x^2\left(\frac{3}{7\sqrt{2}}+x\right)\left(\frac{1}{4}5^{1/4}\sqrt{107}\,x^2+2\left(\frac{\sqrt{5}}{8}+x\right)\right)}{2\sqrt{3}}+$$

$$5^{1/8}\sqrt{\frac{3}{7}\left(-\frac{1}{8}+\frac{7\,565^{1/13}}{3}\right)}\;x^2\left(\frac{\sqrt{107}}{4}-2\,5^{1/4}x^2\left(\frac{\sqrt{5}}{8}+x\right)\right)$$

UUU = f k + g h

$$5^{1/8}\sqrt{\frac{3}{7}\left(-\frac{1}{8}+\frac{7\,565^{1/13}}{3}\right)}\;x^2\left(\frac{1}{4}5^{1/4}\sqrt{107}\,x^2+2\left(\frac{\sqrt{5}}{8}+x\right)\right)+$$

$$\frac{5^{1/8}\sqrt{7}\,x^2\left(\frac{3}{7\sqrt{2}}+x\right)\left(\frac{\sqrt{107}}{4}-2\,5^{1/4}x^2\left(\frac{\sqrt{5}}{8}+x\right)\right)}{2\sqrt{3}}$$

Expand[P − (TTT^2 + UUU^2)]

0

(*As stated in Section D a product of two sums of squares is another sums of squares so P = TTT^2+ UUU^2= (f h − g k)^2+ (f k + g h)^2= F^2+*G^2)

Appendix C. Section 2 Examples

(* Example 2.1- Given a Polynomial of degree 3, find a type II representation for P.*)

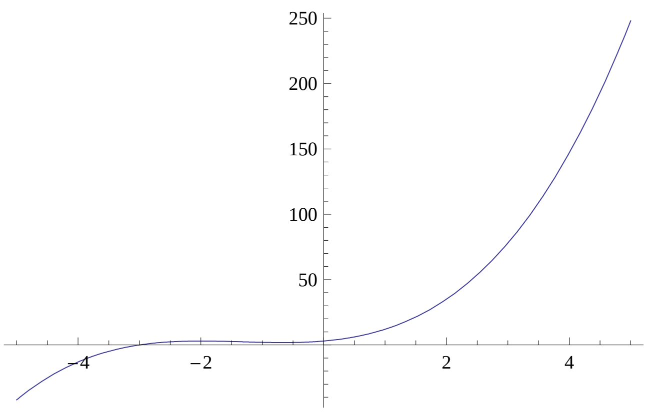
P = 3 + 4 x + 4 x^2 + x^3

$3 + 4x + 4x^2 + x^3$

Factor[P]

$(3 + x)\left(1 + x + x^2\right)$

Plot[P, {x, -5, 5}]



(*Step One: Get the quadratic in SOS form)

Expand[(x^2 + x + 1)]

$1 + x + x^2$

a = 1

1

b = 1

1

c = 1

$$\left(\left(\sqrt{a}\right)\left(x+\frac{b}{2\,a}\right)\right)^2+\left(\sqrt{\frac{4\,a\,c-b\char`^2}{4\,a}}\right)^2$$

$$\frac{3}{4}+\left(\frac{1}{2}+x\right)^2$$

(* The trick is to get the P in form of the lemma $f^2+g^2+x\left(h^2+k^2\right)$)
(* So we provide some manipulation to do so taking the original problem
P = (3+x) $\left(1+x+x^2\right)$ which is the same as $3\left(1+x+x^2\right)+x\left(1+x+x^2\right)$ so,
below I verified directly verified P =
$f^2+g^2+x\left(h^2+k^2\right)$ however it isn't too easy to see so I'll explain in the next few steps.

Expand[P − (3 (1 + x + x^2) + x (1 + x + x^2))]

0

(*In order to to figure out the $x\left(h^2+k^2\right)$ portion we subtract out $\left(f^2+g^2\right)=3\left(1+x+x^2\right)$,
which in turn will give us $x\left(h^2+k^2\right)$

Expand[P − (f^2 + g^2)]

$x+x^2+x^3$

(* Factoring this portion that is left order
it immediately gives us the $x\left(h^2+k^2\right)$ portion of the lemma.

Factor[%]

$x\left(1+x+x^2\right)$

(*Below is the $3\left(1+x+x^2\right)$ in SOS form.)

$$f=\sqrt{\frac{3}{4}}$$

$$\frac{\sqrt{3}}{2}$$

$$g=\sqrt{\left(\frac{1}{2}+x\right)^2}$$

$$\sqrt{\left(\frac{1}{2}+x\right)^2}$$

$$h=\sqrt{3}$$

$$\sqrt{3}$$

k = 0

0

A = f h - g k

$\frac{3}{2}$

B = f k + g h

$\sqrt{3}\sqrt{\left(\frac{1}{2}+x\right)^2}$

(*Below we verify that P = f^2+ g^2 +x (h^2+k^2))

f = A

$\frac{3}{2}$

g = B

$\sqrt{3}\sqrt{\left(\frac{1}{2}+x\right)^2}$

h = $\sqrt{\left(\frac{1}{2}+x\right)^2}$

$\sqrt{\left(\frac{1}{2}+x\right)^2}$

k = $\sqrt{\frac{3}{4}}$

$\frac{\sqrt{3}}{2}$

P_i = Expand[f^2 + g^2 + x (h^2 + k^2)]

$3 + 4\,x + 4\,x^2 + x^3$

Expand[P - P_i]

0

```
(* Example 2.2 Given two polynomials P and Q, which are given as
 the following. Find the Type II x Type II expression of their product. *)
```

P = $(x-3)^2 + x\left(x^2+4\right)$

$(-3+x)^2 + x\left(4+x^2\right)$

Q = $\left(x^2+1\right) + x\left(x^2+8x+17\right)$

$1+x^2+x\left(17+8x+x^2\right)$

```
(*Convert to SOS form b^2-4a c < 0.*)
```

Expand$\left[\left(17+8x+x^2\right)\right]$

$17+8x+x^2$

a = 1

1

b = 8

8

c = 17

17

$$\left(\left(\sqrt{a}\right)\left(x+\frac{b}{2a}\right)\right)^2 + \left(\sqrt{\frac{4ac-b^{\wedge}2}{4a}}\right)^2$$

$1+(4+x)^2$

f = $x-3$

$-3+x$

g = 0

0

h = x

x

k = 2

$\text{Expand}\left[P - \left(f^2 + g^2 + x\left(h^2 + k^2\right)\right)\right]$

0

(*P is of type II*)

$a = x$

x

$b = 1$

1

$c = 1$

1

$d = 4 + x$

$4 + x$

$\text{Expand}\left[Q - \left(a^2 + b^2 + x\left(c^2 + d^2\right)\right)\right]$

0

(*Q is of type II*)

$P_i = \left(\left(f^2 + g^2\right)\left(a^2 + b^2\right)\right) + \left(x^2\left(h^2 + k^2\right)\left(c^2 + d^2\right)\right)$

$(-3 + x)^2\left(1 + x^2\right) + x^2\left(4 + x^2\right)\left(1 + (4 + x)^2\right)$

$Q_i = \left(\left(f^2 + g^2\right)\left(c^2 + d^2\right) + \left(a^2 + b^2\right)\left(h^2 + k^2\right)\right)$

$\left(1 + x^2\right)\left(4 + x^2\right) + (-3 + x)^2\left(1 + (4 + x)^2\right)$

$\text{Expand}\left[\left((Q)(P)\right) - \left(P_i + x\,Q_i\right)\right]$

0

(*Example 2.3- Given a polynomial of degree 7,find a type II representation for P.

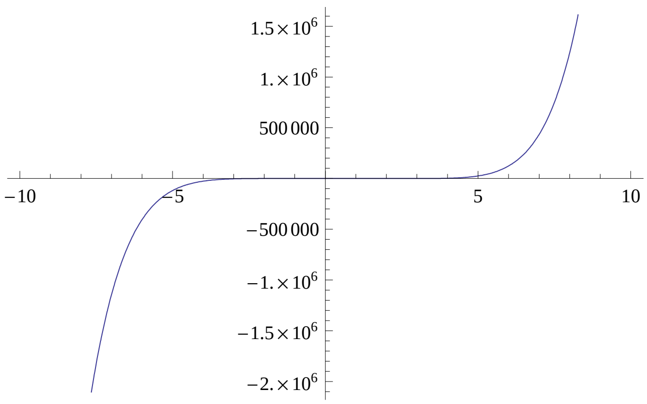
P = 36 + 84 x + 49 x^2 - 3 x^3 - 2 x^4 - 2 x^5 - 3 x^6 + x^7

$36 + 84x + 49x^2 - 3x^3 - 2x^4 - 2x^5 - 3x^6 + x^7$

Factor[P]

$(-3 + x)^2 (1 + x)^3 (4 + x^2)$

Plot[P, {x, -10, 10}]



(*Group the first two polynomials that have even
powers or are SOS form and combine them together into one SOS *)

Z = Expand[(x^2 + 4) (x - 3)^2]

$36 - 24x + 13x^2 - 6x^3 + x^4$

f = (-3 + x)

$-3 + x$

g = 0

0

h = $\sqrt{4}$

2

k = x

x

A = f h - g k

$2(-3 + x)$

```
B = f k + g h
```

$(-3+x)\,x$

```
(*Here you verify that Z = A^2+B^2 *)
Expand[Z - (A^2 + B^2)]
```

0

```
(*Subtract out the portion f^2+ g^2 leaving you with x (h^2+k^2)*)
X = Expand[P - (A^2 + B^2)]
```

$108\,x + 36\,x^2 + 3\,x^3 - 3\,x^4 - 2\,x^5 - 3\,x^6 + x^7$

```
(*Factor the polynominal we see that its a bunch of polynomials
  multiplied by x:        x((4+x^2)(3+3 x+x^2)(-3+x)^2) = x (h^2+k^2)   *)
Factor[X]
```

$(-3+x)^2\,x\,(4+x^2)\,(3+3\,x+x^2)$

```
(*Convert the polynomial with no real roots*)
FullSimplify[3 + 3 x + x^2]
```

$3 + x\,(3+x)$

```
a = 1
```

1

```
b = 3
```

3

```
c = 3
```

3

$$\left(\left(\sqrt{a}\right)\left(x+\frac{b}{2\,a}\right)\right)^2+\left(\sqrt{\frac{4\,a\,c-b\text{^}2}{4\,a}}\right)^2$$

$$\frac{3}{4}+\left(\frac{3}{2}+x\right)^2$$

```
(*Group in the form of h^2+k^2)
f = A
```

$2\,(-3+x)$

g = B

$(-3+x)\,x$

h = $\sqrt{\frac{3}{4}}$

$\frac{\sqrt{3}}{2}$

k = $\left(\frac{3}{2}+x\right)$

$\frac{3}{2}+x$

AA = f h - g k

$\sqrt{3}\,(-3+x) - (-3+x)\,x\left(\frac{3}{2}+x\right)$

BB = f k + g h

$\frac{1}{2}\sqrt{3}\,(-3+x)\,x + 2\,(-3+x)\left(\frac{3}{2}+x\right)$

(*h = AA, k = BB, X = x$\left(AA^2+BB^2\right)$ = x$\left(h^2+k^2\right)$

Expand$\left[X - x\left(AA^2+BB^2\right)\right]$

0

(*Now the final part is to apply the lemma $f^2+g^2+x\left(h^2+k^2\right)$ verfying that P = $f^2+g^2+x\left(h^2+k^2\right)$

f = A

$2\,(-3+x)$

g = B

$(-3+x)\,x$

h = AA

$\sqrt{3}\,(-3+x) - (-3+x)\,x\left(\frac{3}{2}+x\right)$

k = BB

$\frac{1}{2}\sqrt{3}\,(-3+x)\,x + 2\,(-3+x)\left(\frac{3}{2}+x\right.$

```
Pᵢ = Expand[f^2 + g^2 + x (h^2 + k^2)]
```

$36 + 84 x + 49 x^2 - 3 x^3 - 2 x^4 - 2 x^5 - 3 x^6 + x^7$

```
Expand[P - Pᵢ]
```

0

```
(*So  P= f^2+ g^2 +x (h^2+k^2
```

Appendix D. Section 3 Examples

```
(* Example 3.1-The polynomial is of this
  form: A[q1... qk(x-yi)^ni... (x-ym)^nm (x-xi)^si...(x-xp)^sp](x-xj)^si(x-xk)^ni,
Where A is a constant >0. The qi's are quadratics that cannot be factored,
so they have no real roots. The exponents ni are even and the exponents si are odd. Notice
  that in this case as long as the constant A is positive xj≤ 0 and each xk≥ 0. *)
```

$$A\left[q_1 \ldots q_k (x-y_i)^{n_i} \ldots (x-y_m)^{n_m} (x-x_i)^{s_i} \ldots (x-x_p)^{s_p}\right](x-x_j)^{s_i}(x-x_k)^{n_i}$$

P = 32 400 – 7200 x + 303 684 x^2 – 69 048 x^3 + 109 848 x^4 – 38 796 x^5 + 9852 x^6 – 5064 x^7 + 216 x^8 + 108 x^9

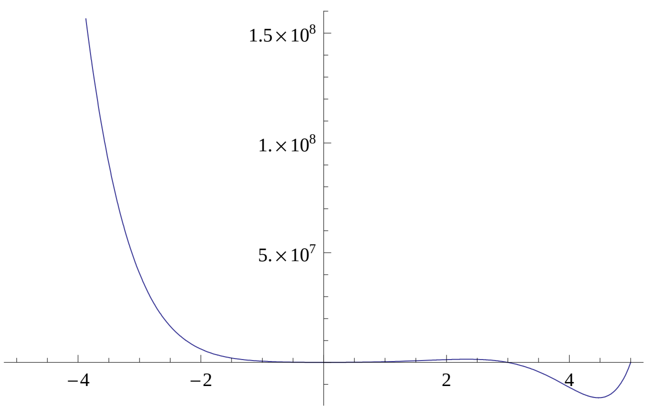
$32\,400 - 7200 x + 303\,684 x^2 - 69\,048 x^3 + 109\,848 x^4 - 38\,796 x^5 + 9852 x^6 - 5064 x^7 + 216 x^8 + 108 x^9$

Factor[P]

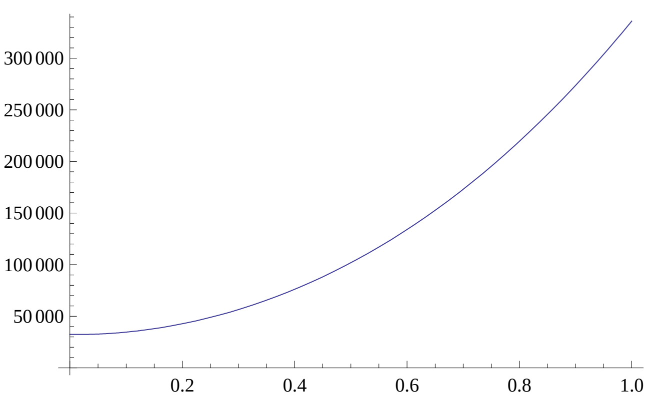
$12 (-5 + x) (-3 + x) (9 + x) (4 + x^2) (5 + x + x^2) (1 + 9 x^2)$

(*Plotting the Polynomial shows that on the interval [0,1] it is strictly positive.*)

Plot[P, {x, -5, 5}]



Plot[P, {x, 0, 1}]



**(*First group the first polynomials into a sum of squares,
here we have polynomials that are already in sum of squares form
so we combine them into one sum of squares and group them together.*)**

```
Z = Expand[(x^2 + 4) (9 x^2 + 1)]
```

$4 + 37 x^2 + 9 x^4$

```
f = x
```

x

```
g = 2
```

2

```
h = √9 x
```

3 x

```
k = 1
```

1

```
A = f h - g k
```

$-2 + 3 x^2$

```
B = f k + g h
```

7 x

```
(*Verifty that Z and A^2 + B^2*)
```

```
Expand[Z - (A^2 + B^2)]
```

0

```
(*Convert into sum of squares form, b^2-4 a c < 0 *)
```

```
Expand[(5 + x + x^2)]
```

$5 + x + x^2$

```
a = 1
```

1

```
b = 1
```

1

```
c = 5
```

5

$$\left(\left(\sqrt{a}\right)\left(x + \frac{b}{2 a}\right)\right)^2 + \left(\sqrt{\frac{4 a c - b\hat{}2}{4 a}}\right)^2$$

$$\frac{19}{4} + \left(\frac{1}{2} + x\right)^2$$

```
Expand[(5 + x + x^2) - %]
```

0

```
(*Apply the Lemma from theorem 1*)

X = Expand[(4 + x^2) (5 + x + x^2) (1 + 9 x^2)]
```

$20 + 4x + 189x^2 + 37x^3 + 82x^4 + 9x^5 + 9x^6$

```
f = A
```

$-2 + 3x^2$

```
g = B
```

$7x$

$$h = \sqrt{\frac{19}{4}}$$

$\frac{\sqrt{19}}{2}$

$$k = \frac{1}{2} + x$$

$\frac{1}{2} + x$

```
AA = f h - g k
```

$-7x\left(\frac{1}{2} + x\right) + \frac{1}{2}\sqrt{19}\left(-2 + 3x^2\right)$

```
BB = f k + g h
```

$\frac{7\sqrt{19}\,x}{2} + \left(\frac{1}{2} + x\right)\left(-2 + 3x^2\right)$

```
(*Verify that AA^2+BB^2=X*)

Expand[X - (AA^2 + BB^2)]
```

0

(*Now apply the Lemma from Theorem 3 that states that $(x-x_o)$, where $x_o \leq 0$*)

```
B0 = x + 9
```

$9 + x$

$\mathbf{x_o = -9}$

-9

$\boldsymbol{\alpha = -x_o}$

9

$\mathbf{B} = \sqrt{\alpha}$

3

$\mathbf{H = Y}$

Y

$\mathbf{a = 1}$

1

$\mathbf{b = 2\,B}$

6

$\mathbf{c = -1}$

-1

$$\left(\frac{-b + \sqrt{b^2 - 4\,a\,c}}{2\,a}\right)$$

$$\frac{1}{2}\left(-6 + 2\sqrt{10}\right)$$

Expand[%]

$-3 + \sqrt{10}$

$$\left(\frac{-b - \sqrt{b^2 - 4\,a\,c}}{2\,a}\right)$$

$$\frac{1}{2}\left(-6 - 2\sqrt{10}\right)$$

Expand[%]

$-3 - \sqrt{10}$

Y = %

$-3 - \sqrt{10}$

H = Y

$-3-\sqrt{10}$

B = $\sqrt{\alpha}$

3

$\mathbf{Y^2 + 6\,Y}$

$6\left(-3-\sqrt{10}\right)+\left(-3-\sqrt{10}\right)^2$

FullSimplify[%]

1

Expand $\left[((\mathbf{B0})) - \left(\left(\mathbf{Y^2 + 6\,Y}\right)\mathbf{x + 9}\right)\right]$

0

Expand[(B0) - ((Y x + B)^2 + x (1 - x) H^2)]

0

(*So x+9= $\left(6\left(-3-\sqrt{10}\right)+\left(-3-\sqrt{10}\right)^2\right)x+\left(\sqrt{9}\right)^2$ ***)**

(*So now combine Type III with Type III since now x +9 is in Type III form.*)

(* B_i **is in the form** $(x-x_o)$**,** C_i **is a constant, and X are all now Type III*)**

CC = 12

12

ZZ = Expand[B0 CC (X)]

$2160+672\,x+20\,460\,x^2+6264\,x^3+9300\,x^4+1956\,x^5+1080\,x^6+108\,x^7$

l = $\sqrt{\mathbf{CC}}$ **AA**

$2\sqrt{3}\left(-7\,x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3\,x^2\right)\right)$

m = $\sqrt{\mathbf{CC}}$ **BB**

$2\sqrt{3}\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3\,x^2\right)\right)$

n = 0

0

o = 0

0

Expand[CC X − (l² + m² + x (1 − x) (n² + o²))]

0

a = Y x + B

$3 + \left(-3 - \sqrt{10}\right) x$

b = 0

0

c = 0

0

d = H

$-3 - \sqrt{10}$

Expand[B0 − ((a² + b²) + x (1 − x) (c² + d²))]

0

RR = ((l² + m²) (a² + b²)) + (x² (1 − x)² (n² + o²) (c² + d²))

$$\left(3 + \left(-3 - \sqrt{10}\right) x\right)^2 \left(12 \left(-7 x \left(\frac{1}{2} + x\right) + \frac{1}{2} \sqrt{19} \left(-2 + 3 x^2\right)\right)^2 + 12 \left(\frac{7 \sqrt{19} x}{2} + \left(\frac{1}{2} + x\right) \left(-2 + 3 x^2\right)\right)^2\right)$$

Factor[RR]

$$12 \left(-3 + 3 x + \sqrt{10} x\right)^2 \left(4 + x^2\right) \left(5 + x + x^2\right) \left(1 + 9 x^2\right)$$

QQ = ((l² + m²) (c² + d²) + (a² + b²) (n² + o²))

$$\left(-3 - \sqrt{10}\right)^2 \left(12 \left(-7 x \left(\frac{1}{2} + x\right) + \frac{1}{2} \sqrt{19} \left(-2 + 3 x^2\right)\right)^2 + 12 \left(\frac{7 \sqrt{19} x}{2} + \left(\frac{1}{2} + x\right) \left(-2 + 3 x^2\right)\right)^2\right)$$

Factor[QQ]

$$12 \left(3 + \sqrt{10}\right)^2 \left(4 + x^2\right) \left(5 + x + x^2\right) \left(1 + 9 x^2\right)$$

(*Find the type I representation of RR and QQ.*)

f = AA

$$-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)$$

g = BB

$$\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)$$

h = $\sqrt{CC}\left(-3+3x+\sqrt{10}\,x\right)$

$$2\sqrt{3}\left(-3+3x+\sqrt{10}\,x\right)$$

k = 0

0

A4 = f k - g h

$$-2\sqrt{3}\left(-3+3x+\sqrt{10}\,x\right)\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)\right)$$

B4 = f h + g k

$$2\sqrt{3}\left(-3+3x+\sqrt{10}\,x\right)\left(-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)\right)$$

Expand[RR - $\left(A4^2+B4^2\right)$]

0

f = AA

$$-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)$$

g = BB

$$\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)$$

h = $\sqrt{CC}\left(3+\sqrt{10}\right)$

$$2\sqrt{3}\left(3+\sqrt{10}\right)$$

k = 0

0

F0 = f k - g h

$$-2\sqrt{3}\left(3+\sqrt{10}\right)\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)\right)$$

G0 = f h + g k

$$2\sqrt{3}\left(3+\sqrt{10}\right)\left(-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)\right)$$

Expand[QQ - (F0^2 + G0^2)]

0

Expand[ZZ - (A4^2 + B4^2 + x (1 - x) (F0^2 + G0^2))]

0

A0 = Expand[(-5 + x) (-3 + x)]

$15-8x+x^2$

h = A0 /. x → 1/x

$$15+\frac{1}{x^2}-\frac{8}{x}$$

g = x^2 h

$$\left(15+\frac{1}{x^2}-\frac{8}{x}\right)x^2$$

q = g /. x → s + 1

$$(1+s)^2\left(15+\frac{1}{(1+s)^2}-\frac{8}{1+s}\right)$$

Factor[q]

$(2+3s)(4+5s)$

Expand[%]

$8+22s+15s^2$

Factor[q - (15 ((-(-2/3) + s) (-(-4/5) + s)))]

f $= \sqrt{15}$

$\sqrt{15}$

g = 0

0

h = 0

0

k = 0

0

$x_o = \frac{-2}{3}$

$-\frac{2}{3}$

$((s^2 (h^2 + k^2) - x_o (f^2 + g^2)) + s ((f^2 + g^2) - x_o (h^2 + k^2)))$

$10 + 15\,s$

f $= \sqrt{10}$

$\sqrt{10}$

g = 0

0

h $= \sqrt{15}$

$\sqrt{15}$

k = 0

0

$x_o = \frac{-4}{5}$

$-\frac{4}{5}$

$((s^2 (h^2 + k^2) - x_o (f^2 + g^2)) + s ((f^2 + g^2) - x_o (h^2 + k^2)))$

$8 + 22\,s + 15$

R = $\sqrt{15\,s^2}$

$\sqrt{15}\;\sqrt{s^2}$

S = $\sqrt{8}$

$2\sqrt{2}$

U = $\sqrt{22}$

$\sqrt{22}$

V = 0

0

q = Expand[$R^2 + S^2 + s\,(U^2 + V^2)$]

$8 + 22\,s + 15\,s^2$

q = q /. s → (x − 1)

$8 + 22\,(-1 + x) + 15\,(-1 + x)^2$

P0 = $w^2 \left(q \text{ /. } x \to \left(\frac{1}{w}\right)\right)$

$\left(8 + 22\left(-1 + \frac{1}{w}\right) + 15\left(-1 + \frac{1}{w}\right)^2\right) w^2$

P0 − A0 /. x → w

$-15 + 8\,w - w^2 + \left(8 + 22\left(-1 + \frac{1}{w}\right) + 15\left(-1 + \frac{1}{w}\right)^2\right) w^2$

Expand[%]

0

(*So the final form of the A_i in Type III form is $\left(\sqrt{15}^{\,2}(1-x)^2 + \left(x\sqrt{8}\right)^2 + x(1-x)\left(\sqrt{22}\right)^2\right.$*)

l = $\sqrt{15}\,(1 - x)$

$\sqrt{15}\,(1 - x)$

m = $x\sqrt{8}$

$2\sqrt{2}\,x$

n = $\sqrt{22}$

$\sqrt{22}$

o = 0

0

Expand[A0 – (l^2 + m^2 + x (1 – x) (n^2 + o^2))]

0

(*Apply the formula once more Type III * Type III = Type III*)

a = A4

$$-2\sqrt{3}\left(-3+3x+\sqrt{10}\,x\right)\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)\right)$$

b = B4

$$2\sqrt{3}\left(-3+3x+\sqrt{10}\,x\right)\left(-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)\right)$$

c = F0

$$-2\sqrt{3}\left(3+\sqrt{10}\right)\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)\right)$$

d = G0

$$2\sqrt{3}\left(3+\sqrt{10}\right)\left(-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)\right)$$

RRR = ((l^2 + m^2) (a^2 + b^2)) + (x^2 $(1 – x)^2$ (n^2 + o^2) (c^2 + d^2))

$$22(1-x)^2x^2\left(12\left(3+\sqrt{10}\right)^2\left(-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)\right)^2+12\left(3+\sqrt{10}\right)^2\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)\right)^2\right)+$$
$$\left(15(1-x)^2+8x^2\right)\left(12\left(-3+3x+\sqrt{10}\,x\right)^2\left(-7x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3x^2\right)\right)^2+\right.$$
$$\left.12\left(-3+3x+\sqrt{10}\,x\right)^2\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3x^2\right)\right)^2\right)$$

Factor[RRR]

$$12\left(4+x^2\right)\left(5+x+x^2\right)\left(1+9x^2\right)$$
$$\left(135-540x-90\sqrt{10}\,x+1450x^2+402\sqrt{10}\,x^2-1820x^3-582\sqrt{10}\,x^3+855x^4+270\sqrt{10}\,x^4\right)$$

QQQ = $((l^2+m^2)(c^2+d^2)+(a^2+b^2)(n^2+o^2))$

$$\left(15\,(1-x)^2+8\,x^2\right)\left(12\left(3+\sqrt{10}\right)^2\left(-7\,x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3\,x^2\right)\right)^2+12\left(3+\sqrt{10}\right)^2\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3\,x^2\right)\right)^2\right)+$$
$$22\left(12\left(-3+3\,x+\sqrt{10}\,x\right)^2\left(-7\,x\left(\frac{1}{2}+x\right)+\frac{1}{2}\sqrt{19}\left(-2+3\,x^2\right)\right)^2+12\left(-3+3\,x+\sqrt{10}\,x\right)^2\left(\frac{7\sqrt{19}\,x}{2}+\left(\frac{1}{2}+x\right)\left(-2+3\,x^2\right)\right)^2\right)$$

Factor[QQQ]

$$36\left(4+x^2\right)\left(5+x+x^2\right)\left(1+9\,x^2\right)\left(161+30\sqrt{10}-322\,x-104\sqrt{10}\,x+285\,x^2+90\sqrt{10}\,x^2\right)$$

Expand[P - ((RRR + QQQ (1 - x) x))]

0

```
(*P has a type III expression;
RRR and QQQ are not in SOS form but positive everywhere for all x,
further implementation is needed to derive those
 expressions which will not be done due to time restrictions.*)
```

```
(* Example 3.2-The polynomial is of this
  form: A[q1... qk(x-yi)^ni... (x-ym)^nm (x-xi)^si...(x-xp)^sp](x-xj)^ni(x-xk)^si,
Where A is a constant <0. The qi's are quadratics that cannot be factored,
so they have no real roots. The exponents ni are even and the exponents si
  are odd. Notice that in this case as long as the constant A is positive xj≤
 0 and each xk≥ 0. Note that some of these factors are missing. *)
```

A = -1

-1

P = -22 + 13 x - x^2

$-22 + 13\,x - x^2$

Factor[P]

$-(-11 + x)\,(-2 + x)$

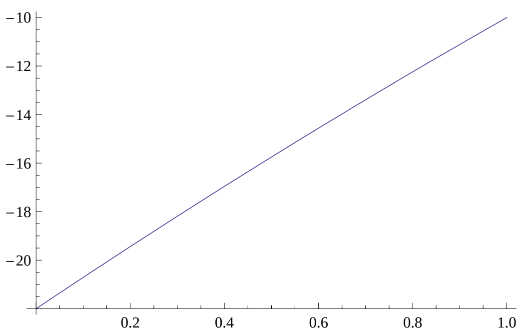
Expand[P - ((-(x - 2)) (-(x - 11)) (-1))]

0

Factor[P]

$-(-11 + x)\,(-2 + x)$

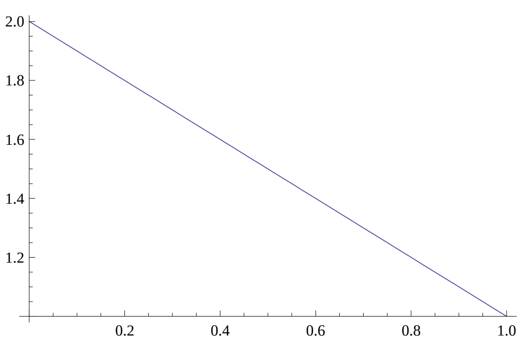
Plot[(P), {x, 0, 1}]



A1 = -(-2 + x)

$2 - x$

Plot[A1, {x, 0, 1}]



$x_o = 2$

2

$B = \sqrt{x_o}$

$\sqrt{2}$

A =.

H = A

A

(*Apply Lemma 3.6 equation (3.5)*)

$0 == 2 A B + A^2 + 1$

$0 == 1 + 2\sqrt{2}\, A + A^2$

a = 1

1

b = 2 B

$2\sqrt{2}$

c = 1

1

$$\left(\frac{-b + \sqrt{b^2 - 4 a c}}{2 a}\right)$$

$$\frac{1}{2}\left(2 - 2\sqrt{2}\right)$$

Expand[%]

$1 - \sqrt{2}$

$$\left(\frac{-b - \sqrt{b^2 - 4 a c}}{2 a}\right)$$

$$\frac{1}{2}\left(-2 - 2\sqrt{2}\right)$$

Expand[%]

$-1 - \sqrt{2}$

$$A = \left(\frac{-b + \sqrt{b^2 - 4 a c}}{2 a}\right)$$

$$\frac{1}{2}\left(2 - 2\sqrt{2}\right)$$

```
H = A
```

$\frac{1}{2}\left(2-2\sqrt{2}\right)$

```
B = Sqrt[xo]
```

$\sqrt{2}$

```
(*Verify the expression *)
```

$A^2 + 2\,B\,A$

$\sqrt{2}\left(2-2\sqrt{2}\right)+\frac{1}{4}\left(2-2\sqrt{2}\right)^2$

```
FullSimplify[%]
```

-1

$\text{Expand}\left[(-(x-x_o))-\left(\left(A^2+2\,B\,A\right)x+x_o\right)\right]$

0

$\text{Expand}[-(x-x_o)-((A\,x+B)\text{^}2+x\,(1-x)\,H\text{^}2)]$

0

```
Expand[A1 - (((A x + B)^2 + x (1 - x) H^2))]
```

0

```
A0 = A x + B
```

$\sqrt{2}+\frac{1}{2}\left(2-2\sqrt{2}\right)x$

```
A01 = H
```

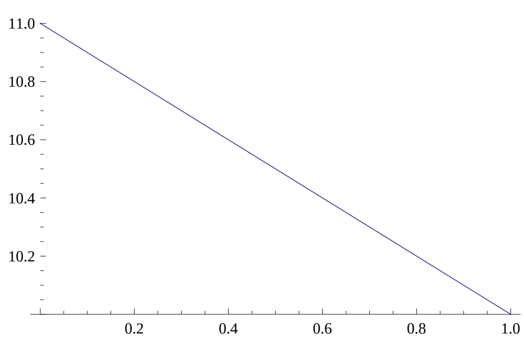
$\frac{1}{2}\left(2-2\sqrt{2}\right)$

```
(**********************************************************************)
```

```
B1 = -(-11 + x)
```

$11 - x$

```
Plot[B1, {x, 0, 1}]
```



$x_o = 11$

11

$B = \sqrt{x_o}$

$\sqrt{11}$

A =.

H = A

A

```
(*Apply Lemma 3.6 equation (3.5)*)
```

$0 == 2\,A\,B + A^2 + 1$

$0 == 1 + 2\sqrt{11}\,A + A^2$

a = 1

1

b = 2 B

$2\sqrt{11}$

c = 1

1

$$\left(\frac{-b + \sqrt{b^2 - 4\,a\,c}}{2\,a}\right)$$

$$\frac{1}{2}\left(2\sqrt{10} - 2\sqrt{11}\right)$$

Expand[%]

$\sqrt{10} - \sqrt{11}$

$$\left(\frac{-b - \sqrt{b^2 - 4\,a\,c}}{2\,a}\right)$$

$$\frac{1}{2}\left(-2\sqrt{10} - 2\sqrt{11}\right)$$

Expand[%]

$-\sqrt{10} - \sqrt{11}$

$$A = \left(\frac{-b + \sqrt{b^2 - 4\,a\,c}}{2\,a}\right)$$

$$\frac{1}{2}\left(2\sqrt{10} - 2\sqrt{11}\right)$$

```
H = A
```

$$\frac{1}{2}\left(2\sqrt{10}-2\sqrt{11}\right)$$

```
B = Sqrt[xo]
```

$$\sqrt{11}$$

```
(*Verify the expression is equal to 1 *)
```

```
A^2 + 2 B A
```

$$\sqrt{11}\left(2\sqrt{10}-2\sqrt{11}\right)+\frac{1}{4}\left(2\sqrt{10}-2\sqrt{11}\right)^2$$

```
FullSimplify[%]
```

$$-1$$

```
Expand[(-(x - xo)) - ((A^2 + 2 B A) x + xo)]
```

0

```
Expand[-(x - xo) - ((A x + B)^2 + x (1 - x) H^2)]
```

0

```
Expand[B1 - (((A x + B)^2 + x (1 - x) H^2))]
```

0

```
B0 = A x + B
```

$$\sqrt{11}+\frac{1}{2}\left(2\sqrt{10}-2\sqrt{11}\right)x$$

```
B01 = H
```

$$\frac{1}{2}\left(2\sqrt{10}-2\sqrt{11}\right)$$

```
(*Apply Multiplication Lemma For Type III polynomials*)
```

```
a = A0
```

$$\sqrt{2}+\frac{1}{2}\left(2-2\sqrt{2}\right)x$$

```
b = 0
```

0

```
c = A01
```

$$\frac{1}{2}\left(2-2\sqrt{2}\right)$$

d = 0

0

f = B0

$$\sqrt{11} + \frac{1}{2} \left(2 \sqrt{10} - 2 \sqrt{11}\right) x$$

g = 0

0

h = 0

0

k = B01

$$\frac{1}{2} \left(2 \sqrt{10} - 2 \sqrt{11}\right)$$

R = ((f^2 + g^2) (a^2 + b^2)) + (x^2 (1 − x)^2 (h^2 + k^2) (c^2 + d^2))

$$\frac{1}{16} \left(2 - 2 \sqrt{2}\right)^2 \left(2 \sqrt{10} - 2 \sqrt{11}\right)^2 (1 - x)^2 x^2 + \left(\sqrt{2} + \frac{1}{2} \left(2 - 2 \sqrt{2}\right) x\right)^2 \left(\sqrt{11} + \frac{1}{2} \left(2 \sqrt{10} - 2 \sqrt{11}\right) x\right)^2$$

Factor[R]

$$-2 \left(-11 + 44 x - 11 \sqrt{2} x - 2 \sqrt{110} x - 113 x^2 + 54 \sqrt{2} x^2 - 8 \sqrt{55} x^2 + 9 \sqrt{110} x^2 + 138 x^3 - 85 \sqrt{2} x^3 + 16 \sqrt{55} x^3 - 13 \sqrt{110} x^3 - 63 x^4 + 42 \sqrt{2} x^4 - 8 \sqrt{55} x^4 + 6 \sqrt{110} x^4\right)$$

Q = ((f^2 + g^2) (c^2 + d^2) + (a^2 + b^2) (h^2 + k^2))

$$\frac{1}{4} \left(2 \sqrt{10} - 2 \sqrt{11}\right)^2 \left(\sqrt{2} + \frac{1}{2} \left(2 - 2 \sqrt{2}\right) x\right)^2 + \frac{1}{4} \left(2 - 2 \sqrt{2}\right)^2 \left(\sqrt{11} + \frac{1}{2} \left(2 \sqrt{10} - 2 \sqrt{11}\right) x\right)^2$$

Factor[Q]

$$75 - 22 \sqrt{2} - 4 \sqrt{110} - 150 x + 86 \sqrt{2} x - 16 \sqrt{55} x + 14 \sqrt{110} x + 126 x^2 - 84 \sqrt{2} x^2 + 16 \sqrt{55} x^2 - 12 \sqrt{110} x^2$$

**(*Mathematica doesn't factor Q and R,
however they are positive for all x. The polynomials Q and R have type I representation, due to time restrictions this will not be explored.*)**

(*Remember to include the last − 1 at the end into the equation to balance since theres one − 1 in P and now one −1 in the Type III representation of P =((R+Q x(1−x))(−1))*)

Expand[P − ((R + Q x (1 − x)) (−1))]

0

```
(* Example 3.3-Corollary 3.10 take p(u)= u^3,
positive [2,4] and give a Type III representation.*)
```

```
P = u^3
```

u^3

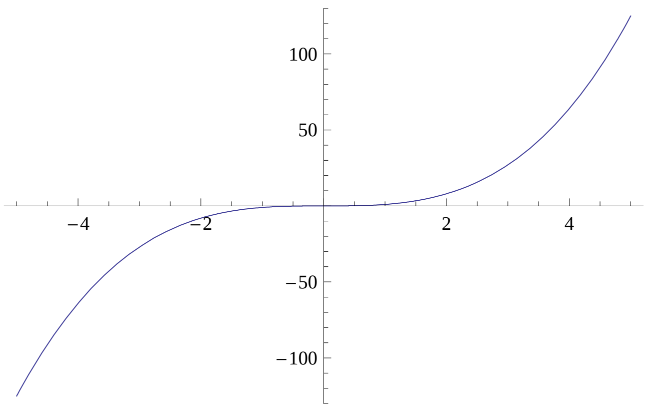
```
a = 2
```

2

```
b = 4
```

4

```
Plot[P, {u, -5, 5}]
```




```
(*Apply the formula from Corollary 3.10*)
```

```
u = a + x (b - a)
```

$2 + 2x$

```
(*Factoring the polynomial shows a odd degree, we use theorem 3.1 case 2.*)
```

```
Factor[Q]
```

$8 (1 + x)^3$

```
Expand[Q - (8 (1 + x)^2 (1 + x))]
```

0

```
(*Next we get the type III representation of the polynomial Q*)
```

```
xo = -1
```

-1

$\alpha = -x_o$

1

$B = \sqrt{\alpha}$

1

A =.

H = A

A

(*Use equation (3.4) from Lemma 3.6*)

$0 == 2\,A\,B + A^2 - 1$

$0 == -1 + 2\,A + A^2$

a = 1

1

b = 2 B

2

c = -1

-1

(*Apply the quadratic formula.*)

$$\left(\frac{-b + \sqrt{b^2 - 4\,a\,c}}{2\,a}\right)$$

$$\frac{1}{2}\left(-2 + 2\sqrt{2}\right)$$

Expand[%]

$-1 + \sqrt{2}$

$$\left(\frac{-b - \sqrt{b^2 - 4\,a\,c}}{2\,a}\right)$$

$$\frac{1}{2}\left(-2 - 2\sqrt{2}\right)$$

```
Expand[%]
```

$-1-\sqrt{2}$

```
A = (-b + Sqrt[b^2 - 4 a c])/(2 a)
```

$\frac{1}{2}\left(-2+2\sqrt{2}\right)$

```
H = A
```

$\frac{1}{2}\left(-2+2\sqrt{2}\right)$

```
B = Sqrt[α]
```

1

```
(*Verify that the expression is equal to 1*)
A^2 + 2 B A
```

$-2+2\sqrt{2}+\frac{1}{4}\left(-2+2\sqrt{2}\right)^2$

```
FullSimplify[%]
```

1

```
(*A^2+2B A =1 so the above expression should be zero.*)
Expand[((x - x_o)) - ((A^2 + 2 B A) x - x_o)]
```

0

```
(*Type III representation below.*)
Expand[(x - x_o) - ((A x + B)^2 + x (1 - x) H^2)]
```

0

```
(*It is concluded that original linear term x+1 has a type III representation.*)
f = 0
```

0

```
g = A x + B
```

$1+\frac{1}{2}\left(-2+2\sqrt{2}\right)x$

```
h = H
```

$$\frac{1}{2}\left(-2+2\sqrt{2}\right)$$

```
k = 0
```

0

```
Expand[(x + 1) - (f^2 + g^2 + x (1 - x) (h^2 + k^2))]
```

0

```
l = Sqrt[8] (1 + x)
```

$$2\sqrt{2}\,(1+x)$$

```
m = 0
```

0

```
n = 0
```

0

```
o = 0
```

0

```
(*Now apply Lemma 3.4 regarding multiplication of Type III polynomials.*)
F = ((f^2 + g^2) (l^2 + m^2)) + (x^2 (1 - x)^2 (h^2 + k^2) (n^2 + o^2))
```

$$8\,(1+x)^2\left(1+\frac{1}{2}\left(-2+2\sqrt{2}\right)x\right)^2$$

```
Factor[F]
```

$$8\,(1+x)^2\left(1-x+\sqrt{2}\,x\right)^2$$

```
G = (f^2 + g^2) (n^2 + o^2) + (l^2 + m^2) (h^2 + k^2)
```

$$2\left(-2+2\sqrt{2}\right)^2(1+x)^2$$

```
Expand[P - (F + G x (1 - x))]
```

0

```
(*Use Lemma 1.7 to get two sums of squares. *)
f = Sqrt[8] (1 + x)
```

$$2\sqrt{2}\,(1+$$

```
g = 0
0
h = (1 - x + Sqrt[2] x)
1 - x + Sqrt[2] x
k = 0
0
X = f k - g h
0
W = f h + g k
2 Sqrt[2] (1 + x) (1 - x + Sqrt[2] x)
Expand[F - (X^2 + W^2)]
0
(*****************************************************)
f = Sqrt[2] (-2 + 2 Sqrt[2])
Sqrt[2] (-2 + 2 Sqrt[2])
g = 0
0
h = (1 + x)
1 + x
k = 0
0
V = f k - g h
0
U = f h + g k
Sqrt[2] (-2 + 2 Sqrt[2]) (1 + x)
```

```
Expand[U]
```

$4-2\sqrt{2}+4x-2\sqrt{2}\,x$

```
Expand[G - (V^2 + U^2)]
```

0

```
Expand[Q - (X^2 + W^2 + x (1 - x) (V^2 + U^2))]
```

0

```
(*Now with the type III expression for Q, on [0,1],
we continue using corollary 3.10 to find the type III representation on[2,4]*)
```

```
a = 2
```

2

```
b = 4
```

4

```
u =.
```

$x = \frac{u-a}{b-a}$

$\frac{1}{2}(-2+u)$

$ZZ = X^2 + W^2 + x\,(1-x)\,(V^2+U^2)$

$$8\left(1+\frac{1}{2}(-2+u)\right)^2\left(1+\frac{2-u}{2}+\frac{-2+u}{\sqrt{2}}\right)^2+\left(-2+2\sqrt{2}\right)^2\left(1+\frac{2-u}{2}\right)\left(1+\frac{1}{2}(-2+u)\right)^2(-2+u)$$

$$RR = \left(\sqrt{8}\left(1+\frac{1}{2}(-2+u)\right)\left(1+\frac{2-u}{2}+\frac{-2+u}{\sqrt{2}}\right)\right)$$

$$2\sqrt{2}\left(1+\frac{1}{2}(-2+u)\right)\left(1+\frac{2-u}{2}+\frac{-2+u}{\sqrt{2}}\right)$$

```
SS = X
```

0

```
TT = V
```

(*The $\frac{1}{(b-a)}$ gets absorbed into each polynomial TT and UU*)

```
UU = 1/(b - a) U
```

$$\frac{\left(-2+2\sqrt{2}\right)\left(1+\frac{1}{2}(-2+u)\right)}{\sqrt{2}}$$

(*Prove that ZZ = Q = P is equal to the expression below*)

```
ExpandAll[ZZ - (RR^2 + SS^2 + (u - a) (b - u) (TT^2 + UU^2))]
```

0

```
ExpandAll[P - (RR^2 + SS^2 + (u - a) (b - u) (TT^2 + UU^2))]
```

0

(*So the orginal P = u^3, has a type III representation on [2,4].*)

Appendix E. Section 4 Examples

```
(*Example 4.1- sum of squares in 2 variables, show that p(x,y) is a sum of squares*)
```

p = 7 + 2 x + 14 x^2 + 14 y + 28 x y + 49 y^2

$7 + 2x + 14x^2 + 14y + 28xy + 49y^2$

v = {1, x, y}

{1, x, y}

P = {{a, b, c}, {b, d, e}, {c, e, f}}

{{a, b, c}, {b, d, e}, {c, e, f}}

MatrixForm[P]

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Together[p - v.P.v]

$7 - a + 2x - 2bx + 14x^2 - dx^2 + 14y - 2cy + 28xy - 2exy + 49y^2 - fy^2$

a = 7

7

b = 1

1

c = 7

7

d = 14

14

e = 14

14

f = 49

49

p - v.P.v

$x + 14x^2 + 7y + 28xy + 49y^2 - x(1 + 14x + 14y) - y(7 + 14x + 49y)$

Factor[%]

P

{{7, 1, 7}, {1, 14, 14}, {7, 14, 49}}

MatrixForm[%]

$$\begin{pmatrix} 7 & 1 & 7 \\ 1 & 14 & 14 \\ 7 & 14 & 49 \end{pmatrix}$$

Det[P]

2891

CholeskyDecomposition[P]

$$\left\{\left\{\sqrt{7}, \frac{1}{\sqrt{7}}, \sqrt{7}\right\}, \left\{0, \sqrt{\frac{97}{7}}, 13\sqrt{\frac{7}{97}}\right\}, \left\{0, 0, 7\sqrt{\frac{59}{97}}\right\}\right\}$$

A = %

$$\left\{\left\{\sqrt{7}, \frac{1}{\sqrt{7}}, \sqrt{7}\right\}, \left\{0, \sqrt{\frac{97}{7}}, 13\sqrt{\frac{7}{97}}\right\}, \left\{0, 0, 7\sqrt{\frac{59}{97}}\right\}\right\}$$

MatrixForm[A]

$$\begin{pmatrix} \sqrt{7} & \frac{1}{\sqrt{7}} & \sqrt{7} \\ 0 & \sqrt{\frac{97}{7}} & 13\sqrt{\frac{7}{97}} \\ 0 & 0 & 7\sqrt{\frac{59}{97}} \end{pmatrix}$$

tA = Transpose[A]

$$\left\{\left\{\sqrt{7}, 0, 0\right\}, \left\{\frac{1}{\sqrt{7}}, \sqrt{\frac{97}{7}}, 0\right\}, \left\{\sqrt{7}, 13\sqrt{\frac{7}{97}}, 7\sqrt{\frac{59}{97}}\right\}\right\}$$

MatrixForm[tA]

$$\begin{pmatrix} \sqrt{7} & 0 & 0 \\ \frac{1}{\sqrt{7}} & \sqrt{\frac{97}{7}} & 0 \\ \sqrt{7} & 13\sqrt{\frac{7}{97}} & 7\sqrt{\frac{59}{97}} \end{pmatrix}$$

```
(*Double check that P = Transpose[A].A: *)

P - tA.A

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

(* to express p as a sum of squares, we now use (v.tA).(A.v) *)

(v.tA).(A.v)
```

$$\frac{2891\,y^2}{97} + \left(\sqrt{\frac{97}{7}}\,x + 13\sqrt{\frac{7}{97}}\,y\right)^2 + \left(\sqrt{7} + \frac{x}{\sqrt{7}} + \sqrt{7}\,y\right)^2$$

```
(*this is a sum of squares expression;
we double check that it equals p*)

q = %
```

$$\frac{2891\,y^2}{97} + \left(\sqrt{\frac{97}{7}}\,x + 13\sqrt{\frac{7}{97}}\,y\right)^2 + \left(\sqrt{7} + \frac{x}{\sqrt{7}} + \sqrt{7}\,y\right)^2$$

```
Factor[p - q]

0

(*Another way to check that a real
 symmetric matrix is positive is to
 verify that all of its eigenvalues are nonnegative: *)

Eigenvalues[P]
```

$\{\text{Root}[-2891 + 881\,\#1 - 70\,\#1^2 + \#1^3\,\&,\ 3],$
$\text{Root}[-2891 + 881\,\#1 - 70\,\#1^2 + \#1^3\,\&,\ 2],\ \text{Root}[-2891 + 881\,\#1 - 70\,\#1^2 + \#1^3\,\&,\ 1]\}$

```
N[%]

{54.9159, 9.60073, 5.48334}

(*This means that there are
   vectors a,b,c such that P.a = 54.91592601638876` a,
  P.b = 9.600729026932235` b, Pc = 5.483344956679007`} c. Further,
any vector w can be expressed
  as w = ra + sb + tc,
so Pw = 54.91592601638876`(ra) + 9.600729026932235`(sb) + 5.483344956679007`(tc).
    This shows that P is like
   stretching by positive multipliers
   in three directions
```

```
(* Example 4.2- Consider a polynomial of two variables x,
y of degree 4. We wish to express P(x,y) as a sum of squares.*)
```

```
P = 6 + 8 x + 16 x^2 + 16 x^3 + 16 x^4 + 8 y + 28 x y + 24 x^2 y + 10 y^2 + 9 x^2 y^2
```

$6 + 8 x + 16 x^2 + 16 x^3 + 16 x^4 + 8 y + 28 x y + 24 x^2 y + 10 y^2 + 9 x^2 y^2$

```
(*Notice mathematica doesn't know how to factor this polynomial.*)

Factor[P]
```

$6 + 8 x + 16 x^2 + 16 x^3 + 16 x^4 + 8 y + 28 x y + 24 x^2 y + 10 y^2 + 9 x^2 y^2$

```
(*The vector v, must count up to half of the total degree.*)

v = {1, x, y, x^2, x y, y^2}
```

$\{1, x, y, x^2, x y, y^2\}$

```
(*The table is created to generated to compare to coefficients of variables*)

M = Table[g[i, j], {i, 6}, {j, 6}]
```

```
{{g[1, 1], g[1, 2], g[1, 3], g[1, 4], g[1, 5], g[1, 6]},
 {g[2, 1], g[2, 2], g[2, 3], g[2, 4], g[2, 5], g[2, 6]},
 {g[3, 1], g[3, 2], g[3, 3], g[3, 4], g[3, 5], g[3, 6]},
 {g[4, 1], g[4, 2], g[4, 3], g[4, 4], g[4, 5], g[4, 6]},
 {g[5, 1], g[5, 2], g[5, 3], g[5, 4], g[5, 5], g[5, 6]},
 {g[6, 1], g[6, 2], g[6, 3], g[6, 4], g[6, 5], g[6, 6]}}
```

```
(*The code below makes the matrix M its own transpose M = M^T*)

For[i = 1, i ≤ 6, i = i + 1,
 For[j = 1, j ≤ i, j = j + 1,
  g[i, j] = g[j, i]]]

MatrixForm[M]
```

$$\begin{pmatrix} g[1,1] & g[1,2] & g[1,3] & g[1,4] & g[1,5] & g[1,6] \\ g[1,2] & g[2,2] & g[2,3] & g[2,4] & g[2,5] & g[2,6] \\ g[1,3] & g[2,3] & g[3,3] & g[3,4] & g[3,5] & g[3,6] \\ g[1,4] & g[2,4] & g[3,4] & g[4,4] & g[4,5] & g[4,6] \\ g[1,5] & g[2,5] & g[3,5] & g[4,5] & g[5,5] & g[5,6] \\ g[1,6] & g[2,6] & g[3,6] & g[4,6] & g[5,6] & g[6,6] \end{pmatrix}$$

```
(*Solve p = v.M.v for M*)

Factor[v.M.v]
```

$g[1, 1] + 2 x g[1, 2] + 2 y g[1, 3] + 2 x^2 g[1, 4] + 2 x y g[1, 5] + 2 y^2 g[1, 6] + x^2 g[2, 2] + 2 x y g[2, 3] + 2 x^3 g[2, 4] + 2 x^2 y g[2, 5] + 2 x y^2 g[2, 6] + y^2 g[3, 3] + 2 x^2 y g[3, 4] + 2 x y^2 g[3, 5] + 2 y^3 g[3, 6] + x^4 g[4, 4] + 2 x^3 y g[4, 5] + 2 x^2 y^2 g[4, 6] + x^2 y^2 g[5, 5] + 2 x y^3 g[5, 6] + y^4 g[6, 6]$

```
Expand[P - %]
```

$6 + 8 x + 16 x^2 + 16 x^3 + 16 x^4 + 8 y + 28 x y + 24 x^2 y + 10 y^2 + 9 x^2 y^2 - g[1, 1] - 2 x g[1, 2] - 2 y g[1, 3] - 2 x^2 g[1, 4] - 2 x y g[1, 5] - 2 y^2 g[1, 6] - x^2 g[2, 2] - 2 x y g[2, 3] - 2 x^3 g[2, 4] - 2 x^2 y g[2, 5] - 2 x y^2 g[2, 6] - y^2 g[3, 3] - 2 x^2 y g[3, 4] - 2 x y^2 g[3, 5] - 2 y^3 g[3, 6] - x^4 g[4, 4] - 2 x^3 y g[4, 5] - 2 x^2 y^2 g[4, 6] - x^2 y^2 g[5, 5] - 2 x y^3 g[5, 6] - y^4 g[$

```
(*Now try to guess the coefficients for  the terms starting with constants, x, x^2,etc.*)
g[1, 1] = 6
6
g[1, 2] = 4
4
g[1, 3] = 4
4
g[2, 2] = 16 - 2 g[1, 4]
16 - 2 g[1, 4]
g[2, 3] = (14 - g[1, 5])
14 - g[1, 5]
g[3, 3] = 10 - 2 g[1, 6]
10 - 2 g[1, 6]
g[2, 4] = 8
8
g[3, 4] = 12 - g[2, 5]
12 - g[2, 5]
g[3, 5] = -g[2, 6]
-g[2, 6]
g[3, 6] = 0
0
g[4, 4] = 16
16
g[4, 5] = 0
0
g[5, 5] = -2 g[4, 6] + 9
9 - 2 g[4, 6]
g[5, 6] = 0
0
```

```
g[6, 6] = 0

0

MatrixForm[M]

  6          4               4              g[1, 4]        g[1, 5]      g[1, 6]
  4      16 - 2 g[1, 4]   14 - g[1, 5]        8            g[2, 5]      g[2, 6]
  4       14 - g[1, 5]   10 - 2 g[1, 6]  12 - g[2, 5]     -g[2, 6]        0
g[1, 4]       8           12 - g[2, 5]       16              0          g[4, 6]
g[1, 5]    g[2, 5]          -g[2, 6]          0          9 - 2 g[4, 6]    0
g[1, 6]    g[2, 6]             0           g[4, 6]           0            0

ExpandAll[P - v.M.v]

0

(*Notice again that the lower corner of the matrix M has
 a zero. This means that the entire row and column must be zero.*)

MatrixForm[M]

  6          4               4              g[1, 4]        g[1, 5]      g[1, 6]
  4      16 - 2 g[1, 4]   14 - g[1, 5]        8            g[2, 5]      g[2, 6]
  4       14 - g[1, 5]   10 - 2 g[1, 6]  12 - g[2, 5]     -g[2, 6]        0
g[1, 4]       8           12 - g[2, 5]       16              0          g[4, 6]
g[1, 5]    g[2, 5]          -g[2, 6]          0          9 - 2 g[4, 6]    0
g[1, 6]    g[2, 6]             0           g[4, 6]           0            0

g[1, 6] = g[2, 6] = g[4, 6] = 0

0

MatrixForm[M]

  6          4               4            g[1, 4]     g[1, 5]  0
  4      16 - 2 g[1, 4]   14 - g[1, 5]       8        g[2, 5]  0
  4       14 - g[1, 5]       10         12 - g[2, 5]     0     0
g[1, 4]       8          12 - g[2, 5]       16           0     0
g[1, 5]    g[2, 5]           0               0           9     0
  0           0              0               0           0     0

(*Define the different determinants of each corners.*)

T1 = Det[M[[{1, 2}, {1, 2}]]]

80 - 12 g[1, 4]

T2 = Det[M[[{1, 2, 3}, {1, 2, 3}]]]

-184 - 88 g[1, 4] + 136 g[1, 5] - 6 g[1, 5]^2

T3 = Det[M[[{1, 2, 3, 4}, {1, 2, 3, 4}]]]

-4224 + 256 g[1, 4] - 156 g[1, 4]^2 + 20 g[1, 4]^3 + 1024 g[1, 5] + 160 g[1, 4] g[1, 5] -
 28 g[1, 4]^2 g[1, 5] - 96 g[1, 5]^2 + g[1, 4]^2 g[1, 5]^2 + 832 g[2, 5] - 304 g[1, 4] g[2, 5] +
 16 g[1, 4]^2 g[2, 5] + 96 g[1, 5] g[2, 5] - 8 g[1, 4] g[1, 5] g[2, 5] - 80 g[2, 5]^2 + 12 g[1, 4] g[2, 5
```

T4 = Det[M[[{1, 2, 3, 4, 5}, {1, 2, 3, 4, 5}]]]

$$-38016 + 2304\, g[1, 4] - 1404\, g[1, 4]^2 + 180\, g[1, 4]^3 + 9216\, g[1, 5] + 1440\, g[1, 4]\, g[1, 5] - 252\, g[1, 4]^2\, g[1, 5] - 32\, g[1, 5]^2 + 32\, g[1, 4]\, g[1, 5]^2 + 9\, g[1, 4]^2\, g[1, 5]^2 - 256\, g[1, 5]^3 + 16\, g[1, 5]^4 + 7488\, g[2, 5] - 2736\, g[1, 4]\, g[2, 5] + 144\, g[1, 4]^2\, g[2, 5] - 32\, g[1, 5]\, g[2, 5] + 104\, g[1, 4]\, g[1, 5]\, g[2, 5] - 32\, g[1, 5]^2\, g[2, 5] + 24\, g[1, 4]\, g[1, 5]^2\, g[2, 5] - 16\, g[1, 5]^3\, g[2, 5] - 560\, g[2, 5]^2 + 12\, g[1, 4]\, g[2, 5]^2 + 10\, g[1, 4]^2\, g[2, 5]^2 + 128\, g[1, 5]\, g[2, 5]^2 - 28\, g[1, 4]\, g[1, 5]\, g[2, 5]^2 + 16\, g[1, 5]^2\, g[2, 5]^2 - 144\, g[2, 5]^3 + 8\, g[1, 4]\, g[2, 5]^3 - 8\, g[1, 5]\, g[2, 5]^3 + 6\, g[2, 5]^4$$

(*Since there is only three variables left define them as s, t, and u.*)

g[1, 4] = s

s

g[1, 5] = t

t

g[2, 5] = u

u

Factor[T1]

$-4\,(-20 + 3\,s)$

Factor[T2]

$-2\,(92 + 44\,s - 68\,t + 3\,t^2)$

Factor[T3]

$$-4224 + 256\,s - 156\,s^2 + 20\,s^3 + 1024\,t + 160\,s\,t - 28\,s^2\,t - 96\,t^2 + s^2\,t^2 + 832\,u - 304\,s\,u + 16\,s^2\,u + 96\,t\,u - 8\,s\,t\,u - 80\,u^2 + 12\,s\,u^2$$

Factor[T4]

$$-38016 + 2304\,s - 1404\,s^2 + 180\,s^3 + 9216\,t + 1440\,s\,t - 252\,s^2\,t - 32\,t^2 + 32\,s\,t^2 + 9\,s^2\,t^2 - 256\,t^3 + 16\,t^4 + 7488\,u - 2736\,s\,u + 144\,s^2\,u - 32\,t\,u + 104\,s\,t\,u - 32\,t^2\,u + 24\,s\,t^2\,u - 16\,t^3\,u - 560\,u^2 + 12\,s\,u^2 + 10\,s^2\,u^2 + 128\,t\,u^2 - 28\,s\,t\,u^2 + 16\,t^2\,u^2 - 144\,u^3 + 8\,s\,u^3 - 8\,t\,u^3 + 6\,u^4$$

Solve[T1 == 0, s]

$\left\{\left\{s \to \frac{20}{3}\right\}\right\}$

N[%]

$\{\{s \to 6.66667\}\}$

(*Solving for the determinant of T1 shows that s,
must be less then 6.6667, in this case we chose s =4. *)

s = 4

```
MatrixForm[M]
```

$$\begin{pmatrix} 6 & 4 & 4 & 4 & t & 0 \\ 4 & 8 & 14-t & 8 & u & 0 \\ 4 & 14-t & 10 & 12-u & 0 & 0 \\ 4 & 8 & 12-u & 16 & 0 & 0 \\ t & u & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
T1
```

32

```
(*T1 is now positive, now continue the process, we solve for t and u
 using the same method as above in order to have a positive matrix M.  *)
```

```
Solve[T2 == 0, t]
```

$$\left\{\left\{t \to \frac{2}{3}\left(17 - 2\sqrt{22}\right)\right\}, \left\{t \to \frac{2}{3}\left(17 + 2\sqrt{22}\right)\right\}\right\}$$

```
N[%]
```

{{t → 5.07945}, {t → 17.5872}}

```
(*t must be between the intervals of [5,17.5], so we pick t =6.*)
```

```
t = 6
```

6

```
MatrixForm[M]
```

$$\begin{pmatrix} 6 & 4 & 4 & 4 & 6 & 0 \\ 4 & 8 & 8 & 8 & u & 0 \\ 4 & 8 & 10 & 12-u & 0 & 0 \\ 4 & 8 & 12-u & 16 & 0 & 0 \\ 6 & u & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
T2
```

64

```
(*T2 is positive, now move on the next determinant T3 and solve for u.*)
```

```
Solve[T3 == 0, u]
```

{{u → 0}, {u → 8}}

```
(*T3 must be in the interval of [0,8] so choose u =0*)
```

```
u = 0
```

```
MatrixForm[M]
```

$$\begin{pmatrix} 6 & 4 & 4 & 4 & 6 & 0 \\ 4 & 8 & 8 & 8 & 0 & 0 \\ 4 & 8 & 10 & 12 & 0 & 0 \\ 4 & 8 & 12 & 16 & 0 & 0 \\ 6 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
T3

0

ExpandAll[P - (v.M.v)]

0

(*Now verify that P is positive using eigenvalues, and cholesky decomposition.*)

Eigenvalues[M]

{Root[-1428 + 576 #1 - 49 #1^2 + #1^3 &, 3],
 Root[-1428 + 576 #1 - 49 #1^2 + #1^3 &, 2], Root[-1428 + 576 #1 - 49 #1^2 + #1^3 &, 1], 0, 0, 0}

N[%]

{32.7382, 12.8735, 3.38826, 0., 0., 0.}

CholeskyDecomposition[M]

CholeskyDecomposition::posdef :
  The matrix {{6,4,4,4,6,0},{4,8,8,8,0,0},{4,8,10,12,0,0},{4,8,12,16,0,0},{6,0,0,0,9,0},{0,0,0,0,0,0}} is not
      sufficiently positive definite to complete the Cholesky decomposition to reasonable accuracy. >>

CholeskyDecomposition[{{6, 4, 4, 4, 6, 0}, {4, 8, 8, 8, 0, 0},
  {4, 8, 10, 12, 0, 0}, {4, 8, 12, 16, 0, 0}, {6, 0, 0, 0, 9, 0}, {0, 0, 0, 0, 0, 0}}]

(*The choleskydecomposition results in a error message however M is definitely positive,
the only problem is the Matrix M has zeros in some of its
 rows and columns and mathematica simply can't produce a outcome.*)

(*The next best step is to break up the matrix M into 4 sections of 3x3 matricies.*)

R = M[[{1, 2, 3}, {1, 2, 3}]]

{{6, 4, 4}, {4, 8, 8}, {4, 8, 10}}

Eigenvalues[R]

{Root[-64 + 92 #1 - 24 #1^2 + #1^3 &, 3],
 Root[-64 + 92 #1 - 24 #1^2 + #1^3 &, 2], Root[-64 + 92 #1 - 24 #1^2 + #1^3 &, 1]}

N[%]

{19.4359
```

(*Eigenvalues shows that R is positive.*)

MatrixForm[R]

$$\begin{pmatrix} 6 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 10 \end{pmatrix}$$

Det[R]

64

CholeskyDecomposition[R]

$$\left\{\left\{\sqrt{6}, 2\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}\right\}, \left\{0, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right\}, \left\{0, 0, \sqrt{2}\right\}\right\}$$

$$\mathbf{X} = \left\{\left\{\sqrt{6}, 2\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}\right\}, \left\{0, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right\}, \left\{0, 0, \sqrt{2}\right\}\right\}$$

$$\left\{\left\{\sqrt{6}, 2\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}\right\}, \left\{0, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right\}, \left\{0, 0, \sqrt{2}\right\}\right\}$$

MatrixForm[CD]

CD

S = M[[{1, 2, 3}, {4, 5, 6}]]

{{4, 6, 0}, {8, 0, 0}, {12, 0, 0}}

W = Inverse[R].S

$$\left\{\left\{0, \frac{3}{2}, 0\right\}, \left\{-1, -\frac{3}{4}, 0\right\}, \{2, 0, 0\}\right\}$$

(*S= R.W*)

S - R.W

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

U = M[[{4, 5, 6}, {4, 5, 6}]]

{{16, 0, 0}, {0, 9, 0}, {0, 0, 0}}

U - Transpose[S].W

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Transpose[X].X - R

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

MatrixForm[X]

$$\begin{pmatrix} \sqrt{6} & 2\sqrt{\frac{2}{3}} & 2\sqrt{\frac{2}{3}} \\ 0 & \frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

X.W

$$\left\{\left\{2\sqrt{\frac{2}{3}},\ \sqrt{6},\ 0\right\},\ \left\{\frac{4}{\sqrt{3}},\ -\sqrt{3},\ 0\right\},\ \left\{2\sqrt{2},\ 0,\ 0\right\}\right\}$$

MatrixForm[%]

$$\begin{pmatrix} 2\sqrt{\frac{2}{3}} & \sqrt{6} & 0 \\ \frac{4}{\sqrt{3}} & -\sqrt{3} & 0 \\ 2\sqrt{2} & 0 & 0 \end{pmatrix}$$

A = $\left\{\left\{\sqrt{6},\ 2\sqrt{\frac{2}{3}},\ 2\sqrt{\frac{2}{3}},\ 2\sqrt{\frac{2}{3}},\ \sqrt{6},\ 0\right\},\ \left\{0,\ \frac{4}{\sqrt{3}},\ \frac{4}{\sqrt{3}},\ \frac{4}{\sqrt{3}},\ -\sqrt{3},\ 0\right\},\ \left\{0,\ 0,\ \sqrt{2},\ 2\sqrt{2},\ 0,\ 0\right\},\ \{0,0,0,0,0,0\},\ \{0,0,0,0,0,0\}\right\}$

$$\left\{\left\{\sqrt{6},\ 2\sqrt{\frac{2}{3}},\ 2\sqrt{\frac{2}{3}},\ 2\sqrt{\frac{2}{3}},\ \sqrt{6},\ 0\right\},\ \left\{0,\ \frac{4}{\sqrt{3}},\ \frac{4}{\sqrt{3}},\ \frac{4}{\sqrt{3}},\ -\sqrt{3},\ 0\right\},\ \left\{0,\ 0,\ \sqrt{2},\ 2\sqrt{2},\ 0,\ 0\right\},\ \{0,0,0,0,0,0\},\ \{0,0,0,0,0,0\}\right\}$$

MatrixForm[A]

$$\begin{pmatrix} \sqrt{6} & 2\sqrt{\frac{2}{3}} & 2\sqrt{\frac{2}{3}} & 2\sqrt{\frac{2}{3}} & \sqrt{6} & 0 \\ 0 & \frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A0 = Transpose[A]

$$\left\{\left\{\sqrt{6},\ 0,\ 0,\ 0,\ 0\right\},\ \left\{2\sqrt{\frac{2}{3}},\ \frac{4}{\sqrt{3}},\ 0,\ 0,\ 0\right\},\ \left\{2\sqrt{\frac{2}{3}},\ \frac{4}{\sqrt{3}},\ \sqrt{2},\ 0,\ 0\right\},\ \left\{2\sqrt{\frac{2}{3}},\ \frac{4}{\sqrt{3}},\ 2\sqrt{2},\ 0,\ 0\right\},\ \left\{\sqrt{6},\ -\sqrt{3},\ 0,\ 0,\ 0\right\},\ \{0,\ 0$$

```
(*M = A^T.A*)

MatrixForm[A0]
```

$$\begin{pmatrix} \sqrt{6} & 0 & 0 & 0 & 0 \\ 2\sqrt{\frac{2}{3}} & \frac{4}{\sqrt{3}} & 0 & 0 & 0 \\ 2\sqrt{\frac{2}{3}} & \frac{4}{\sqrt{3}} & \sqrt{2} & 0 & 0 \\ 2\sqrt{\frac{2}{3}} & \frac{4}{\sqrt{3}} & 2\sqrt{2} & 0 & 0 \\ \sqrt{6} & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
M - A0.A

{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}

Factor[(v.A0) (A.v)]
```

$$\left\{\frac{2}{3}\left(3+2x+2x^2+2y+3xy\right)^2,\ \frac{1}{3}\left(4x+4x^2+4y-3xy\right)^2,\ 2\left(2x^2+y\right)^2,\ 0,\ 0\right\}$$

$$\mathbf{\frac{2}{3}\left(3+2x+2x^2+2y+3xy\right)^2+\frac{1}{3}\left(4x+4x^2+4y-3xy\right)^2+2\left(2x^2+y\right)^2}$$

$$2\left(2x^2+y\right)^2+\frac{1}{3}\left(4x+4x^2+4y-3xy\right)^2+\frac{2}{3}\left(3+2x+2x^2+2y+3xy\right)^2$$

```
ExpandAll[P - %]

0

(*P is a sum of squares and positive for each x,
```