TRUNCATED K-MOMENT PROBLEMS IN SEVERAL VARIABLES

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ABSTRACT. Let $\beta \equiv \beta^{(2n)}$ be an N-dimensional real multi-sequence of degree $2n$, with associated moment matrix $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$, and let $r := \operatorname{rank} \mathcal{M}(n)$. We prove that if $\mathcal{M}(n)$ is positive semidefinite and admits a rank-preserving moment matrix extension $\mathcal{M}(n+1)$, then $\mathcal{M}(n+1)$ has a unique representing measure μ, which is r-atomic, with $\operatorname{supp} \mu$ equal to $\mathcal{V}(\mathcal{M}(n+1))$, the algebraic variety of $\mathcal{M}(n+1)$. Further, β has an r-atomic (minimal) representing measure supported in a semi-algebraic set $K_{\mathcal{Q}}$ subordinate to a family $\mathcal{Q} \equiv \{q_i\}_{i=1}^m \subseteq \mathbb{R}[t_1, ..., t_N]$ if and only if $\mathcal{M}(n)$ is positive semidefinite and admits a rank-preserving extension $\mathcal{M}(n+1)$ for which the associated localizing matrices $\mathcal{M}_{q_i}(n + [\frac{1+\deg q_i}{2}])$ are positive semidefinite $(1 \le i \le m)$; in this case, μ (as above) satisfies $\operatorname{supp} \mu \subseteq K_{\mathcal{Q}}$, and μ has precisely $\operatorname{rank} \mathcal{M}(n) - \operatorname{rank} \mathcal{M}_{q_i}(n + [\frac{1+\deg q_i}{2}])$ atoms in $\mathcal{Z}(q_i) \equiv \{t \in \mathbb{R}^N : q_i(t) = 0\}$, $1 \le i \le m$.

1. Introduction

Given a finite real multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}_+^N, |i| \le 2n}$, and a closed set $K \subseteq \mathbb{R}^N$, the *truncated K-moment problem* for β entails determining whether there exists a positive Borel measure μ on $\mathbb{R}^N$ such that

$$\beta_i = \int_{\mathbb{R}^N} t^i \, d\mu(t), \qquad i \in \mathbb{Z}_+^N, \ |i| \le 2n, \tag{1.1}$$

and

$$\operatorname{supp} \mu \subseteq K; \tag{1.2}$$

a measure μ satisfying (1.1) is a *representing measure* for β; μ is a *K-representing measure* if it satisfies (1.1) and (1.2).

In the sequel, we characterize the existence of a finitely atomic K-representing measure having the fewest possible atoms, in the case when K is semi-algebraic. This is the case where $\mathcal{Q} \equiv \{q_i\}_{i=1}^m \subseteq \mathbb{R}^N[t] \equiv \mathbb{R}[t_1, \dots, t_N]$ and $K = K_{\mathcal{Q}} := \{(t_i, \dots, t_N) \in \mathbb{R}^N : q_i(t_1, \dots, t_N) \ge 0, 1 \le i \le m\}$. Our existence condition (Theorem 1.1 below) is expressed in terms of positivity and extension properties of the *moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}^N(n)(\beta)$ associated to β, and in terms of positivity of the *localizing matrix* $\mathcal{M}_{q_i}$ corresponding to each q_i (see below for terminology and notation). In Theorem 1.2 we provide a procedure for computing the atoms and densities of a minimal representing measure in any truncated moment problem (independent of K).

If μ is a representing measure for β (or, as we often say, a representing measure for $\mathcal{M}(n)$), then $\operatorname{card} \operatorname{supp} \mu \ge \operatorname{rank} \mathcal{M}(n)$; moreover, there exists a $\operatorname{rank} \mathcal{M}(n)$-atomic (minimal) representing

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measure for β if and only if $\mathcal{M}(n)$ is positive semidefinite ($\mathcal{M}(n) \geq 0$) and $\mathcal{M}(n)$ admits a rank-preserving (or *flat*) extension to a moment matrix $\mathcal{M}(n+1)$; in this case, $\mathcal{M}(n+1)$ admits unique successive flat moment matrix extensions $\mathcal{M}(n+2)$, $\mathcal{M}(n+3), \ldots$ (Theorem 2.19). For $1 \leq i \leq m$, suppose $\deg q_i = 2k_i$ or $2k_i - 1$; relative to $\mathcal{M}(n+k_i)$ we have the *localizing matrix* $\mathcal{M}_{q_i}(n+k_i)$ (cf. Section 3).

Our two main results, which follow, characterize the existence of $\operatorname{rank} M(n)$-atomic (minimal) $K_{\mathcal{Q}}$-representing measures for β and show how to compute the atoms and densities of such measures.

Theorem 1.1. *An N-dimensional real sequence $\beta \equiv \beta^{(2n)}$ has a* $\operatorname{rank} \mathcal{M}(n)$*-atomic representing measure supported in $K_{\mathcal{Q}}$ if and only if $\mathcal{M}(n) \geq 0$ and $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$ such that $\mathcal{M}_{q_i}(n+k_i) \geq 0$ $(1 \leq i \leq m)$. In this case, $\mathcal{M}(n+1)$ admits a unique representing measure μ, which is a* $\operatorname{rank} \mathcal{M}(n)$*-atomic (minimal) $K_{\mathcal{Q}}$-representing measure for β; moreover, μ has precisely* $\operatorname{rank} \mathcal{M}(n) - \operatorname{rank} \mathcal{M}_{q_i}(n+k_i)$ *atoms in $\mathcal{Z}(q_i) \equiv \left\{t \in \mathbb{R}^N : q_i(t) = 0\right\}$, $1 \leq i \leq m$.*

The uniqueness statement in Theorem 1.1 actually depends on our next result, which provides a concrete procedure for computing the measure μ. As described in Section 2, the rows and columns of $\mathcal{M}(n)$ are indexed by the lexicographic ordering of the monomials t^i $(i \in \mathbb{Z}_+^N,\ |i| \leq n)$, and are denoted by T^i $(|i| \leq n)$; a dependence relation in the columns of $\mathcal{M}(n)$ may thus be expressed as $p(T) = 0$ for a suitable $p \in \mathbb{R}^N[t]$ with $\deg p \leq n$. We define the *variety* of $\mathcal{M}(n)$ by $\mathcal{V}(\mathcal{M}(n)) := \bigcap_{\substack{p \in \mathbb{R}^N[t], \deg p \leq n \\ p(T)=0}} \mathcal{Z}(p)$, where $\mathcal{Z}(p) := \{t \in \mathbb{R}^N : p(t) = 0\}$. Let $r := \operatorname{rank} \mathcal{M}(n)$ and let $\mathcal{B} \equiv \{T^{i_k}\}_{k=1}^r$ denote a maximal linearly independent set of columns of $\mathcal{M}(n)$. For $\mathcal{V} \equiv \{t_j\}_{j=1}^r \subseteq \mathbb{R}^N$, let $W_{\mathcal{B},\mathcal{V}}$ denote the $r \times r$ matrix whose entry in row k, column j is $t_j^{i_k}$ $(1 \leq k, j \leq r)$.

Theorem 1.2. *If $\mathcal{M}(n) \equiv \mathcal{M}^N(n) \geq 0$ admits a flat extension $\mathcal{M}(n+1)$, then $\mathcal{V} := \mathcal{V}(\mathcal{M}(n+1))$ satisfies* $\operatorname{card} \mathcal{V} = r$ $(\equiv \operatorname{rank} \mathcal{M}(n))$*, and $\mathcal{V} \equiv \{t_j\}_{j=1}^r$ forms the support of the unique representing measure μ for $\mathcal{M}(n+1)$. If $\mathcal{B} \equiv \{T^{i_k}\}_{k=1}^r$ is a maximal linearly independent subset of columns of $\mathcal{M}(n)$, then $W_{\mathcal{B},\mathcal{V}}$ is invertible, and $\mu = \sum_{i=1}^r \rho_j \delta_{t_j}$, where $\rho \equiv (\rho_1, ..., \rho_r)$ is uniquely determined by $\rho^t = W_{\mathcal{B},\mathcal{V}}^{-1} (\beta_{i_1}, ..., \beta_{i_r})^t$.*

Theorem 1.2 describes μ in terms of $\mathcal{V}(\mathcal{M}(n+1))$. To compute the variety of any moment matrix $\mathcal{M}(n)$, we may rely on the following general result. Given $n \geq 1$, write $J \equiv J(n) := \{j \in \mathbb{Z}_+^N : |j| \leq n\}$. Clearly, size $\mathcal{M}(n) = \operatorname{card} J(n) = \binom{N+d}{d} = \dim\{p \in \mathbb{R}^N[t] : \deg p \leq n\}$.

Proposition 1.3. *Let $\mathcal{M}(n) \equiv \mathcal{M}^N(n)$ be a real moment matrix, with columns T^j indexed by $j \in J$, let $r :=$* $\operatorname{rank} \mathcal{M}(n)$*, and let $\mathcal{B} \equiv \{T^i\}_{i \in I}$ be a maximal linearly independent set of columns of $\mathcal{M}(n)$, where $I \subseteq J$ satisfies* $\operatorname{card} I = r$*. For each index $j \in J \setminus I$, let q_j denote the unique polynomial in lin.span $\{t^i\}_{i \in I}$ such that $T^j = q_j(T)$, and let $r_j(t) := t^j - q_j(t)$. Then $\mathcal{V}(\mathcal{M}(n))$ is precisely the set of common zeros of $\{r_j\}_{j \in J \setminus I}$.*

Cases are known where $\beta^{(2n)}$ has no $\operatorname{rank} \mathcal{M}(n)$-atomic $K_{\mathcal{Q}}$-representing measure, but does have a *finitely atomic* $K_{\mathcal{Q}}$-representing measure (cf. [CuFi6], [CuFi9], [Fia3]). It follows from Theorem 1.1 that $\beta^{(2n)}$ has a finitely atomic representing measure supported in $K_{\mathcal{Q}}$ if and only if $\mathcal{M}(n)(\beta)$ admits some positive moment matrix extension $\mathcal{M}(n+j)$, which in turn admits a flat extension $\mathcal{M}(n+j+1)$ for which the unique successive flat extensions $\mathcal{M}(n+j+k)$ satisfy $\mathcal{M}_{q_i}(n+j+k_i) \geq 0$ $(1 \leq i \leq m)$. We may estimate the minimum size of j as follows.

Corollary 1.4. *The N-dimensional real sequence $\beta^{(2n)}$ has a finitely atomic representing measure supported in $K_{\mathcal{Q}}$ if and only if $\mathcal{M}(n)(\beta)$ admits some positive moment matrix extension $\mathcal{M}(n+j)$,* *with* $j \le 2\binom{2n+N}{N} - n$, *which in turn admits a flat extension* $\mathcal{M}(n+j+1)$ *for which* $\mathcal{M}_{q_i}(n+j+k_i) \geq 0 \ \ (1 \le i \le m)$.

If the conditions of Corollary 1.4 hold, then the atoms and densities of a finitely atomic K_Q-representing measure for β may be computed by applying Theorem 1.2 and Proposition 1.3 to the flat extension $\mathcal{M}(n+j+1)$. It is an open problem whether the existence of a representing measure μ for $\beta^{(2n)}$ implies the existence of a finitely atomic representing measure; such is the case, for example, if μ has convergent moments of degree $2n+1$ (cf. [CuFi8, Theorem 1.4], [Put3], [Tch]).

We view Theorem 1.1 as our main result concerning existence of minimal K_Q-representing measures, and Theorem 1.2 primarily as a tool for computing such measures (cf. Example 1.5 below). Note that Theorem 1.2 applies to arbitrary moment problems, not just the K-moment problem. Although Theorem 1.2 can also be regarded as an existence result, it may be very difficult to utilize it in this way in specific examples. To explain this viewpoint, we recall a result of [EFP]. Let ω denote the restriction of planar Lebesgue measure to the closed unit disk $\bar{\mathbb{D}}$ and consider $\beta \equiv \beta^{(6)}[\omega]$ and $\mathcal{M} \equiv \mathcal{M}(3)(\beta)$; then $\operatorname{rank} \mathcal{M} = 10$. Flat extensions $\mathcal{M}(4)$ of $\mathcal{M}$ exist in abundance and correspond to 10-atomic (minimal) cubature rules ν of degree 6 for ω. In [EFP] it is proved that no such rule ν is "inside," i.e., with $\operatorname{supp} \nu \subseteq \bar{\mathbb{D}}$. The proof in [EFP] first characterizes the flat extensions $\mathcal{M}(4)$ in terms of algebraic relations among the "new moments" of degree 7 that appear in such extensions. These relations lead to inequalities which ultimately imply that, in Theorem 1.1, $\mathcal{M}_p(4)$ cannot be positive semi-definite, where $p(x,y) := 1 - x^2 - y^2$. One could also try to establish the nonexistence of 10-atomic inside rules directly from Theorem 1.2, without recourse to Theorem 1.1. In this approach one would first compute general formulas for the new moments of degree 7 in a flat extension $\mathcal{M}(4)$, use these moments to compute the general form of $\mathcal{V}(\mathcal{M}(4))$, and then show that $\mathcal{V}(\mathcal{M}(4))$ cannot be contained in $\bar{\mathbb{D}}$. As a practical matter, however, this plan cannot be carried out; the new moments comprise the solution of a system of 6 quadratic equations in 8 real variables, and at present a program such as *Mathematica* seems unable to solve this system in a tractable form. For a problem such as this, Theorem 1.1 seems indispensable. We illustrate the interplay between Theorem 1.1, Theorem 1.2 and Proposition 1.3 in Example 1.5 below.

For measures in the plane ($N = 2$), Theorem 1.1 is equivalent to [CuFi4, Theorem 1.6], which characterizes the existence of minimal K-representing measures in the semi-algebraic case of the *truncated complex K-moment problem* (with moments relative to monomials of the form $\bar{z}^i z^j$). In [CuFi4] we remarked that [CuFi4, Theorem 1.6] extended to truncated moment problems in any number of real or complex variables. In [Las1], Lasserre developed applications of [CuFi4, Theorem 1.6] to optimization problems in the plane. These applications also extend to $\mathbb{R}^N$ ($N > 2$) (cf. [Las1], [Las2], [Las3]), but they require the above mentioned generalization of [CuFi4, Theorem 1.6] that we provide in Theorem 1.1. Lasserre's work motivated us to revisit our assertion in [CuFi4]; we then realized that there were unforeseen difficulties with the generalization, particularly for the case when N is odd. The purpose of Theorem 1.1 is to provide the desired generalization.

The proofs of Theorem 1.1 and Corollary 1.4 appear in Section 5. In Theorem 5.1 we characterize the existence of minimal K-representing measures in the semi-algebraic case of the truncated complex K-moment problem for measures on $\mathbb{C}^m$. The equivalence of this result to the "even" case of Theorem 1.1 ($N = 2d$) is given in the first part of the proof of Theorem 5.2; this is based on the equivalence of the truncated moment problem for $\mathbb{C}^d$ with the truncated real moment problem for $\mathbb{R}^{2d}$ (cf. Propositions 2.15, 2.16, 2.17 and 2.18). The proof of Theorem 1.1 for $N = 2d-1$, given in the second part of the proof of Theorem 5.2, requires an additional argument, based on the equivalence of a truncated moment problem for $\mathbb{R}^{2d-1}$ with an associated moment problem for $\mathbb{R}^{2d}$.

We prove Theorem 1.2 and Proposition 1.3 in Section 2. Theorem 1.2 is new even for $N = 2$. Previously, for $N = 2$ we knew that the measure μ of Theorems 1.1 and 1.2 could be computed with $\operatorname{supp}\mu = \mathcal{V}(\mathcal{M}(r))$ [CuFi1, p. 33], where $r := \operatorname{rank}\mathcal{M}(n)$ satisfies $r \le \frac{(n+1)(n+2)}{2}$; but for $r > n+1$ this entails iteratively generating the extensions $\mathcal{M}(n+2), ..., \mathcal{M}(r)$. For $N > 2$, we previously had no method for computing μ. In order to prove Theorem 1.2 we first obtain some results concerning truncated complex moment problems on $\mathbb{C}^d$. Let $M(n) \equiv M^d(n)(\gamma)$ denote the moment matrix for a d-dimensional complex multisequence γ of degree $2n$, and let $\mathcal{V}(M(n))$ denote the corresponding algebraic variety. In Theorem 2.4 we prove that if $M(n) \ge 0$ admits a flat extension $M(n+1)$, then the unique successive flat moment matrix extensions $\mathcal{M}(n+2), \mathcal{M}(n+3), ...$ (cf. Theorem 2.2) satisfy $\mathcal{V}(M(n+1)) = \mathcal{V}(M(n+2)) = ...$. This result is used to prove Theorem 2.3, which is the analogue of Theorem 1.2 for the complex moment problem. The proof of Theorem 1.2 is then given in Theorem 2.21, using Theorem 2.3 and the "equivalence" results cited above.

In Section 3 we study the *localizing matrix* $M_p^d(n)$ corresponding to a complex moment matrix $M^d(n)$ and a polynomial $p \in \mathbb{C}_{2n}^d[z, \bar{z}]$; Theorem 3.2 provides a computational formula for $M_p^d(n)$ as a linear combination of certain compressions of $M^d(n)$ corresponding to the monomial terms of p; an analogous formula holds as well for *real localizing matrices* (cf. Theorem 3.6). In Section 4, we show that a flat extension $M^d(n+1)$ of $M^d(n) \ge 0$ induces flat extensions of positive localizing matrices. Indeed, the flat extension $M^d(n+1)$ has unique successive flat extensions $M^d(n+2), M^d(n+3), ...$, and in Theorem 4.1, for $p \in \mathbb{C}^d[z, \bar{z}]$, $\deg p = 2k$ or $2k-1$, we prove that if $M_p^d(n+k) \ge 0$, then $M_p^d(n+k+1)$ is a flat, positive extension of $M_p^d(n+k)$. In proving Theorem 4.1 we follow the same general plan as in the proof of [CuFi4, Theorem 1.6] (for moment problems on $\mathbb{C}$), but we have streamlined the argument somewhat, placing more emphasis on the abstract properties of flat extensions and less emphasis on detailed calculations of the extensions; such calculations unnecessarily complicated the argument given in [CuFi4]. Theorem 4.1 is the main technical result that we need to prove Theorem 1.1.

In the following example, we show the interaction of Theorem 1.1, Theorem 1.2 and Proposition 1.3 in a 3-dimensional cubature problem.

Example 1.5. We consider the cubature problem of degree 2 for volume measure $\mu \equiv \mu_{\mathbb{B}^3}$ on the closed unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$ (cf. [Str]). Thus $\beta \equiv \beta^{(2)} = \left\{\beta_{(i,j,k)}\right\}_{i,j,k\ge 0,\, i+j+k\le 2}$, where $\beta_{(i,j,k)} := \int_{\mathbb{B}^3} x^i y^j z^k \, d\mu$, i.e., $\beta_{(0,0,0)} = \frac{4\pi}{3}$, $\beta_{(1,0,0)} = \beta_{(0,1,0)} = \beta_{(0,0,1)} = 0$, $\beta_{(2,0,0)} = \beta_{(0,2,0)} = \beta_{(0,0,2)} = \frac{4\pi}{15}$, $\beta_{(1,1,0)} = \beta_{(1,0,1)} = \beta_{(0,1,1)} = 0$. The moment matrix $\mathcal{M}^3(1)(\beta)$ has rows and columns indexed by 1, X, Y, Z; for $i \equiv (i_1, i_2, i_3)$, $j \equiv (j_1, j_2, j_3) \in \mathbb{Z}_+^3$ with $|i|, |j| \le 1$, the entry in row $X^{i_1}Y^{i_2}Z^{i_3}$, column $X^{j_1}Y^{j_2}Z^{j_3}$, is $\beta_{(i_1+j_1,\, i_2+j_2,\, i_3+j_3)}$. Thus we have $\mathcal{M} \equiv \mathcal{M}^3(1)(\beta) = \operatorname{diag}\left(\frac{4\pi}{3}, \frac{4\pi}{15}, \frac{4\pi}{15}, \frac{4\pi}{15}\right)$. We will use Theorem 1.1 to construct a $\operatorname{rank}\mathcal{M}$-atomic representing measure for β supported in $K = \mathbb{B}^3$. A moment matrix extension $\mathcal{M}(2)$ of $\mathcal{M}$ admits a block decomposition $\mathcal{M}(2) = \begin{pmatrix} \mathcal{M} & \mathcal{B}(2) \\ \mathcal{B}(2)^t & \mathcal{C}(2) \end{pmatrix}$, where $\mathcal{B}(2)$ includes "new moments" of degree 3 and $\mathcal{C}(2)$ is a moment matrix block of degree 4; the rows and columns of $M(2)$ are indexed by 1, X, Y, Z, X^2, YX, ZX, Y^2, ZY, Z^2 (see Section 2 below). Clearly, $\mathcal{M}$ is positive definite and invertible, so a flat extension $\mathcal{M}(2)$ is determined by a choice of moments of degree 3 such that $\mathcal{B}(2)^t\mathcal{M}^{-1}\mathcal{B}(2)$ has the form of a moment matrix block $\mathcal{C}(2)$ (cf. the remarks following Theorem 2.3). Due to its complexity, we are unable to compute the general solution $\mathcal{B}(2)$ to

$$\mathcal{C}(2) = \mathcal{B}(2)^t\mathcal{M}^{-1}\mathcal{B}(2). \tag{1.3}$$

Instead, we specify certain moments of degree 3 as follows:

$$\begin{aligned} &\beta_{(2,0,1)} = \beta_{(2,1,0)} = \beta_{(1,1,1)} = \beta_{(0,2,1)} = \beta_{(0,1,2)} = 0, \\ &\beta_{(3,0,0)} = \frac{1125\beta_{(1,2,0)}^2 - 16\pi^2}{1125\beta_{(1,2,0)}}, \quad \beta_{(1,0,2)} = -\frac{16\pi^2}{1125\beta_{(1,2,0)}}. \end{aligned} \tag{1.4}$$

(Observe that we have left $\beta_{(1,2,0)}$, $\beta_{(0,3,0)}$ and $\beta_{(0,0,3)}$ free.) With these choices, $\mathcal{B}(2)^t\mathcal{M}^{-1}\mathcal{B}(2)$ is a moment matrix block of degree 4, and $\mathcal{M}(2) \equiv \mathcal{M}(2)\{\beta_{(1,2,0)}, \beta_{(0,3,0)}, \beta_{(0,0,3)}\}$ (defined by (1.3)) is a flat extension of $\mathcal{M}$. To show that β admits a 4-atomic K-representing measure, we consider $p(x,y,z) = 1-(x^2+y^2+z^2)$, so that $K = K_p$ (where by K_p we mean $K_{\mathcal{Q}}$ with $\mathcal{Q} \equiv \{p\}$). Since $\deg p = 2$, in Theorem 1.1 we have $n = k = 1$; it thus suffices to show that the flat extension $\mathcal{M}(2)$ corresponding to (1.4) satisfies $\mathcal{M}_p(2) \geq 0$. As we describe in Section 3 below, $\mathcal{M}_p(2) = \mathcal{M}_1(2) - (\mathcal{M}_{x^2}(2) + \mathcal{M}_{y^2}(2) + \mathcal{M}_{z^2}(2))$, where $\mathcal{M}_1(2) = \mathcal{M}$, $\mathcal{M}_{x^2}(2)$ is the compression of $\mathcal{M}(2)$ to rows and columns indexed by X, X^2, YX, ZX, $\mathcal{M}_{y^2}(2)$ is the compression of $\mathcal{M}(2)$ to rows and columns indexed by Y, YX, Y^2, ZY, and $\mathcal{M}_{z^2}(2)$ is the compression of $\mathcal{M}(2)$ to rows and columns indexed by Z, ZX, ZY, Z^2. From these observations, and using (1.3)–(1.4), it is straightforward to verify that

$$\mathcal{M}_p(2) = \begin{pmatrix} \frac{8\pi}{15} & -2\beta_{(3,0,0)} & -\beta_{(0,3,0)} & -\beta_{(0,0,3)} \\ -2\beta_{(3,0,0)} & f(\beta_{(1,2,0)}) & -\frac{15\beta_{(1,2,0)}\beta_{(0,3,0)}}{4\pi} & \frac{4\pi\beta_{(0,0,3)}}{75\beta_{(1,2,0)}} \\ -\beta_{(0,3,0)} & -\frac{15\beta_{(1,2,0)}\beta_{(0,3,0)}}{4\pi} & g(\beta_{(1,2,0)}, \beta_{(0,3,0)}) & 0 \\ -\beta_{(0,0,3)} & \frac{4\pi\beta_{(0,0,3)}}{75\beta_{(1,2,0)}} & 0 & h(\beta_{(1,2,0)}, \beta_{(0,0,3)}) \end{pmatrix},$$

where

$$\begin{aligned} f(r) &:= -\frac{(1125r^2 - 300\pi r + 16\pi^2)(1125r^2 + 300\pi r + 16\pi^2)}{168750\pi r^2}, \\ g(r,s) &:= -\frac{2250r^2 + 1125s^2 - 64\pi^2}{300\pi}, \\ h(r,t) &:= -\frac{-72000\pi^2 r^2 + 512\pi^4 + 1265625r^2t^2}{337500\pi r^2}, \end{aligned}$$

and that $\mathcal{M}_p(2)$ is positive semi-definite if $\beta_{(0,3,0)} = \beta_{(0,0,3)} = 0$ and $\frac{2}{15}\sqrt{\frac{2}{5}}\pi \leq \beta_{(1,2,0)} \leq \frac{4}{15}\sqrt{\frac{2}{5}}\pi$. Under these conditions, Theorem 1.1 now implies the existence of a unique 4-atomic (minimal) representing measure η for $\mathcal{M}(2)$, each of whose atoms lies in the closed unit ball. Theorem 1.2 implies that $\operatorname{supp}\eta = \mathcal{V} := \mathcal{V}(\mathcal{M}(2))$. To compute the atoms of η via Proposition 1.3, observe that in the column space of $\mathcal{M}(2)$ we have the following linear dependence relations: $X^2 = \frac{1}{5}1 + \frac{1125\beta_{(1,2,0)}^2 - 16\pi^2}{300\pi\beta_{(1,2,0)}}X$, $XY = \frac{15\beta_{(1,2,0)}}{4\pi}Y$, $XZ = -\frac{4\pi}{75\beta_{(1,2,0)}}Z$, $Y^2 = \frac{1}{5}1 + \frac{15\beta_{(1,2,0)}}{4\pi}X$, $YZ = 0$, and $Z^2 = \frac{1}{5}1 - \frac{4\pi}{75\beta_{(1,2,0)}}X$; thus, $\mathcal{V}$ is determined by the polynomials corresponding to these relations. A calculation shows that $\mathcal{V} = \{P_i\}_{i=0}^3$, where $P_i \equiv (x_i, y_i, z_i)$ satisfies

$$P_0 = \left(\frac{15\beta_{(1,2,0)}}{4\pi}, -\frac{s}{4\sqrt{5}\pi}, 0 \right), \ P_1 = \left(\frac{15\beta_{(1,2,0)}}{4\pi}, \frac{s}{4\sqrt{5}\pi}, 0 \right),$$
$$P_2 = \left(-\frac{4\pi}{75\beta_{(1,2,0)}}, 0, -\frac{s}{75\beta_{(1,2,0)}} \right), \ P_3 = \left(-\frac{4\pi}{75\beta_{(1,2,0)}}, 0, \frac{s}{75\beta_{(1,2,0)}} \right),$$

with $s := \sqrt{1125\beta_{(1,2,0)}^2 + 16\pi^2}$. The measure η is thus of the form $\eta = \sum_{i=0}^3 \rho_i \delta_{P_i}$. To compute the densities ρ_i using Theorem 1.2, consider the basis $\mathcal{B} := \{1, X, Y, Z\}$ for $\mathcal{C}_{\mathcal{M}(1)}$ and let

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{pmatrix}.$$

Following Theorem 1.2, $\rho \equiv (\rho_0, \rho_1, \rho_2, \rho_3)$ is uniquely determined by

$$\rho^t = W^{-1} \left(\beta_{(0,0,0)}, \beta_{(1,0,0)}, \beta_{(0,1,0)}, \beta_{(0,0,1)} \right)^t = W^{-1} \left(\frac{4\pi}{3}, 0, 0, 0 \right)^t,$$

and thus

$$\rho_0 = \rho_1 = \frac{32\pi^3}{3(1125\beta_{(1,2,0)}^2 + 16\pi^2)}, \ \rho_2 = \rho_3 = \frac{750\beta_{(1,2,0)}^2 \pi}{1125\beta_{(1,2,0)}^2 + 16\pi^2}.$$

For a concrete numerical example, we can take $\beta_{(1,2,0)} = \frac{4}{15}\sqrt{\frac{2}{5}}\pi$, and obtain $\rho_0 = \rho_1 = \frac{2}{9}\pi$, $\rho_2 = \rho_3 = \frac{4}{9}\pi$, and $P_0 = \left(\sqrt{\frac{2}{5}}, -\sqrt{\frac{3}{5}}, 0\right)$, $P_1 = \left(\sqrt{\frac{2}{5}}, \sqrt{\frac{3}{5}}, 0\right)$, $P_2 = \left(-\sqrt{\frac{1}{10}}, 0, -\sqrt{\frac{3}{10}}\right)$, and $P_3 = \left(-\sqrt{\frac{1}{10}}, 0, \sqrt{\frac{3}{10}}\right)$. Note that

$$\mathcal{M}_p(2) = \begin{pmatrix} \frac{8\pi}{15} & -\frac{4}{15}\sqrt{\frac{2}{5}}\pi \\ -\frac{4}{15}\sqrt{\frac{2}{5}}\pi & \frac{4\pi}{75} \end{pmatrix} \bigoplus (0) \bigoplus (\frac{4\pi}{25}),$$

so $\operatorname{rank} \mathcal{M}(1) - \operatorname{rank} \mathcal{M}_p(2) = 2$, and (as Theorem 1.1 predicts) there are two points, P_0 and P_1, that lie on the unit sphere.

We pause to locate Theorem 1.1 within the extensive literature on the K-moment problem (cf. [Akh], [BeCJ], [BeMa], [Fug], [KrNu], [Rez], [ShTa], [StSz2]). A classical theorem of M. Riesz [Rie, Section 5] provides a solution to the full K-moment problem on $\mathbb{R}$, as follows. Given a real sequence $\beta \equiv \{\beta_i\}_{i=0}^\infty$ and a closed set $K \subseteq \mathbb{R}$, there exists a positive Borel measure μ on $\mathbb{R}$ such that $\beta_i = \int t^i \, d\mu \ \ (i \geq 0)$ and $\operatorname{supp} \mu \subseteq K$ if and only if each polynomial $p \in \mathbb{C}[t]$, $p(t) = \sum_{i=0}^N a_i t^i$, with $p|_K \geq 0$, satisfies $\sum_{i=0}^N a_i \beta_i \geq 0$. For a general closed set $K \subseteq \mathbb{R}$ there is no concrete description of the case $p|_K \geq 0$, so it may be very difficult to verify the Riesz hypothesis for a particular β.

In [Hav], Haviland extended Riesz's theorem to $\mathbb{R}^N$ $(N > 1)$ and also showed that for several semi-algebraic sets K, the Riesz hypothesis can be checked by concrete positivity tests. Indeed, by combining the generalized Riesz hypothesis with concrete descriptions of non-negative polynomials on $\mathbb{R}$, $[0, +\infty]$, $[a, b]$, or the unit circle, Haviland recovered classical solutions to the full moment problems of Hamburger, Stieltjes, Hausdorff, and Herglotz [Hav]. More recently, for the case of the closed unit disk, Atzmon [Atz] found a concrete solution to the full K-moment problem using subnormal operator theory, and Putinar [Put1] subsequently presented an alternate solution using hyponormal operator theory.

In [Cas], Cassier initiated the study of the K-moment problem for compact subsets of $\mathbb{R}^N$. For the case when K is compact and semi-algebraic, Schmüdgen [Sch] used real algebraic geometry to solve the full K-moment problem in terms of concrete positivity tests. Using infinite moment matrices, we may paraphrase Schmüdgen's theorem as follows: a full multi-sequence $\beta \equiv \beta^{(\infty)} = \{\beta_i\}_{i \in \mathbb{Z}_+^N}$ has a representing measure supported in a compact semi-algebraic set $K_{\mathcal{Q}}$ if and only if $\mathcal{M}^N(\infty)(\beta) \geq 0$ and $\mathcal{M}_q^N(\infty)(\beta) \geq 0$ for every polynomial q that is a product of distinct q_i. Schmüdgen's approach, using real algebra, is to concretely describe the polynomials nonnegative on $K_{\mathcal{Q}}$ (as above) and to then apply the Riesz-Haviland criterion. Putinar and Vasilescu [PuVa] subsequently provided a reduced set of testing polynomials q (see also [Dem]). Recently, Powers and Scheiderer [PoSc] characterized the non-compact semi-algebraic sets $K_{\mathcal{Q}}$ for which a generalized Schmüdgen-type theorem is valid. Indeed, recent advances in real algebra make it possible to concretely describe the polynomials nonnegative on certain noncompact semi-algebraic sets ([KuMa], [PoRe1], [PoRe2], [PoSc], [Pre], [Put2], [Sche]), so as to establish moment theorems via the previously intractable Riesz-Haviland approach.

There is at present no viable analogue of the Riesz-Haviland criterion for truncated moment problems. Theorem 1.1 is motivated by the above results for the full K-moment problem and also by a recent result of J. Stochel [Sto2] which shows that the truncated K-moment problem is actually more general than the full K-moment problem. Stochel's result in [Sto2] is stated for the complex multidimensional moment problem, but we may paraphrase it for the real moment problem as follows.

Theorem 1.6. *(cf.* [Sto2]*). Let K be a closed subset of $\mathbb{R}^N$ $(N > 1)$. A real multisequence $\beta \equiv \beta^{(\infty)} = \{\beta_i\}_{i \in \mathbb{Z}_+^N}$ has a K-representing measure if, and only if, for each $n > 0$, $\beta^{(2n)} \equiv \{\beta_i\}_{i \in \mathbb{Z}_+^N,\, |i| \leq 2n}$ has a K-representing measure.*

For the semi-algebraic case ($K = K_{\mathcal{Q}}$), Theorem 1.1 addresses the existence of finitely atomic K-representing measures for $\beta^{(2n)}$ with the fewest atoms possible. Concerning the existence of a flat extension $\mathcal{M}^N(n+1)$ in Theorem 1.1, there is at present no satisfactory general test available, so in this sense Theorem 1.1 is "abstract." However, in certain special cases, concrete solutions to the flat extension problem have been found ([CuFi2], [CuFi3]). For example, consider the case of the parabolic moment problem, where $q(x, y) = 0$ represents a parabola in $\mathbb{R}^2$. Theorem 1.1 implies that $\beta^{(2n)}$ has a rank $\mathcal{M}^2(n)$-atomic representing measure supported in $\mathcal{Z}(q)$ if and only if $\mathcal{M}^2(n)(\beta)$ is positive and admits a flat extension $\mathcal{M}^2(n+1)$ for which $\mathcal{M}_q^2(n+1) = 0$. In [CuFi7] we obtained the following concrete characterization of this case.

Theorem 1.7. *(*[CuFi7, Theorem 2.2]*) Let $q(x, y) = 0$ denote a parabola in $\mathbb{R}^2$. The following statements are equivalent for $\beta \equiv \beta^{(2n)}$:*
(i) β has a representing measure supported in $\mathcal{Z}(q)$;
(ii) β has a (minimal) $\operatorname{rank} \mathcal{M}^2(n)(\beta)$*-atomic representing measure supported in $\mathcal{Z}(q)$ (cf. Theorem 1.1);*
(iii) $\mathcal{M}^2(n)(\beta)$ is positive and recursively generated (cf. Section 2), there is a column dependence relation $q(X, Y) = 0$, and $\operatorname{card} \mathcal{V}(\mathcal{M}^2(n)(\beta)) \geq \operatorname{rank} \mathcal{M}^2(n)(\beta)$.

Analogues of Theorem 1.7 for all other curves of degree 2 appear in [CuFi5], [CuFi6], [CuFi9], [Fia3]. The full moment problem on a curve of degree 2 had previously been concretely solved in [Sto1] (cf. [StSz1]); an alternate solution appears in [PoSc].

Acknowledgment. Example 1.5 was obtained using calculations with the software tool *Mathematica* [Wol].

2. Moment matrices

Let $\mathbb{C}_r^d[z,\bar{z}]$ denote the space of polynomials with complex coefficients in the indeterminates $z \equiv (z_1,...,z_d)$ and $\bar{z} \equiv (\bar{z}_1,...,\bar{z}_d)$, with total degree at most r; thus $\dim \mathbb{C}_r^d[z,\bar{z}] = \eta(d,r) := \begin{pmatrix} r+2d \\ 2d \end{pmatrix}$. For $i \equiv (i_1,...,i_d) \in \mathbb{Z}_+^d$, let $|i| := i_1 + ... + i_d$ and let $z^i := z_1^{i_1} \cdots z_d^{i_d}$. Given a complex sequence $\gamma \equiv \gamma^{(s)} = \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+^d}$, $|i|+|j| \le s$, the *truncated complex moment problem* for γ entails determining necessary and sufficient conditions for the existence of a positive Borel measure ν on $\mathbb{C}^d$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\nu \ \ (\equiv \int \bar{z}_1^{i_1} \cdots \bar{z}_d^{i_d} z_1^{j_1} \cdots z_d^{j_d} \, d\nu(z_1,...,z_d,\bar{z}_1,...,\bar{z}_d)) \ \ (|i|+|j| \le s). \tag{2.1}$$

A measure ν as in (2.1) is a *representing measure* for $\gamma^{(s)}$; if $K \subseteq \mathbb{C}^d$ is a closed set and $\operatorname{supp}\nu \subseteq K$, then ν is a *K-representing measure* for $\gamma^{(s)}$.

In the sequel we focus on the case when s is even, say $s = 2n$. In this case, the moment data $\gamma^{(2n)}$ determine the *moment matrix* $M(n) \equiv M^d(n)(\gamma)$ that we next describe. The size of $M(n)$ is $\eta(d,n)$, with rows and columns $\{\bar{Z}^i Z^j\}_{i,j\in\mathbb{Z}_+^d, |i|+|j|\le n}$, indexed by the lexicographic ordering of the monomials in $\mathbb{C}_n^d[z,\bar{z}]$; for $d=2, n=2$, this ordering is $1, Z_1, Z_2, \bar{Z}_1, \bar{Z}_2, Z_1^2, Z_1Z_2, \bar{Z}_1Z_1, \bar{Z}_2Z_1, Z_2^2, \bar{Z}_1Z_2, \bar{Z}_2Z_2, \bar{Z}_1^2, \bar{Z}_1\bar{Z}_2, \bar{Z}_2^2$. The entry of $M(n)$ in row $\bar{Z}^i Z^j$, column $\bar{Z}^k Z^\ell$ is $\gamma_{i+\ell,k+j}$ $(|i|+|j|, |k|+|\ell| \le n)$. By a *representing measure for* $M(n)$ we mean a representing measure for γ.

For $p \in \mathbb{C}_n^d[z,\bar{z}]$, $p(z,\bar{z}) \equiv \sum_{r,s\in\mathbb{Z}_+^d, |r|+|s|\le n} a_{rs}\bar{z}^r z^s$, we set $\hat{p} := (a_{rs})$; $\hat{p}$ is the coefficient vector of p relative to the basis for $\mathbb{C}_n^d[z,\bar{z}]$ consisting of the monomials $\{\bar{z}^i z^j\}_{i,j\in\mathbb{Z}_+^d, |i|+|j|\le n}$ in lexicographic order. We recall the *Riesz functional* $\Lambda \equiv \Lambda_\gamma : \mathbb{C}_{2n}^d[z,\bar{z}] \to \mathbb{C}$, defined by $\Lambda(\sum_{r,s\in\mathbb{Z}_+^d, |r|+|s|\le 2n} b_{rs}\bar{z}^r z^s) := \sum_{r,s\in\mathbb{Z}_+^d, |r|+|s|\le 2n} b_{rs}\gamma_{rs}$. The matrix $M^d(n)(\gamma)$ is uniquely determined by

$$\left\langle M^d(n)(\gamma)\hat{f}, \hat{g} \right\rangle := \Lambda_\gamma(f\bar{g}), f, g \in \mathbb{C}_n^d[z,\bar{z}]. \tag{2.2}$$

If γ has a representing measure ν, then $\Lambda_\gamma(f\bar{g}) = \int f\bar{g} \, d\nu$; in particular, $\left\langle M^d(n)(\gamma)\hat{f}, \hat{f} \right\rangle = \int |f|^2 \, d\nu \ge 0$, so $M^d(n)(\gamma)$ is positive semidefinite in this case.

Corresponding to $p \in \mathbb{C}_n^d[z,\bar{z}]$, $p(z,\bar{z}) \equiv \sum a_{rs}\bar{z}^r z^s$ (as above), we may define an element in $\mathcal{C}_{M(n)}$, the column space of $M(n)$, by $p(Z,\bar{Z}) := \sum a_{rs}\bar{Z}^r Z^s$; the following result will be used in the sequel to locate the support of a representing measure.

Proposition 2.1. *(*[CuFi1, (7.4)]*) Suppose ν is a representing measure for $\gamma^{(2n)}$, let $p \in \mathbb{C}_n^d[z,\bar{z}]$, and let $\mathcal{Z}(p) := \{z \in \mathbb{C}^d : p(z,\bar{z}) = 0\}$. Then* $\operatorname{supp}\nu \subseteq \mathcal{Z}(p)$ *if and only if* $p(Z,\bar{Z}) = 0$.

It follows from Proposition 2.1 that if $\gamma^{(2n)}$ has a representing measure, then $M^d(n)(\gamma)$ is *recursively generated* in the following sense:

$$p, q, pq \in \mathbb{C}_n^d[z,\bar{z}], p(Z,\bar{Z}) = 0 \Rightarrow (pq)(Z,\bar{Z}) = 0. \tag{2.3}$$

We define the *variety* of $M(n)$ by $\mathcal{V}(M(n)) := \bigcap_{p\in\mathbb{C}_n^d[z,\bar{z}], p(Z,\bar{Z})=0} \mathcal{Z}(p)$; we sometimes refer to $\mathcal{V}(M(n))$ as $\mathcal{V}(\gamma)$. Proposition 2.1 implies that if ν is a representing measure for $\gamma^{(2n)}$, then $\operatorname{supp}\nu \subseteq \mathcal{V}(\gamma)$ and, moreover, that

$$\operatorname{card}\mathcal{V}(\gamma) \ge \operatorname{card}\operatorname{supp}\nu \ge \operatorname{rank} M^d(n)(\gamma) \ \ (\text{cf. [CuFi1, (7.6)]}). \tag{2.4}$$

The following result characterizes the existence of "minimal," i.e., $\operatorname{rank} M(n)$-atomic, representing measures.

Theorem 2.2. *([CuFi1, Corollary 7.9 and Theorem 7.10]) $\gamma^{(2n)}$ has a $\operatorname{rank} M^d(n)(\gamma)$-atomic representing measure if and only if $M(n) \equiv M^d(n)(\gamma)$ is positive semidefinite and $M(n)$ admits an extension to a moment matrix $M(n+1) \equiv M^d(n+1)(\tilde{\gamma})$ satisfying $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$. In this case, $M(n+1)$ admits unique successive rank-preserving positive moment matrix extensions $M(n+2), M(n+3), \ldots$, and there exists a $\operatorname{rank} M(n)$-atomic representing measure for $M(\infty)$.*

Various concrete sufficient conditions are known for the existence of the rank-preserving extension $M(n+1)$ described in Theorem 2.2, particularly when $d = 1$ (moment problems in the plane) [CuFi1], [CuFi2], [CuFi3], [CuFi5], [CuFi6], [CuFi7], [CuFi9]; for general d, an important sufficient condition is that $M^d(n)(\gamma)$ is positive semidefinite and *flat*, i.e., $\operatorname{rank} M^d(n)(\gamma) = \operatorname{rank} M^d(n-1)(\gamma)$ [CuFi1, Theorem 7.8].

We now present the complex version of Theorem 1.2.

Theorem 2.3. *If $M(n) \equiv M^d(n) \geq 0$ admits a rank-preserving extension $M(n+1)$, then $\mathcal{V} := \mathcal{V}(M(n+1))$ satisfies $\operatorname{card} \mathcal{V} = r$ $(\equiv \operatorname{rank} M(n))$, and $\mathcal{V} \equiv \{\omega_j\}_{j=1}^r$ forms the support of the unique representing measure ν for $M(n+1)$. If $\mathcal{B} \equiv \{\bar{Z}^{i_k} Z^{j_k}\}_{k=1}^r$ is a maximal linearly independent subset of columns of $M(n)$, then the $r \times r$ matrix $W_{\mathcal{B},\mathcal{V}}$ (whose entry in row m, column k is $\bar{\omega}_k^{i_m} \omega_k^{j_m}$) is invertible, and $\nu = \sum_{j=1}^r \rho_j \delta_{\omega_j}$, where $\rho \equiv (\rho_1, \ldots, \rho_r)$ is uniquely determined by $\rho^t = W_{\mathcal{B},\mathcal{V}}^{-1} (\gamma_{i_1,j_1}, \ldots, \gamma_{i_r,j_r})^t$.*

Toward the proof of Theorem 2.3, we begin with some remarks concerning positive matrix extensions. Let $\tilde{A} \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a block matrix. A result of Smul'jan [Smu] shows that $\tilde{A} \geq 0$ if and only if $A \geq 0$ and there exists a matrix W such that $B = AW$ and $C \geq W^* A W$. In this case, $W^* A W$ is independent of W satisfying $B = AW$, and the matrix $[A; B] := \begin{pmatrix} A & B \\ B^* & W^* A W \end{pmatrix}$ is positive and satisfies $\operatorname{rank}[A; B] = \operatorname{rank} A$; conversely, any rank-preserving positive extension $\tilde{A}$ of A is of this form. We refer to such a rank-preserving extension as a *flat extension* of A. Now, a moment matrix $M^d(n+1)$ admits a block decomposition $M(n+1) = \begin{pmatrix} M(n) & B(n+1) \\ B(n+1)^* & C(n+1) \end{pmatrix}$; thus a positive moment matrix $M(n)$ admits a flat (positive) moment matrix extension $M(n+1)$ if and only if there is a choice of moments of degree $2n+1$ and a matrix W such that $B(n+1) = M(n)W$ and $W^* M(n) W$ has the form of a moment matrix block $C(n+1)$, i.e., $[M(n); B(n+1)]$ is a moment matrix.

Consider again a positive extension $\tilde{A}$ of A (as above). The Extension Principle ([CuFi1, Proposition 3.9], [Fia1, Proposition 2.4]) implies that each linear dependence relation in the columns of A extends to the columns of $\begin{pmatrix} A \\ B^* \end{pmatrix}$ in $\tilde{A}$. In the case when $M(n+1)$ is a positive extension of $M(n)$, it follows that $\mathcal{V}(M(n+1)) \subseteq \mathcal{V}(M(n))$; we will use this relation frequently in the sequel, without further reference.

Now recall from Theorem 2.2 that if $M(n) \geq 0$ admits a flat extension $M(n+1)$, then $M(n+1)$ admits a unique flat extension $M(n+2)$. Indeed, every column of $M(n+1)$ of total degree $n+1$ is a linear combination of columns corresponding to monomials of total degree at most n; we can write this as

$$\bar{Z}^i Z^j = p_{i,j}(Z, \bar{Z}) \quad (p_{i,j} \in \mathbb{C}_n^d[z, \bar{z}]; \ |i| + |j| = n+1). \tag{2.5}$$

Then the unique flat extension $M(n+2)$ is given by

$$\bar{Z}^i Z^j = \begin{cases} (z_\ell p_{i,j-\varepsilon(\ell)})(Z,\bar{Z}) & \text{if } j_\ell \geq 1 \text{ for some } \ell = 1,...,d \\ (\bar{z}_k p_{i-\varepsilon(k),0})(Z,\bar{Z}) & \text{if } j=0 \text{ and } i_k \geq 1 \text{ for some } k=1,...,d \end{cases} \tag{2.6}$$

($|i|+|j|=n+2$), where $\varepsilon(\ell) := (0,...,0,\overset{\ell}{1},0,...,0)$. ($\bar{Z}^i Z^j$ is independent of the choice of j_ℓ or i_k; cf. [CuFi1, Theorem 7.8].)

Suppose $M(n) \geq 0$ admits a flat extension $M(n+1)$; the following result implies that the unique rank-preserving extensions $M(n+2), M(n+3), ...,$ are also variety-preserving; this is a key ingredient in the proof of Theorem 1.2 and may be of independent interest.

Theorem 2.4. *Assume that $M(n) \equiv M^d(n) \geq 0$ admits a flat extension $M(n+1)$. Then $\mathcal{V}(M(n+2)) = \mathcal{V}(M(n+1))$.*

Proof. Recall that $\mathcal{V}(M(n+2)) \subseteq \mathcal{V}(M(n+1))$; to prove the reverse inclusion, it suffices to show that if $\omega \in \mathcal{V}(M(n+1))$, and $f \in \mathbb{C}^d_{n+2}[z,\bar{z}]$ satisfies $f(Z,\bar{Z})=0$ in $\mathcal{C}_{M(n+2)}$, then $f(\omega,\bar{\omega})=0$. As discussed above, the flat extension $M(n+2)$ admits a decomposition

$$M(n+2) = \begin{pmatrix} M(n+1) & M(n+1)W \\ W^*M(n+1) & W^*M(n+1)W \end{pmatrix}.$$

Write $f = g+h$, where $g \in \mathbb{C}^d_{n+1}[z,\bar{z}]$, and $h(z,\bar{z}) \equiv \sum_{|i|+|j|=n+2} h_{i,j}\bar{z}^i z^j$. Recall that $\hat{f} \in \mathbb{C}^{\eta(d,n+2)}$ and $\hat{g} \in \mathbb{C}^{\eta(d,n+1)}$ denote the coefficient vectors of f and g relative to the bases of monomials in lexicographic order. Let $\tilde{h} \in \mathbb{C}^{\eta(d,n+2)-\eta(d,n+1)}$ denote the coefficient vector of h relative to the monomials of degree $n+2$ in lexicographic order; thus $\hat{f} = \begin{pmatrix} \hat{g} \\ \tilde{h} \end{pmatrix}$. Now,

$$f(Z,\bar{Z}) = M(n+2)\hat{f} = \begin{pmatrix} M(n+1)\hat{g} + M(n+1)W\tilde{h} \\ W^*M(n+1)\hat{g} + W^*M(n+1)W\tilde{h} \end{pmatrix},$$

so $f(Z,\bar{Z})=0$ implies

$$M(n+1)(\hat{g}+W\tilde{h}) = 0. \tag{2.7}$$

We seek to associate $\hat{g}+W\tilde{h}$ with the coefficient vector $\hat{q}$ of some polynomial $q \in \mathbb{C}^d_{n+1}[z,\bar{z}]$, and to this end we first describe an explicit formula for W.

Recall that $M(n+1)W = B(n+2)$, and that the columns of $B(n+2)$ are associated with the monomials $\bar{z}^i z^j$ $((|i|+|j|=n+2)$. For $(i,j) \in \mathbb{Z}^d_+ \times \mathbb{Z}^d_+$ with $|i|+|j|=n+2$, the (i,j)-th column of $B(n+2)$ is, on one hand $M(n+1)W\widetilde{\bar{z}^i z^j}$, while on the other hand it equals $\left[(z_\ell p_{i,j-\varepsilon(\ell)})(Z,\bar{Z})\right]_{\eta(d,n+1)}$ or $\left[(\bar{z}_k p_{i-\varepsilon(k),0})(Z,\bar{Z})\right]_{\eta(d,n+1)}$, by (2.6). Since the polynomials $z_\ell p_{i,j-\varepsilon(\ell)}$ and $\bar{z}_k p_{i-\varepsilon(k),0}$ belong to $\mathbb{C}^d_{n+1}[z,\bar{z}]$, we can write

$$\left[(z_\ell p_{i,j-\varepsilon(\ell)})(Z,\bar{Z})\right]_{\eta(d,n+1)} = M(n+1)(z_\ell p_{i,j-\varepsilon(\ell)})\hat{}$$

and

$$\left[(\bar{z}_k p_{i-\varepsilon(k),0})(Z,\bar{Z})\right]_{\eta(d,n+1)} = M(n+1)(\bar{z}_k p_{i-\varepsilon(k),0})\hat{}.$$

It follows at once that W can be given by

$$W\widetilde{\bar{z}^i z^j} = \begin{cases} (z_\ell p_{i,j-\varepsilon(\ell)})\hat{} & \text{if } j_\ell \geq 1 \text{ for some } \ell=1,...,d \\ (\bar{z}_k p_{i-\varepsilon(k),0})\hat{} & \text{if } j=0 \text{ and } i_k \geq 1 \text{ for some } k=1,...,d \end{cases} \quad (|i|+|j|=n+2). \tag{2.8}$$

We now consider $W\tilde{h}$. Since $\tilde{h} \equiv \sum_{|i|+|j|=n+2} h_{i,j} \widetilde{\bar{z}^i z^j}$, it follows from (2.8) that

$$\begin{aligned} W\tilde{h} &= \sum_{|i|+|j|=n+2, j\neq 0} h_{i,j} (z_\ell p_{i,j-\varepsilon(\ell)})\hat{} + \sum_{|i|=n+2} h_{i,0} (\bar{z}_k p_{i-\varepsilon(k),0})\hat{} \\ &= [\sum_{|i|+|j|=n+2, j\neq 0} h_{i,j} z_\ell p_{i,j-\varepsilon(\ell)} + \sum_{|i|=n+2} h_{i,0} \bar{z}_k p_{i-\varepsilon(k),0}]\hat{}. \end{aligned}$$

Now we set

$$\begin{aligned} q(z,\bar{z}) &:= g(z,\bar{z}) + \sum_{|i|+|j|=n+2, j\neq 0} h_{i,j} (z_\ell p_{i,j-\varepsilon(\ell)})(z,\bar{z}) \\ &+ \sum_{|i|=n+2} h_{i,0} (\bar{z}_k p_{i-\varepsilon(k),0})(z,\bar{z}) \in \mathbb{C}^d_{n+1}[z,\bar{z}]. \end{aligned}$$

Observe that in $\mathcal{C}_{M(n+1)}$,

$$\begin{aligned} q(Z,\bar{Z}) &= M(n+1)\hat{q} = M(n+1)\hat{g} + M(n+1)[\sum_{|i|+|j|=n+2, j\neq 0} h_{i,j} z_\ell p_{i,j-\varepsilon(\ell)} \\ &+ \sum_{|i|=n+2} h_{i,0} \bar{z}_k p_{i-\varepsilon(k),0}]\hat{} \\ &= M(n+1)\hat{g} + M(n+1)W\tilde{h} = M(n+1)(\hat{g} + W\tilde{h}) = 0 \ \text{(by (2.7))}. \end{aligned}$$

Thus, $q \in \mathbb{C}^d_{n+1}[z,\bar{z}]$ and $q(Z,\bar{Z}) = 0$. Since $\omega \in \mathcal{V}(M(n+1))$, we must have $q(\omega,\bar{\omega}) = 0$. Therefore,

$$\begin{aligned} 0 &= g(\omega,\bar{\omega}) + \sum_{|i|+|j|=n+2, j\neq 0} h_{i,j} \omega_\ell p_{i,j-\varepsilon(\ell)}(\omega,\bar{\omega}) \\ &+ \sum_{|i|=n+2} h_{i,0} \bar{\omega}_k p_{i-\varepsilon(k),0}(\omega,\bar{\omega}). \end{aligned} \tag{2.9}$$

Let $r_{i,j}(z,\bar{z}) := \bar{z}^i z^j - p_{i,j}(z,\bar{z})$ $(|i|+|j| = n+1)$. Clearly each $r_{i,j} \in \mathbb{C}^d_{n+1}[z,\bar{z}]$ and $r_{i,j}(Z,\bar{Z}) = 0$ by (2.5), so $r_{i,j}(\omega,\bar{\omega}) = 0$ $(|i|+|j| = n+1)$. Multiplying $r_{i,j}(\omega,\bar{\omega}) = 0$ by either ω_ℓ or $\bar{\omega}_k$, it follows that

$$\begin{cases} \bar{\omega}^i \omega^j = (z_\ell p_{i,j-\varepsilon(\ell)})(\omega,\bar{\omega}) & (|i|+|j| = n+2,\ j_\ell \geq 1 \text{ for some } \ell = 1,...,d \\ \bar{\omega}^i = (\bar{z}_k p_{i-\varepsilon(k),0})(\omega,\bar{\omega}) & (|i| = n+2,\ j = 0,\ i_k \geq 1 \text{ for some } k = 1,...,d \end{cases}.$$

Now (2.9) becomes

$$\begin{aligned} 0 &= g(\omega,\bar{\omega}) + \sum_{|i|+|j|=n+2, j\neq 0} h_{i,j} \bar{\omega}^i \omega^j + \sum_{|i|=n+2} h_{i,0} \bar{\omega}^i \\ &= g(\omega,\bar{\omega}) + h(\omega,\bar{\omega}) = f(\omega,\bar{\omega}). \end{aligned}$$

Thus, $f(\omega,\bar{\omega}) = 0$, as desired. □

Lemma 2.5. *Assume that $M(n) \equiv M^d(n) \geq 0$ admits an r-atomic representing measure ν, where $r := \operatorname{rank} M(n)$, and let $\mathcal{V} := \operatorname{supp} \nu$. If $\mathcal{B} \equiv \{\bar{Z}^{i_k} Z^{j_k}\}_{k=1}^r$ is a maximal linearly independent subset of columns of $M(n)$, then $W_{\mathcal{B},\mathcal{V}}$ is invertible (cf. Theorem 2.3).*

Proof. Let $R_1, ..., R_r$ denote the rows of $W_{\mathcal{B},\mathcal{V}}$, and assume that $W_{\mathcal{B},\mathcal{V}}$ is singular. Then there exists scalars $c_1, ..., c_r \in \mathbb{C}$, not all zero, such that $c_1 R_1 + ... + c_r R_r = 0$. Let $p(z,\bar{z}) := c_1 z^{i_1} \bar{z}^{j_1} + ... + c_r z^{i_r} \bar{z}^{j_r}$. Clearly, $p|_{\operatorname{supp}\nu} \equiv 0$, so Proposition 2.1 implies that $p(Z,\bar{Z}) = 0$. Then $c_1 Z^{i_1} \bar{Z}^{j_1} + ... + c Z^{i_r} \bar{Z}^{j_r} = 0$ in $\mathcal{C}_{M(n)}$, contradicting the fact that $\mathcal{B}$ is linearly independent. □

Proof of Theorem 2.3. Let $r := \operatorname{rank} M(n)$; we first show that $\mathcal{V} \equiv \mathcal{V}(M(n+1))$ satisfies $\operatorname{card} \mathcal{V} = r$. Theorem 2.2 implies that $M(n+1)$ admits a unique flat extension $M(\infty)$ and that $M(\infty)$ admits an r-atomic representing measure η. Write $\operatorname{supp} \eta \equiv \{\omega_1, ..., \omega_r\}$, and define $p \in \mathbb{C}^d_{2r}[z, \bar{z}]$ by $p(z, \bar{z}) := \prod_{i=1}^r \|z - \omega_i\|^2$ (where, for $z \equiv (z_1, ..., z_d)$, $\|z\|^2 := \sum_{j=1}^d \bar{z}_j z_j \in \mathbb{C}^d_2[z, \bar{z}]$). Clearly, $\mathcal{Z}(p) = \operatorname{supp} \eta$, and since η is a representing measure for $M(2r)$, Proposition 2.1 implies $p(Z, \bar{Z}) = 0$ in $\mathcal{C}_{M(2r)}$. Thus $\mathcal{V}(M(2r)) \subseteq \mathcal{Z}(p)$ and $\operatorname{card} \mathcal{V}(M(2r)) \leq \operatorname{card} \mathcal{Z}(p) = r$. To show that $\operatorname{card} \mathcal{V} = r$, we consider two cases. If $2r \leq n$, then, since η is a representing measure for $M(n+1)$, $\operatorname{supp} \eta \subseteq \mathcal{V}(M(n+1)) \subseteq \mathcal{V}(M(n)) \subseteq \mathcal{V}(M(2r)) \subseteq \mathcal{Z}(p) = \operatorname{supp} \eta$, whence $\operatorname{supp} \eta = \mathcal{V}$ and $\operatorname{card} \mathcal{V} = r$. If $2r \geq n+1$, repeated application of Theorem 2.4 implies that $\mathcal{V} \equiv \mathcal{V}(M(n+1)) = \mathcal{V}(M(n+2)) = ... = \mathcal{V}(M(2r))$, and since η is a representing measure for $M(n+1)$, (2.4) implies

$$r = \operatorname{rank} M(n+1) \leq \operatorname{card} \mathcal{V}(M(n+1)) = ... = \operatorname{card} \mathcal{V}(M(2r)). \tag{2.10}$$

Now, from above, $\operatorname{card} \mathcal{V}(M(2r)) \leq r$, so (2.10) implies that $\operatorname{card} \mathcal{V} = r$ in this case too.

Now let ν be a representing measure for $M(n+1)$. Then $r = \operatorname{rank} M(n+1) \leq \operatorname{card} \operatorname{supp} \nu \leq \operatorname{card} \mathcal{V} = r$, and since $\operatorname{supp} \nu \subseteq \mathcal{V}$, it follows that $\operatorname{supp} \nu = \mathcal{V}$, whence $\nu = \sum_{i=1}^r \rho_i \delta_{\omega_i}$, for some densities $\rho_1, ..., \rho_r$. Since ν is a representing measure for $M(n)$, $\rho \equiv (\rho_1, ..., \rho_r)$ satisfies $W_{\mathcal{B},\mathcal{V}} \rho^t = (\gamma_{i_1,j_1}, ..., \gamma_{i_r,j_r})^t$, and since $W_{\mathcal{B},\mathcal{V}}$ is invertible by Lemma 2.5, ρ is uniquely determined. Thus ν is the unique representing measure for $M(n+1)$. $\square$

In [CuFi1, Theorem 7.7] we proved that a finite rank positive infinite moment matrix $M \equiv M^d(\infty)$ has a $\operatorname{rank} M$-atomic representing measure; for $d = 1$ we established uniqueness in [CuFi1, Theorem 4.7]. We can now establish uniqueness for arbitrary d.

Corollary 2.6. *A finite rank positive moment matrix $M \equiv M^d(\infty)$ has a unique representing measure ν, and* $\operatorname{card} \operatorname{supp} \nu = \operatorname{rank} M$.

Proof. Following [CuFi1, Theorem 7.7], let η be a $\operatorname{rank} M$-atomic representing measure for M. Let j be the smallest integer such that $\operatorname{rank} M(j) = \operatorname{rank} M(j+1)$. Theorem 2.3 implies that $M(j+1)$ has a unique representing measure ν, whence $\eta = \nu$ and $\operatorname{card} \operatorname{supp} \nu = \operatorname{rank} M$. $\square$

Remark 2.7. The measure ν in Corollary 2.6 may be computed using Theorem 2.3; indeed, $\operatorname{supp} \nu = \mathcal{V}(M(j+1))$.

In order to study moment problems on $\mathbb{R}^N$, we next introduce real moment matrices. Let $\mathbb{C}^N[t] \equiv \mathbb{C}[t_1, ..., t_N]$ denote the space of complex polynomials in N real variables, and let $\mathbb{C}^N_s[t]$ denote the polynomials of degree at most s; then $\dim \mathbb{C}^N_s[t] = \begin{pmatrix} N+s \\ s \end{pmatrix}$. For $t \equiv (t_1, ..., t_N) \in \mathbb{R}^N$ and $i \equiv (i_1, ..., i_N) \in \mathbb{Z}^N_+$, we set $t^i := t_1^{i_1} \cdots t_N^{i_N}$. Given a real sequence $\beta \equiv \beta^{(r)} = \{\beta_i\}_{i \in \mathbb{Z}^N_+, |i| \leq r}$, the truncated moment problem for β concerns conditions for the existence of a positive Borel measure μ on $\mathbb{R}^N$ satisfying

$$\beta_i = \int t^i \, d\mu(t) \ (\equiv \int t_1^{i_1} \cdots t_N^{i_N} \, d\mu(t_1, ..., t_N)) \ \ (|i| \leq r). \tag{2.11}$$

A measure μ satisfying (2.11) is a *representing measure* for β; if, in addition, $K \subseteq \mathbb{R}^N$ is closed and $\operatorname{supp} \mu \subseteq K$, then μ is a *K-representing measure* for β.

Let $r = 2n$; in this case $\beta^{(2n)}$ corresponds to a *real moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}^N(n)(\beta)$, defined as follows. Let $\mathcal{B} \equiv \{t^i\}_{i \in \mathbb{Z}^N_+, |i| \leq n}$ denote the basis of monomials in $\mathbb{C}^N[t]$, ordered lexicographically; e.g., for $N = 3$, $n = 2$, this ordering is $1, t_1, t_2, t_3, t_1^2, t_1 t_2, t_1 t_3, t_2^2, t_2 t_3, t_3^2$. The size of $\mathcal{M}(n)$ is $\dim \mathbb{C}^N_n[t]$ $(= \begin{pmatrix} N+n \\ n \end{pmatrix})$, with rows and columns indexed as $\{T^i\}_{i \in \mathbb{Z}^N_+, |i| \leq n}$, following the same lexicographic order as above. The entry of $\mathcal{M}(n)$ in row T^i, column T^j is β_{i+j}, $i, j \in \mathbb{Z}_+^N$, $|i|+|j| \le 2n$. Note that for $N = 1$, $\mathcal{M}^N(n)(\beta)$ is the Hankel matrix (β_{i+j}) associated with the classical Hamburger moment problem $(K = \mathbb{R})$ (cf. [Akh]).

For $p \in \mathbb{C}_n^N[t]$, $p(t) \equiv \sum_{i \in \mathbb{Z}_+^N, |i| \le n} a_i t^i$, we let $\tilde{p} := (a_i)$ denote the coefficient vector of p relative to $\mathcal{B}$. The Riesz functional $\Lambda_\beta : \mathbb{C}_{2n}^N[t] \to \mathbb{C}$ is defined by $\Lambda_\beta(\sum b_r t^r) := \sum b_r \beta_r$. Thus, $\mathcal{M}^N(n)(\beta)$ is uniquely determined by

$$\left\langle \mathcal{M}^N(n)(\beta)\tilde{f}, \tilde{g} \right\rangle := \Lambda_\beta(f\bar{g}) \ \ (f, g \in \mathbb{C}_n^N[t]). \tag{2.12}$$

If $\beta^{(2n)}$ has a representing measure μ, then $\Lambda_\beta(f\bar{g}) = \int f\bar{g}\, d\mu$, so $\mathcal{M}^N(n)(\beta)$ is positive semidefinite.

For $p \equiv \sum_{r \in \mathbb{Z}_+^N, |r| \le n} a_r t^r$, we define an element in $\mathcal{C}_{\mathcal{M}(n)}$ (the column space of $\mathcal{M}(n)$) by $p(T) := \sum_{r \in \mathbb{Z}_+^N, |r| \le n} a_r T^r$. Let $\mathcal{V}(\mathcal{M}(n)) := \bigcap_{\substack{p \in C_n^N[t] \\ p(T)=0}} \mathcal{Z}(p)$ denote the *variety* of $\mathcal{M}(n)$; we also denote this variety by $\mathcal{V}(\beta)$. Let $J \equiv J(n) := \{j \in \mathbb{Z}_+^N : |j| \le n\}$; thus $\operatorname{card} J(n) = \operatorname{size} \mathcal{M}(n)$. Let $s := \operatorname{size} \mathcal{M}(n) - \operatorname{rank} \mathcal{M}(n)$; the following result, which proves Proposition 1.3, identifies s polynomials in $\mathbb{R}_n^N[t]$ whose common zeros comprise $\mathcal{V}(\mathcal{M}(n))$.

Proposition 2.8. *Let $\mathcal{M}(n)$ be a real moment matrix, with columns T^j indexed by $j \in J$, let $r := \operatorname{rank} \mathcal{M}(n)$, and let $\mathcal{B} \equiv \{T^i\}_{i \in I}$ be a maximal linearly independent set of columns of $\mathcal{M}(n)$, where $I \subseteq J$ satisfies $\operatorname{card} I = r$. For each index $j \in J \setminus I$, let q_j denote the unique polynomial in lin.span $\{t^i\}_{i \in I}$ such that $T^j = q_j(T)$, and let $r_j(t) := t^j - q_j(t)$. Then $\mathcal{V}(\mathcal{M}(n))$ is precisely the set of common zeros of $\{r_j\}_{j \in J \setminus I}$.*

Proof. Clearly $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n)) \subseteq \bigcap_{j \in J} \mathcal{Z}(r_j)$. For the reverse inclusion, set $\mathbb{R}_n^N[t] := \{p \in \mathbb{R}^N[t] : \deg p \le n\}$ and let $\Phi : \mathbb{R}_n^N[t] \to \mathcal{C}_{\mathcal{M}(n)}$ denote the map $p \mapsto p(Z, \bar{Z})$. Φ is linear and surjective, so $\dim \ker \Phi = \dim \mathbb{R}_n^N[t] - \dim \mathcal{C}_{\mathcal{M}(n)} = \operatorname{card} J - \operatorname{card} I$. Observe now that for $j \in J \setminus I$, since $T^j = q_j(T)$ we have $r_j \in \ker \Phi$. Moreover, for $j \in J \setminus I$, the monomial t^j only appears in r_j, so it is straightforward to verify that $\{r_j\}_{j \in J \setminus I}$ is a linearly independent subset of $\mathbb{R}_n^N[t]$. It follows at once that $\{r_j\}_{j \in J \setminus I}$ is a basis for $\ker \Phi$, whence $\bigcap_{j \in J} \mathcal{Z}(r_j) \subseteq \bigcap_{p \in \ker \Phi} \mathcal{Z}(p) = \mathcal{V}$. $\square$

Remark 2.9. Proposition 2.8 admits an exact analogue for complex moment matrices.

We omit the proofs of the following results, which are analogous to the corresponding proofs for $M^d(n)(\gamma)$.

Proposition 2.10. *Suppose μ is a representing measure for $\beta^{(2n)}$. For $p \in \mathbb{C}_n^N[t]$, $\operatorname{supp} \mu \subseteq \mathcal{Z}(p) := \{t \in \mathbb{R}^N : p(t) = 0\}$ if and only if $p(T) = 0$.*

Corollary 2.11. *If $\beta^{(2n)}$ has a representing measure, then $\mathcal{M}^N(n)(\beta)$ is recursively generated, i.e., if $p, q, pq \in \mathbb{C}_n^N[t]$ and $p(T) = 0$, then $(pq)(T) = 0$.*

Corollary 2.12. *If μ is a representing measure for $\beta^{(2n)}$, then $\operatorname{supp} \mu \subseteq \mathcal{V}(\beta)$ and $\operatorname{card} \mathcal{V}(\beta) \ge \operatorname{card} \operatorname{supp} \mu \ge \operatorname{rank} \mathcal{M}^N(n)(\beta)$.*

We devote the remainder of this section to describing an equivalence between truncated moment problems on $\mathbb{R}^{2d}$ and $\mathbb{C}^d$. In the sequel, $\mathcal{C}^{(n)}$ denotes the ordered basis for $\mathbb{C}_n^d[z, \bar{z}]$ consisting of the monomials, ordered lexicographically by degree. We denote the coefficient vector of $p \in \mathbb{C}_n^d[z, \bar{z}]$ relative to $\mathcal{C}^{(n)}$ by $\hat{p}$; thus $\mathcal{K}^{(n)} := \{\hat{p} : p \in \mathbb{C}_n^d[z, \bar{z}]\} \cong \mathbb{C}^\eta \cong \mathbb{C}_n^d[z, \bar{z}]$. For $0 \le j \le n$, let $\mathcal{K}_j$ denote the subspace of $\mathcal{K}^{(n)}$ spanned by elements $\widehat{\bar{z}^r z^s}$ with $|r| + |s| = j$; thus $\mathcal{K}^{(n)} = \mathcal{K}^{(n-1)} \bigoplus \mathcal{K}_n \equiv \mathcal{K}_0 \bigoplus ... \bigoplus \mathcal{K}_n$, and $\dim \mathcal{K}_j = \binom{j-1+2d}{2d-1}$ $(0 \le j \le n)$.

Next, let $\mathbb{C}_n^{2d}[t] \equiv \mathbb{C}_n[t_1, ..., t_{2d}]$ denote the vector space over $\mathbb{C}$ of polynomials in real indeterminates $t_1, ..., t_{2d}$ with total degree at most n. For $i \equiv (i_1, ..., i_{2d}) \in \mathbb{Z}_+^{2d}$, $|i| \le n$, let $t^i := t_1^{i_1} \cdot ... \cdot t_{2d}^{i_{2d}}$; thus $q \in \mathbb{C}_n^{2d}[t]$ may be expressed as $q(t) \equiv \sum_{|i| \le n} b_i t^i$. Note that $\dim \mathbb{C}_n^{2d}[t] = \eta(d, n)$. In the sequel, $\mathcal{B}^{(n)}$ denotes the ordered basis for $\mathbb{C}_n^{2d}[t]$ consisting of the monomials, ordered lexicographically by degree; for $d = n = 2$, this ordering is $1, t_1, t_2, t_3, t_4, t_1^2, t_1t_2, t_1t_3, t_1t_4, t_2^2, t_2t_3, t_2t_4, t_3^2, t_3t_4, t_4^2$. Now we set $x_i := t_i$ $(1 \le i \le d)$ and $y_i := t_{i+d}$ $(1 \le i \le d)$, so that $\mathbb{C}_n^{2d}[t] = \mathbb{C}_n^d[x, y] := \mathbb{C}_n[x_1, ..., x_d; y_1, ..., y_d]$; with this notation, for $d = n = 2$ the basis $\mathcal{B}^{(2)}$ assumes the form $1, x_1, x_2, y_1, y_2, x_1^2$, $x_1x_2, y_1x_1, y_2x_1, x_2^2$, $y_1x_2, y_2x_2, y_1^2, y_1y_2, y_2^2$. We denote the coefficient vector of $q \in \mathbb{C}_n^d[x, y]$ relative to $\mathcal{B}^{(n)}$ by $\tilde{q}$; thus $\mathcal{H}^{(n)} := \{\tilde{q} : q \in \mathbb{C}_n^{2d}[x, y]\} \cong \mathbb{C}^\eta \cong \mathbb{C}_n^{2d}[t]$. For $0 \le j \le n$, let $\mathcal{H}_j$ denote the subspace of $\mathcal{H}^{(n)}$ spanned by elements $\widetilde{y^r x^s}$ with $|r| + |s| = j$; thus $\mathcal{H}^{(n)} = \mathcal{H}^{(n-1)} \bigoplus \mathcal{H}_n \equiv \mathcal{H}_0 \bigoplus ... \bigoplus \mathcal{H}_n$, and $\dim \mathcal{H}_j = \binom{j-1+2d}{2d-1}$ $(0 \le j \le n)$.

For $0 \le j \le n$, we define a linear map $L_j : \mathcal{K}_j \to \mathcal{H}_j$ by $L_j(\widehat{\bar{z}^k z^\ell}) := [(x - iy)^k (x + iy)^\ell]\tilde{}$ $(|k| + |\ell| = j)$. Since $(x - iy)^k (x + iy)^\ell \equiv (x_1 - iy_1)^{k_1} \cdot ... \cdot (x_d - iy_d)^{k_d} (x_1 + iy_1)^{\ell_1} \cdot ... \cdot (x_d + iy_d)^{\ell_d}$, the Binomial Theorem shows that $L_j(\widehat{\bar{z}^k z^\ell})$ is indeed an element of $\mathcal{H}_j$. We now define $L \equiv L^{(n)} : \mathcal{K}^{(n)} \to \mathcal{H}^{(n)}$ by $L := \bigoplus_{k=0}^n L_k$ $(= L^{(n-1)} \bigoplus L_n)$. For $d = n = 2$, we have

$$L_0 = (1), \; L_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 & -2i & 0 & 0 \\ 0 & i & 0 & -i & 0 & i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & i & 0 & i & 0 & -i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 2i & 0 & 0 & 0 & 0 & -2i \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

and $L^{(2)} = L_0 \bigoplus L_1 \bigoplus L_2 = L^{(1)} \bigoplus L_2$. To clarify the properties of L we introduce the map $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}^d \times \mathbb{C}^d$ defined by $\psi(x, y) := (z, \bar{z})$, where $z \equiv x + iy$, $\bar{z} \equiv x - iy \in \mathbb{C}^d$. Clearly ψ is injective, and we let
$\tau : \operatorname{Ran} \psi \to \mathbb{R}^d \times \mathbb{R}^d$ denote the inverse map, $\tau(z, \bar{z}) := (\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i})$.

Lemma 2.13. *(i)* $L\hat{p} := \widetilde{p \circ \psi}$ *(*$p \in \mathbb{C}_n^d[z, \bar{z}]$*).*
(ii) L *is invertible, with* $L^{-1}(\tilde{q}) = (q \circ \tau)\hat{}$.

Proof. (i) For $p \in \mathbb{C}_n^d[z, \bar{z}]$, write $p(z, \bar{z}) \equiv \sum_{|k|+|\ell| \le n} a_{k\ell} \bar{z}^k z^\ell$. Then

$$\begin{aligned} L(\hat{p}) &= \sum_{|k|+|\ell| \le n} a_{k\ell} L(\widehat{\bar{z}^k z^\ell}) = \sum_{|k|+|\ell| \le n} a_{k\ell} [(x - iy)^k (x + iy)^\ell]\tilde{} \\ &= \sum_{|k|+|\ell| \le n} a_{k\ell} [\bar{z}^k z^\ell \circ \psi]\tilde{} = [(\sum_{|k|+|\ell| \le n} a_{k\ell} \bar{z}^k z^\ell) \circ \psi]\tilde{} = \widetilde{p \circ \psi}. \end{aligned}$$

(ii) A calculation shows that $R_j := L_j^{-1} : \mathcal{H}_j \to \mathcal{K}_j$ is given by $R_j(\widetilde{y^r x^s}) := [(\frac{z-\bar{z}}{2i})^r(\frac{z+\bar{z}}{2})^s]\hat{} \equiv [(\frac{z_1-\bar{z}_1}{2i})^{r_1}\cdot\ldots\cdot(\frac{z_d-\bar{z}_d}{2i})^{r_d}(\frac{z_1+\bar{z}_1}{2})^{s_1}\cdot\ldots\cdot(\frac{z_d+\bar{z}_d}{2})^{s_d}]\hat{}$; thus $L^{-1} : \mathcal{H}^{(n)} \to \mathcal{K}^{(n)}$ satisfies $L^{-1}(\tilde{q}) = (q\circ\tau)\hat{}$. $\square$

Our next goal is to associate to a complex sequence $\gamma \equiv \gamma^{(2n)} = \{\gamma_{rs}\}_{r,s\in\mathbb{Z}_+^d,|r|+|s|\le 2n}$, with $\gamma_{00} > 0$ and $\gamma_{rs} = \bar{\gamma}_{sr}$, an "equivalent" real sequence $\beta \equiv \beta^{(2n)} = \{\beta_j\}_{j\in\mathbb{Z}_+^{2d},|j|\le 2n}$, with $\beta_0 = \gamma_{00}$. We require the following lemma.

Lemma 2.14. *Let $p \equiv \sum a_{rs}\bar{z}^r z^s \in \mathbb{C}_{2n}^d[z,\bar{z}]$ and assume that p is real-valued. Then $\Lambda_\gamma(p)$ is real.*

Proof. Recall that $\Lambda_\gamma(p) = \sum a_{rs}\gamma_{rs}$. Then $\overline{\Lambda_\gamma(p)} = \sum \bar{a}_{rs}\bar{\gamma}_{rs} = \sum \bar{a}_{rs}\gamma_{sr} = \Lambda_\gamma(\bar{p}) = \Lambda_\gamma(p)$, so $\Lambda_\gamma(p)$ is real. $\square$

For $j \in \mathbb{Z}_+^{2d}$, $|j| \le 2n$, set $\pi_x(j) := (j_1, ..., j_d)$ and $\pi_y(j) := (j_{d+1}, ..., j_{2d})$. For γ as above, we now set $\beta_j := \Lambda_\gamma(y^{\pi_y(j)}x^{\pi_x(j)})$, where, for $z \in \mathbb{C}^d$, $x := \frac{z+\bar{z}}{2}$ and $y := \frac{z-\bar{z}}{2i}$. Since the operand of Λ_γ is real-valued (as an element of $\mathbb{C}_{2n}^d[z,\bar{z}]$), Lemma 2.14 implies $\beta_j \in \mathbb{R}$. We now set $\mathcal{R}(\gamma) := \beta$; note that

$$\beta_j = \Lambda_\beta(t^j) = \Lambda_\beta(y^{\pi_y(j)}x^{\pi_x(j)}) = \Lambda_\gamma((\frac{z-\bar{z}}{2i})^{\pi_y(j)}(\frac{z+\bar{z}}{2})^{\pi_x(j)}). \tag{2.13}$$

Proposition 2.15. $\mathcal{M}(n)(\mathcal{R}(\gamma)) = L^{*-1}M(n)(\gamma)L^{-1}$.

Proof. It suffices to show that for $k, \ell, r, s \in \mathbb{Z}_+^d$, with $|k| + |\ell|, |r| + |s| \le n$, and for $\beta = \mathcal{R}(\gamma)$, we have $\left\langle \mathcal{M}(n)(\beta)\widetilde{y^k x^\ell}, \widetilde{y^r x^s}\right\rangle = \left\langle L^{*-1}M(n)(\gamma)L^{-1}\widetilde{y^k x^\ell}, \widetilde{y^r x^s}\right\rangle$. Now,

$$\begin{aligned}\left\langle L^{*-1}M(n)(\gamma)L^{-1}\widetilde{y^k x^\ell}, \widetilde{y^r x^s}\right\rangle &= \left\langle M(n)(\gamma)L^{-1}\widetilde{y^k x^\ell}, L^{-1}\widetilde{y^r x^s}\right\rangle \\ &= \left\langle M(n)(\gamma)[(\frac{z-\bar{z}}{2i})^k(\frac{z+\bar{z}}{2})^\ell]\hat{}, [(\frac{z-\bar{z}}{2i})^r(\frac{z+\bar{z}}{2})^s]\hat{}\right\rangle \\ &\quad \text{(by Lemma 2.13)} \\ &= \Lambda_\gamma((\frac{z-\bar{z}}{2i})^{k+r}(\frac{z+\bar{z}}{2})^{\ell+s}). \end{aligned} \tag{2.14}$$

Choosing $j \in \mathbb{Z}_+^{2d}$ so that $\pi_x(j) = \ell + s$ and $\pi_y(j) = k + r$, we have $|j| = (|k| + |\ell|) + (|r| + |s|) \le 2n$, so (2.13) shows that the expression in (2.14) is equal to $\Lambda_\beta(y^{k+r}x^{\ell+s}) = \left\langle \mathcal{M}(n)(\beta)\widetilde{y^k x^\ell}, \widetilde{y^r x^s}\right\rangle$, as desired. $\square$

Next, we define an inverse to $\mathcal{R}$. Given a real sequence $\beta \equiv \beta^{(2n)} = \{\beta_j\}_{j\in\mathbb{Z}_+^{2d},|j|\le 2n}$, with $\beta_0 > 0$, we will associate to β a complex sequence $\gamma \equiv \gamma^{(2n)}$. For $k, \ell \in \mathbb{Z}_+^d$, $|k| + |\ell| \le 2n$, let

$$\begin{aligned}\gamma_{k\ell} &:= \Lambda_\beta((x - iy)^k(x + iy)^\ell) \\ &= \Lambda_\beta((t_1 - it_{d+1})^{k_1}\cdot\ldots\cdot(t_d - it_{2d})^{k_d}(t_1 + it_{d+1})^{\ell_1}\cdot\ldots\cdot(t_d + it_{2d})^{\ell_d}).\end{aligned}$$

Clearly, $\gamma_{00} = \Lambda_\beta(1) = \beta_0 > 0$, and $\gamma_{\ell k} = \bar{\gamma}_{k\ell}$. We set $\mathcal{S}(\beta) := \gamma$; we omit the proof of the following result, which is dual to that in Proposition 2.15.

Proposition 2.16. $M(n)(\mathcal{S}(\beta)) = L^*\mathcal{M}(n)(\beta)L$.

Taken together, Propositions 2.15 and 2.16 show that $(\mathcal{R}\circ\mathcal{S})(\beta) = \beta$ and $(\mathcal{S}\circ\mathcal{R})(\gamma) = \gamma$. We are now in position to formulate the equivalence between the real and complex truncated moment problems, as expressed in the following two results.

Proposition 2.17. *Given $\gamma \equiv \gamma^{(2n)}$, let $\beta \equiv \beta^{(2n)} := \mathcal{R}(\gamma)$.*
(i) $\mathcal{M}(n)(\beta) = L^{*-1}M(n)(\gamma)L^{-1}$.
(ii) $\mathcal{M}(n)(\beta) \geq 0 \Leftrightarrow M(n)(\gamma) \geq 0$.
(iii) $\operatorname{rank} \mathcal{M}(n)(\beta) = \operatorname{rank} M(n)(\gamma)$.
(iv) $\mathcal{M}(n)(\beta)$ is positive and admits a flat extension $\mathcal{M}(n+1)$ if and only if $M(n)(\gamma)$ is positive and admits a flat extension $M(n+1)$.
(v) For $q \in \mathbb{C}_n[x,y]$, $q(X,Y) = L^{*-1}((q \circ \tau)(Z, \bar{Z}))$.
(vi) For $q \in \mathbb{C}_n[x,y]$, $\Lambda_\beta(q) = \Lambda_\gamma(q \circ \tau)$
(vii) If ν is a representing measure for γ, then $\mu := \nu \circ \psi$ is a representing measure for β, of the same measure class and cardinality of support; moreover, $\operatorname{supp} \mu = \tau(\operatorname{supp} \nu)$.

Proof. (i) This is Proposition 2.15.
(ii) This follows from (i) and the invertibility of L (Lemma 2.13).
(iii) This also follows from (i) and the invertibility of L.
(iv) Suppose $M(n)(\gamma)$ is positive and admits a flat extension

$$M(n+1)(\tilde{\gamma}) \equiv \begin{pmatrix} M(n)(\gamma) & B(n+1) \\ B(n+1)^* & C(n+1) \end{pmatrix}.$$

Proposition 2.15 (using $n+1$) implies that $\mathcal{M} := (L^{(n+1)*})^{-1}M(n+1)(\tilde{\gamma})(L^{(n+1)})^{-1}$ is of the form $M(n+1)(\mathcal{R}(\tilde{\gamma}))$, while (i) and the direct sum structure of $(L^{(n+1)})^{-1}$ show that

$$\mathcal{M} = \begin{pmatrix} (L^{(n)*})^{-1}M(n)(\gamma)(L^{(n)})^{-1} & * \\ * & * \end{pmatrix} = \begin{pmatrix} \mathcal{M}(n)(\mathcal{R}(\gamma)) & * \\ * & * \end{pmatrix}.$$

Since $\operatorname{rank} \mathcal{M} = \operatorname{rank} M(n+1)(\tilde{\gamma}) = \operatorname{rank} M(n)(\gamma) = \operatorname{rank} \mathcal{M}(n)(\beta)$, it follows that $\mathcal{M}$ is a flat extension of $\mathcal{M}(n)(\beta)$ (≥ 0). The converse is proved similarly, using Proposition 2.16; we omit the details.

(v)
$$\begin{aligned} q(X,Y) &\equiv \mathcal{M}(n)(\beta)\tilde{q} = L^{*-1}M(n)(\gamma)L^{-1}\tilde{q} \text{ (by (i))} \\ &= L^{*-1}M(n)(\gamma)\widehat{q \circ \tau} \text{ (by Lemma 2.13)} \\ &= L^{*-1}(q \circ \tau)(Z, \bar{Z}). \end{aligned}$$

(vi) Straightforward from (2.13).
(vii) For $j \in \mathbb{Z}_+^{2d}$, $|j| \leq 2n$,

$$\begin{aligned} \int t^j \, d\mu &= \int y^{\pi_y(j)} x^{\pi_x(j)} \, d(\nu \circ \psi)(x,y) \\ &= \int (\frac{z - \bar{z}}{2i})^{\pi_y(j)} (\frac{z + \bar{z}}{2})^{\pi_x(j)} \, d\nu(z, \bar{z}) \\ &= \Lambda_\gamma((\frac{z - \bar{z}}{2i})^{\pi_y(j)} (\frac{z + \bar{z}}{2})^{\pi_x(j)}) \\ &= \beta_j \text{ (by (2.13))}; \end{aligned}$$

thus, μ is a representing measure for β, and the other properties of μ are clear. $\square$

We omit the proof of the following result, which is dual to Proposition 2.17.

Proposition 2.18. *Given $\beta \equiv \beta^{(2n)}$, let $\gamma \equiv \gamma^{(2n)} := \mathcal{S}(\beta)$.*
(i) $M(n)(\gamma) = L^*\mathcal{M}(n)(\beta)L$.
(ii) $M(n)(\gamma) \geq 0 \Leftrightarrow \mathcal{M}(n)(\beta) \geq 0$.
(iii) $\operatorname{rank} M(n)(\gamma) = \operatorname{rank} \mathcal{M}(n)(\beta)$.

(iv) $M(n)(\gamma)$ is positive and admits a flat extension $M(n+1)$ if and only if $\mathcal{M}(n)(\beta)$ is positive and admits a flat extension $\mathcal{M}(n+1)$.
(v) For $p \in \mathbb{C}_n[z,\bar{z}]$, $p(Z,\bar{Z}) = L^*((p\circ\psi)(X,Y))$.
(vi) For $p \in \mathbb{C}_n[z,\bar{z}]$, $\Lambda_\gamma(p) = \Lambda_\beta(p\circ\psi)$.
(vii) If μ is a representing measure for β, then $\nu := \mu\circ\tau$ is a representing measure for γ, of the same measure class and cardinality of support; moreover, $\operatorname{supp}\nu = \psi(\operatorname{supp}\mu)$.

Throughout the sequel, whenever we have equivalent sequences γ and β (as described by the preceding results), the context always indicates whether we have $\beta = \mathcal{R}(\gamma)$ or $\gamma = \mathcal{S}(\beta)$, so we do not explicitly refer to $\mathcal{R}$ or $\mathcal{S}$.

We next present an analogue of Theorem 2.2 for truncated moment problems on $\mathbb{R}^N$.

Theorem 2.19. *Let $\beta \equiv \beta^{(2n)}$ and let $r := \operatorname{rank}\mathcal{M}^N(n)(\beta)$. If μ is an r-atomic representing measure for β, then $\mathcal{M}^N(n+1)[\mu]$ is a flat (positive) extension of $\mathcal{M}(n) \equiv \mathcal{M}^N(n)(\beta)$. Conversely, if $\mathcal{M}(n)$ is positive semidefinite and admits a flat extension $\mathcal{M}(n+1) \equiv \mathcal{M}^N(n+1)(\tilde{\beta})$, then $\mathcal{M}(n+1)$ admits unique flat positive moment matrix extensions $\mathcal{M}^N(n+2)(\tilde{\beta})$, $\mathcal{M}^N(n+3)(\tilde{\beta})$,..., and there exists an r-atomic representing measure for $\mathcal{M}^N(\infty)(\tilde{\beta})$ (i.e., a representing measure for $\tilde{\beta}^{(\infty)}$).*

Proof. Suppose μ is an r-atomic representing measure for β, i.e., $\mathcal{M}^N(n)(\beta) = \mathcal{M}^N(n)[\mu]$. Since μ is also a representing measure for $\mathcal{M}^N(n+1)[\mu]$, Corollary 2.12 implies that $r = \operatorname{card}\operatorname{supp}\mu \geq \operatorname{rank}\mathcal{M}^N(n+1)[\mu] \geq \operatorname{rank}\mathcal{M}^N(n)[\mu] = r$, so $\mathcal{M}^N(n+1)[\mu]$ is a flat (positive) extension of $\mathcal{M}(n)$.

For the converse, we assume that $\mathcal{M}^N(n)(\beta)$ is positive and admits a flat extension $\mathcal{M}^N(n+1)(\tilde{\beta})$. We consider first the case when N is even, say $N = 2d$. In this case, let $\gamma \equiv \gamma^{(2n)} = \mathcal{S}(\beta)$. Proposition 2.18 implies that $M^d(n)(\gamma)$ is positive and admits a flat extension $M^d(n+1)(\tilde{\gamma})$. Theorem 2.2 now implies that $M^d(n+1)(\tilde{\gamma})$ admits unique successive flat (positive) extensions $M^d(n+2)(\tilde{\gamma}), M^d(n+3)(\tilde{\gamma}), ...,$ and that $\tilde{\gamma}^{(\infty)}$ admits an r-atomic representing measure ν. Proposition 2.17 (and the direct sum structure of $L^{(n+j)}$ $(j \geq 0)$) now imply that $\mathcal{M}^{2d}(n+1)(\tilde{\beta})$ admits unique successive flat extensions $\{\mathcal{M}^{2d}(n+j)(\tilde{\beta})\}_{j\geq 2}$, defined by $\mathcal{M}^{2d}(n+j)(\tilde{\beta}) := (L^{(n+j)*})^{-1}M^d(n+j)(\tilde{\gamma})(L^{(n+j)})^{-1}$. Proposition 2.17 further implies that ν corresponds to an r-atomic representing measure μ for $\tilde{\beta}^{(\infty)}$.

We now consider the case $N = 2d-1$. For $x \in \mathbb{R}^{2d-1}$, $t \in \mathbb{R}$, $i \in \mathbb{Z}_+^{2d-1}$, $j \in \mathbb{Z}_+$, we set $\breve{x} := (x,t) \in \mathbb{R}^{2d}$ and $\breve{\imath} := (i,j) \in \mathbb{Z}_+^{2d}$, so that $\breve{x}^{\breve{\imath}} = x^i t^j$. Corresponding to $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i\in\mathbb{Z}_+^{2d-1}}, |i| \leq 2n$, we define a sequence $\breve{\beta} \equiv \breve{\beta}^{(2n)} = \{\breve{\beta}_{\breve{\imath}}\}_{\breve{\imath}\in\mathbb{Z}_+^{2d},|\breve{\imath}|\leq 2n}$ as follows:

$$\breve{\beta}_{\breve{\imath}} := \begin{cases} \beta_i & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases} . \tag{2.15}$$

Corresponding to $\mathcal{M} \equiv \mathcal{M}^{2d-1}(n)(\beta)$ we define the moment matrix $\breve{\mathcal{M}} \equiv \mathcal{M}^{2d}(n)(\breve{\beta})$. Since $\breve{\mathcal{M}}$ is unitarily equivalent to a matrix of the form $\mathcal{M} \bigoplus 0$, we have $\operatorname{rank}\breve{\mathcal{M}} = \operatorname{rank}\mathcal{M}$, and $\breve{\mathcal{M}} \geq 0$ if and only if $\mathcal{M} \geq 0$. Suppose $\mathcal{M} \equiv \mathcal{M}^{2d-1}(n)(\beta) \geq 0$ and suppose $\mathcal{M}(n+1) \equiv \mathcal{M}^{2d-1}(n+1)(\tilde{\beta})$ is a flat extension of $\mathcal{M}$. We claim that $\breve{\mathcal{M}}(n+1) \equiv [M(n+1)]\hat{}$ is a flat extension of $\breve{\mathcal{M}}$. Since $\mathcal{M}(n+1) \geq 0$, then $\breve{\mathcal{M}}(n+1) \geq 0$, and $\operatorname{rank}\breve{\mathcal{M}}(n+1) = \operatorname{rank}\mathcal{M}(n+1) = \operatorname{rank}\mathcal{M}(n) = \operatorname{rank}\breve{\mathcal{M}}(n)$. Let us denote $\breve{\mathcal{M}}(n+1)$ as $\mathcal{M}^{2d}(n+1)(\lambda)$, for some sequence λ. To show that $\breve{\mathcal{M}}(n+1)$ is an extension of $\breve{\mathcal{M}}(n)$, it suffices to show that if $\breve{\imath}$ satisfies $|\breve{\imath}| \leq 2n$, then $\lambda_{\breve{\imath}} = \breve{\beta}_{\breve{\imath}}$. Indeed, if $\breve{\imath} = (i,j)$ and $j = 0$, then $\lambda_{\breve{\imath}} = \tilde{\beta}_i = \beta_i = \breve{\beta}_{\breve{\imath}}$, while if $j > 0$, then $\lambda_{\breve{\imath}} = 0 = \breve{\beta}_{\breve{\imath}}$. Thus $\breve{\mathcal{M}}(n+1)$ is a flat (positive) extension of $\breve{\mathcal{M}}(n)$.

Since $\breve{\mathcal{M}}(n+1) = \mathcal{M}^{2d}(n+1)(\lambda)$, the "even" case (above) implies that $\breve{\mathcal{M}}(n+1)$ has unique successive flat moment matrix extensions $\mathcal{M}^{2d}(n+j)(\tilde{\lambda})$ $(j \geq 2)$, and that $\tilde{\lambda}^{(\infty)}$ admits a rank $\breve{\mathcal{M}}$-atomic representing measure ν. For $j \geq 2$ and $i \in \mathbb{Z}_+^{2d-1}$ with $|i| \leq 2(n+j)$, we set $\tilde{\beta}_i := \tilde{\lambda}_{(i,0)}$. Then $\mathcal{M}^{2d-1}(n+2)(\tilde{\beta}), \mathcal{M}^{2d-1}(n+3)(\tilde{\beta}), ...,$ define the unique successive flat moment matrix extensions of $\mathcal{M}^{2d-1}(n+1)(\tilde{\beta})$ (indeed, $[\mathcal{M}^{2d-1}(n+1)(\tilde{\beta})]^{\smile} = M^d(n+j)(\tilde{\lambda})$ $(j \geq 1)$). Finally, if $\nu \equiv \sum_{s=1}^r \rho_s \delta_{(x_s,t_s)}$ (with $x_s \in \mathbb{R}^{2d-1}$, $t_s \in \mathbb{R}$, $\rho_s > 0$), then $\mu := \sum_{s=1}^r \rho_s \delta_{x_s}$ is an r-atomic representing measure for $\tilde{\beta}^{(\infty)}$. □

Remark 2.20. We note the following for future reference. In $\breve{\mathcal{M}}(n+1) \equiv \mathcal{M}^{2d}(n+1)(\lambda)$, since $\lambda_{\breve{\imath}} = 0$ whenever $|\breve{\imath}| \leq 2(n+1)$ and $j > 0$, each column that is indexed by a multiple of t is identically 0. Further, since $\tilde{\lambda}^{(\infty)}$ has a representing measure, each of the successive flat extensions $\mathcal{M}^{2d}(n+j)(\tilde{\lambda})$ $(j \geq 2)$ is recursively generated; hence, in $\mathcal{M}^{2d}(n+j)(\tilde{\lambda})$, each column indexed by a multiple of t is identically 0, whence $\tilde{\lambda}_{(i,j)} = 0$ whenever $j > 0$.

We can now give a proof of Theorem 1.2, which we restate here for the reader's convenience.

Theorem 2.21. *If $\mathcal{M}(n) \equiv \mathcal{M}^N(n)(\beta) \geq 0$ admits a flat extension $\mathcal{M}(n+1)$, then $\mathcal{V} := \mathcal{V}(\mathcal{M}(n+1))$ satisfies* $\operatorname{card}\mathcal{V} = r$ $(\equiv \operatorname{rank}\mathcal{M}(n))$*, and $\mathcal{V} \equiv \{t_j\}_{j=1}^r \subseteq \mathbb{R}^N$ forms the support of the unique representing measure μ for $\mathcal{M}(n+1)$. If $\mathcal{B} \equiv \{T^{i_k}\}_{k=1}^r$ is a maximal linearly independent subset of columns of $\mathcal{M}(n)$, then $W_{\mathcal{B},\mathcal{V}}$ is invertible, and $\mu = \sum_{i=1}^r \rho_j \delta_{t_j}$, where $\rho \equiv (\rho_1, ..., \rho_r)$ is uniquely determined by $\rho^t = W_{\mathcal{B},\mathcal{V}}^{-1}(\beta_{i_1}, ..., \beta_{i_r})^t$.*

Proof. We first consider the support of a representing measure μ for $\mathcal{M}(n+1)$ (cf. Theorem 2.19). For $N = 2d$, let γ be the equivalent complex sequence associated to β via Proposition 2.18; Propositions 2.17(v) and 2.18(v) imply that $\mathcal{V}(\mathcal{M}(n+1)(\beta))$ and $\mathcal{V}(M(n+1)(\gamma))$ are identical when regarded as subsets of $\mathbb{R}^{2d}$. The conclusion that $\operatorname{card}\mathcal{V} = r$ and $\operatorname{supp}\mu = \mathcal{V}$ thus follows by a straightforward application of Theorem 2.3 and Propositions 2.17 and 2.18. For $N = 2d-1$, one needs to argue as in the proof of Theorem 2.19, to convert the initial moment problem for β into an equivalent one for $\hat{\beta}$ in $\mathbb{R}^{2d}$ (using (2.15)), and to then appeal to the result for $N = 2d$. We omit the details of this argument, except to note that in the notation of the proof of Theorem 2.19, $\mathcal{V}(\mathcal{M}(n+1)(\beta)) \times \{0\} = \mathcal{V}(\mathcal{M}(n+1)(\lambda))$. As for the uniqueness of μ and the calculation of the densities using $W_{\mathcal{B},\mathcal{V}}$, the proof is very similar to the argument establishing the uniqueness of ν in Theorem 2.3; for this we use an analogue of Lemma 2.5 for the invertibility of $W_{\mathcal{B},\mathcal{V}}$ in the case of real moment matrices. □

Remark 2.22. Theorem 2.4 and Corollary 2.6 admit exact analogues for real moment matrices.

3. Localizing matrices

Let $0 \leq k \leq n$ and let $p \equiv p(z,\bar{z}) \in \mathbb{C}^d[z,\bar{z}]$, $\deg p = 2k$ or $2k-1$. We next define the *localizing matrix* $M_p^d(n) \equiv M_p^d(n)(\gamma)$ whose positivity is directly related to the existence of a representing measure for $\gamma \equiv \gamma^{(2n)}$ with support in $\mathcal{K}_p \equiv \left\{z \in \mathbb{C}^d : p(z,\bar{z}) \geq 0\right\}$. Note that $\dim \mathbb{C}_{n-k}^d[z,\bar{z}] = \eta \equiv \eta(d, n-k) = \binom{n-k+2d}{2d}$; thus $\mathbb{C}^\eta = \left\{\hat{f} : f \in \mathbb{C}_{n-k}^d[z,\bar{z}]\right\}$. We define the $\eta \times \eta$ matrix $M_p^d(n)$ by

$$\left\langle M_p^d(n)\hat{f}, \hat{g}\right\rangle = \Lambda_\gamma(pf\bar{g}) \qquad (f, g \in \mathbb{C}_{n-k}^d[z,\bar{z}]). \tag{3.1}$$

If γ has a representing measure ν supported in $\mathcal{K}_p$, then $\left\langle M_p(n)\hat{f}, \hat{f}\right\rangle = \Lambda_\gamma\left(p|f|^2\right) = \int p|f|^2\, d\nu \geq 0$, whence $M_p^d(n) \geq 0$. Note also the following consequences of (3.1):

$$M_p^d(n)^* = M_{\bar{p}}^d(n); \tag{3.2}$$

if $p = p_1 + p_2$ with $\deg p_i \le \deg p$ $(i = 1, 2)$, then

$$M_p^d(n) = \left[M_{p_1}^d(n)\right]_\eta + \left[M_{p_2}^d(n)\right]_\eta. \tag{3.3}$$

The main result of this section (Theorem 3.2 below) provides a concrete description of $M_p^d(n)$ as a linear combination of certain compressions of $M^d(n)$ corresponding to the monomial terms of p. In order to state this result, we require a preliminary lemma and some additional notation.

Lemma 3.1. *For $r, s \in \mathbb{Z}_+^d$ with $|r| + |s| \le 2k$, there exist $i, j \in \mathbb{Z}_+^d$ such that*

$$\bar{z}^r z^s = \bar{z}^i z^j \bar{z}^{r-i} z^{s-j} \qquad \text{and} \qquad |i| + |j|, |r-i| + |s-j| \le k.$$

Proof. *Case (i)*: $|r|, |s| \le k$; let $i = r$, $j = 0$. *Case (ii)*: $k < |r|$. We have $r = (r_1, \ldots, r_d)$ with $|r| = r_1 + \cdots + r_d > k$. Choose $r' \equiv (r_i', \ldots, r_d') \in \mathbb{Z}_+^d$ so that $0 \le r_i' \le r_i$ $(1 \le i \le d)$ and $|r'| = r_1' + \cdots + r_d' = k$. With $i = r'$, $j = 0$, we have $|r - i| + |s - j| = |r - r'| + |s| = (r_1 - r_1') + \cdots + (r_d - r_d') + |s| = |r| + |s| - |r'| \le 2k - k = k$. *Case (iii)*: $k < |s|$; similar to Case (ii). $\square$

For $p(z, \bar{z})$ as above (with $\delta \equiv \deg p = 2k$ or $2k - 1$), we write $p(z, \bar{z}) \equiv \sum_{r,s \in \mathbb{Z}_+^d,\ |r|+|s| \le \delta} a_{rs} \bar{z}^r z^s$. Lemma 3.1 shows that for each $r, s \in \mathbb{Z}_+^d$ with $|r|+|s| \le \delta$, there are tuples $i \equiv i(r, s, k)$, $j \equiv j(r, s, k)$, $t \equiv t(r, s, k)$, $u \equiv u(r, s, k)$ in $\mathbb{Z}_+^d$, such that $i + t = r$, $j + u = s$, $|i| + |j|, |t| + |u| \le k$. In the sequel, ${}_{[\bar{Z}^u Z^t;1,\eta]} M^d(n)_{[\bar{Z}^i Z^j;1,\eta]}$ denotes the compression of $M^d(n)$ to the first η rows that are indexed by multiples of $\bar{Z}^u Z^t$ and to the first η columns that are indexed by multiples of $\bar{Z}^i Z^j$.

Theorem 3.2. $M_p^d(n) = \sum_{\substack{r,s \in \mathbb{Z}_+^d \\ |r|+|s| \le \delta}} a_{rs\,[\bar{Z}^u Z^t;1,\eta]} M^d(n)_{[\bar{Z}^i Z^j;1,\eta]}.$

For the proof of Theorem 3.2, we require several preliminary results. Let $0 \le k \le n$ and let $r, s \in \mathbb{Z}_+^d$, with $|r| + |s| \le 2k$. From Lemma 3.1, we have $i, j \in \mathbb{Z}_+^d$ with $|i| + |j|, |r - i| + |s - j| \le k$.

Lemma 3.3. *For $f, g \in \mathbb{C}_{n-k}^d[z, \bar{z}]$,*

$$\left\langle M_{\bar{z}^r z^s}^d(n) \hat{f}, \hat{g} \right\rangle = \left\langle M^d(n) \left(\bar{z}^i z^j f\right)\hat{}, \left(z^{r-i} \bar{z}^{s-j} g\right)\hat{} \right\rangle.$$

Proof. Let $|r| + |s| = 2l$ or $2l - 1$; if $l < k$, $M_{\bar{z}^r z^s}^d(n)$ has size $\eta(d, n - l)$; in this case we regard $\mathbb{C}_{n-k}^d[z, \bar{z}]$ as embedded in $\mathbb{C}_{n-l}^d[z, \bar{z}]$ and take coefficient vectors $\hat{f}$, $\hat{g}$ relative to $\mathbb{C}_{n-l}^d[z, \bar{z}]$; in any case, $\bar{z}^i z^j f$ and $z^{r-i} \bar{z}^{s-j} g$ are elements of $\mathbb{C}_n[z, \bar{z}]$, so $\left(\bar{z}^i z^j f\right)\hat{}$ and $\left(z^{r-i} \bar{z}^{s-j} g\right)\hat{}$ are computed relative to $\mathbb{C}_n[z, \bar{z}]$. We have

$$\begin{aligned}
\left\langle M_{\bar{z}^r z^s}^d(n) \hat{f}, \hat{g} \right\rangle &= \Lambda\left(\bar{z}^r z^s f \bar{g}\right) \\
&= \Lambda\left(\bar{z}^i z^j f \cdot \left(z^{r-i} \bar{z}^{s-j} g\right)^{-}\right) \\
&= \left\langle M^d(n) \left(\bar{z}^i z^j f\right)\hat{}, \left(\bar{z}^{s-j} z^{r-i} g\right)\hat{} \right\rangle.
\end{aligned}$$

$\square$

Proposition 3.4. *Let $0 \le k \le n$. Let $r, s, t, u, q, v \in \mathbb{Z}_+^d$ satisfy $|r|+|s| \le 2k$, $|t|+|u|, |q|+|v| \le n-k$. Then*

$$\left\langle M_{\bar{z}^r z^s}^d(n) \widehat{\bar{z}^q z^v}, \widehat{\bar{z}^t z^u} \right\rangle = \gamma_{r+q+u, s+v+t}.$$

Proof. From Lemma 3.1, we have $i,j\in\mathbb{Z}_+^d$ such that $|i|+|j|,|r-i|+|s-j|\le k$. Lemma 3.3 implies that

$$\begin{aligned}\left\langle M^d_{\bar{z}^r z^s}(n)\,\widehat{\bar{z}^q z^v},\widehat{\bar{z}^t z^u}\right\rangle &= \left\langle M^d(n)\left(\bar{z}^i z^j \bar{z}^q z^v\right)^{\wedge},\left(z^{r-i}\bar{z}^{s-j}\bar{z}^t z^u\right)^{\wedge}\right\rangle\\ &= \left\langle M^d(n)\left(\bar{z}^{i+q} z^{j+v}\right)^{\wedge},\left(\bar{z}^{s+t-j} z^{r+u-i}\right)^{\wedge}\right\rangle\\ &= \gamma_{(i+q)+(r+u-i),(j+v)+(s+t-j)}=\gamma_{q+r+u,s+v+t}.\end{aligned}$$

□

Lemma 3.5. *Let* $0\le k\le n$ *and let* $\eta=\eta(d,n-k)$*. Suppose* $p,q,l,m\in\mathbb{Z}_+^d$ *satisfy* $|p|+|q|,|l|+|m|\le k$ *and set*

$$M:={}_{\left[\bar{Z}^m Z^l;1,\eta\right]}M^d(n)_{\left[\bar{Z}^p Z^q;1,\eta\right]}.$$

Then $M=\left[M^d_{\bar{z}^p z^q\cdot\bar{z}^l z^m}(n)\right]_\eta$*, the compression of* $M^d_{\bar{z}z^q\cdot\bar{z}^l z^m}$ *to its first* η *rows and columns.*

Proof. The columns of M are indexed by $\bar{Z}^{p+i}Z^{q+j}$, $i,j\in\mathbb{Z}_+^d$, $|i|+|j|\le n-k$, and the rows are indexed by $\bar{\bar{Z}}^{m+a}Z^{l+b}$, $a,b\in\mathbb{Z}_+^d$, $|a|+|b|\le n-k$. The entry in row $\bar{Z}^{m+a}Z^{l+b}$, column $\bar{Z}^{p+i}Z^{q+j}$ of M is thus

$$\left\langle M^d(n)\left(\bar{z}^{p+i}z^{q+j}\right)^{\wedge},\left(\bar{z}^{m+a}z^{l+b}\right)^{\wedge}\right\rangle=\gamma_{p+i+l+b,q+j+m+a}.$$

The corresponding entry of $M^d_{\bar{z}^p z^q\cdot\bar{z}^l z^m}(n)$, in row $\bar{Z}^a Z^b$, column $\bar{Z}^i Z^j$, is $\left\langle M^d_{\bar{z}^p z^q\cdot\bar{z}^l z^m}(n)\,\widehat{\bar{z}^i z^j},\widehat{\bar{z}^a z^b}\right\rangle$, which, by Proposition 3.4, is also equal to $\gamma_{p+i+l+b,q+j+m+a}$. □

Proof of Theorem 3.2. We have $0\le k\le n$ and

$$p\equiv p(z,\bar{z})=\sum_{r,s\in\mathbb{Z}_+^d,\ |r|+|s|\le\delta}a_{rs}\bar{z}^r z^s,$$

with $\delta\equiv\deg p$ ($=2k$ or $2k-1$). The size of $M_p^d(n)$ is thus $\eta\times\eta$, where $\eta=\eta(d,n-k)$. By (3.3) and the uniqueness of $M_p^d(n)$, we have

$$M_p^d(n)=\sum_{r,s\in\mathbb{Z}_+^d,\ |r|+|s|\le\delta}a_{rs}\left[M^d_{\bar{z}^r z^s}(n)\right]_\eta. \tag{3.4}$$

From Lemma 3.1, for each $r,s\in\mathbb{Z}_+^d$ with $|r|+|s|\le\delta$, we have $i\equiv i(r,s,k),j\equiv j(r,s,k),t\equiv t(r,s,k),u\equiv u(r,s,k)\in\mathbb{Z}_+^d$ with $i+t=r$, $j+u=s$, $|i|+|j|,|t|+|u|\le k$. Lemma 3.5 implies that for each r, s,

$$\begin{aligned}\left[M^d_{\bar{z}^r z^s}(n)\right]_\eta &= \left[M^d_{\bar{z}^i z^j\cdot\bar{z}^t z^u}(n)\right]_\eta\\ &={}_{\left[\bar{Z}^u Z^t;1,\eta\right]}M^d(n)_{\left[\bar{Z}^i Z^j;1,\eta\right]},\end{aligned}$$

so the result follows from (3.4). □

We conclude this section with an analogue of Theorem 3.2 for real moment matrices. Given a real moment matrix $\mathcal{M}^N(n)\equiv\mathcal{M}^N(n)(\beta)$, let $k\le n$, and let $p\in\mathbb{C}[t_1,...,t_N]$, with $\deg p=2k$ or $2k-1$. The *localizing matrix* $\mathcal{M}_p^N(n)$ has size $\tau\equiv\tau(N,n-k):=\begin{pmatrix}n-k+N\\ N\end{pmatrix}$ and is uniquely determined by

$$\left\langle\mathcal{M}_p^N(n)\hat{f},\hat{g}\right\rangle=\Lambda_\beta(pf\bar{g})\quad(f,g\in\mathbb{C}^N_{n-k}[t]); \tag{3.5}$$

if β has a representing measure supported in $K_p := \{t \in \mathbb{R}^N : p(t) \geq 0\}$, then clearly $\mathcal{M}_p^N(n) \geq 0$. Write $p(t) \equiv \sum_{i \in \mathbb{Z}_+^n, |i| \leq \deg p} a_i t^i$. For each i, there exist (non-unique) $r \equiv r(i)$, $s \equiv s(i)$ in $\mathbb{Z}_+^n$ such that $r + s = i$ and $|r|, |s| \leq k$; thus $p(t) = \sum_{i \in \mathbb{Z}_+^n, |i| \leq \deg p} a_i t^{r(i)} t^{s(i)}$. Let ${}_{[T^r;1,\tau]}\mathcal{M}^N(n)_{[T^s;1,\tau]}$ denote the compression of $\mathcal{M}^N(n)$ to the first τ rows that are indexed by multiples of T^r and to the first τ columns that are indexed by multiples of T^s.

Theorem 3.6. $\mathcal{M}_p^N(n) = \sum_{i \in \mathbb{Z}_+^n, |i| \leq \deg p} a_{i[T^r;1,\tau]}\mathcal{M}^N(n)_{[T^s;1,\tau]}$.

The proof of Theorem 3.6 follows by formal repetition of the proof of Theorem 3.2; we omit the details. Example 1.5 illustrates Theorem 3.6 with $N = 3, n = 1, \deg p = 2$.

4. Flat extensions of positive localizing matrices

In this section we present a flat extension theorem for positive localizing matrices, which provides the main tool for proving Theorem 1.1. Suppose $M^d(n)(\gamma)$ is positive and admits a flat extension $M^d(n+1)$; thus, there is a matrix W such that $M^d(n+1)$ admits a block decomposition of the form

$$M^d(n+1) = \begin{pmatrix} M^d(n) & B^d(n+1) \\ B^d(n+1)^* & C^d(n+1) \end{pmatrix}, \tag{4.1}$$

where $B^d(n+1) = M^d(n)W$ and $C^d(n+1) = W^* M^d(n) W$. It follows from Theorem 2.2 that $M^d(n+1)$ admits a unique positive flat extension $M^d(\infty)$ and that $M^d(\infty)$ admits a representing measure ν. In particular, $M^d(n+1) \equiv M^d(n+1)[\nu]$ is positive and recursively generated, and $M^d(n+1)$ admits unique successive positive, flat moment matrix extensions $M^d(n+2) \equiv M^d(n+2)[\nu], M^d(n+3) \equiv M^d(n+3)[\nu], \ldots$. Thus, if $p \in \mathbb{C}^d[z, \bar{z}]$ and $k := [(1 + \deg p)/2] \leq n$, we may consider $M_p^d(n+k)$ and $M_p^d(n+k+1)$.

Theorem 4.1. *Suppose $M^d(n)(\gamma) \geq 0$ admits a flat extension*

$$M^d(n+1) = \begin{pmatrix} M^d(n) & M^d(n)W \\ W^* M^d(n) & W^* M^d(n) W \end{pmatrix}.$$

Let $p \in \mathbb{C}^d[z, \bar{z}]$, with $\deg p = 2k$ or $2k - 1$. If $M_p^d(n+k) \geq 0$, then

$$M_p^d(n+k+1) = \begin{pmatrix} M_p^d(n+k) & M_p^d(n+k)W \\ W^* M_p^d(n+k) & W^* M_p^d(n+k) W \end{pmatrix}; \tag{4.2}$$

in particular, $M_p^d(n+k+1)$ is a flat, positive extension of $M_p(n+k)$.

Remark 4.2. In Theorem 4.1, we are not assuming that $M_p^d(n+k)$ is a moment matrix; rather, in Section 5 we will prove that under the hypotheses of Theorem 4.1, both $M_p^d(n+k)$ and $M_p^d(n+k+1)$ are indeed moment matrices.

The proof of Theorem 4.1 is based on a computational description of $M_p^d(n+k+1)$, and to derive this we require some additional notation. For $m > 0$, let A be a matrix of size $\eta(d, m)$ with rows and columns $\{\bar{Z}^a Z^b\}_{a,b \in \mathbb{Z}_+^d, |a|+|b| \leq m}$, ordered lexicographically. Suppose $i, j \in \mathbb{Z}_+^d$, with $|i| + |j| \leq m$, and suppose there are at least β columns of A that are indexed by multiples of $\bar{Z}^i Z^j$. Suppose $u, t \in \mathbb{Z}_+^d$, with $|u| + |t| \leq m$, and suppose there are at least υ rows of A that are indexed by multiples of $\bar{Z}^u Z^t$. For $\alpha \leq \beta$ and $\rho \leq \upsilon$, let ${}_{[\bar{Z}^u Z^t, \rho, \upsilon]}A_{[\bar{Z}^i Z^j, \alpha, \beta]}$ denote the compression of A to the α-th through β-th consecutive columns indexed by multiples of $\bar{Z}^i Z^j$ and to the ρ-th through υ-th consecutive rows indexed by multiples of $\bar{Z}^u Z^t$. We omit the proof of the following elementary result.

Lemma 4.3. $\left({}_{\left[\bar{Z}^u Z^t;\rho,\upsilon\right]} A_{\left[\bar{Z}^i Z^j;\alpha,\beta\right]}\right)^* = {}_{\left[\bar{Z}^i Z^j;\alpha,\beta\right]} \left(A^*\right)_{\left[\bar{Z}^u Z^t;\rho,\upsilon\right]}$.

(Here, the convention is that rows and columns of A^* are indexed in the same way as the rows and columns of A, as $\left\{\bar{Z}^a Z^b\right\}_{a,b\in\mathbb{Z}_+^d,\ |a|+|b|\le m}$.)

To prove Theorem 4.1, we will first obtain analogues of (4.2) for each monomial term of p. To this end, let $\delta = \deg p$ $(= 2k$ or $2k-1)$; write p as $p\left(z,\bar{z}\right) \equiv \sum\limits_{r,s\in\mathbb{Z}_+^d,\ |r|+|s|\le\delta} a_{rs}\bar{z}^r z^s$. Recall from Section 3 that

$$\eta_2 \equiv \operatorname{size} M_p^d\left(n+k+1\right) = \eta\left(d,\left(n+k+1\right)-k\right) = \eta\left(d,n+1\right) = \operatorname{size} M^d\left(n+1\right)$$

and

$$\eta_1 \equiv \operatorname{size} M_p^d\left(n+k\right) = \eta\left(d,\left(n+k\right)-k\right) = \eta\left(d,n\right) = \operatorname{size} M^d\left(n\right).$$

Let $p_{rs} = \bar{z}^r z^s$; from Lemma 3.1, we have $\bar{z}^r z^s = \bar{z}^{i(r,s,k)} z^{j(r,s,k)}.\bar{z}^{t(r,s,k)} z^{u(r,s,k)}$, where $i \equiv i\left(r,s,k\right), j \equiv j\left(r,s,k\right), t \equiv t\left(r,s,k\right)$, and $u \equiv u\left(r,s,k\right) \in \mathbb{Z}_+^d$ satisfy $r = i+t,\ s = j+u,\ |i|+|j|, |t|+|u| \le k$. Lemma 3.5 (applied with n replaced by $n+k+1$) shows that

$$\left[M_{p_{rs}}^d\left(n+k+1\right)\right]_{\eta_2} = {}_{\left[\bar{Z}^u Z^t;1,\eta_2\right]} M^d\left(n+k+1\right)_{\left[\bar{Z}^i Z^j;1,\eta_2\right]}; \tag{4.3}$$

similarly,

$$\left[M_{p_{rs}}^d\left(n+k\right)\right]_{\eta_1} = {}_{\left[\bar{Z}^u Z^t;1,\eta_1\right]} M^d\left(n+k\right)_{\left[\bar{Z}^i Z^j;1,\eta_1\right]}. \tag{4.4}$$

We next use (4.3) and (4.4) to relate $\left[M_{p_{rs}}^d\left(n+k+1\right)\right]_{\eta_2}$ to $\left[M_{p_{rs}}^d\left(n+k\right)\right]_{\eta_1}$ via a block decomposition of $\left[M_{p_{rs}}^d\left(n+k+1\right)\right]_{\eta_2}$. From (4.3), note that the columns of $\left[M_{p_{rs}}^d\left(n+k+1\right)\right]_{\eta_2}$ are compressions of the first η_2 columns of $M^d\left(n+k+1\right)$ that are indexed by multiples of $\bar{Z}^i Z^j$; these monomials are ordered as $\left\{\bar{Z}^{i+i_q} Z^{j+j_q}\right\}_{q=1}^{\eta_2}$, where $\left\{\bar{Z}^{i_q} Z^{j_q}\right\}_{q=1}^{\eta_2}$ is the lexicographic ordering of the first η_2 monomials in $\mathbb{C}^d\left[z,\bar{z}\right]$. In particular, from (4.4) we see that the first η_1 of these monomials also index the columns of $\left[M_{p_{rs}}^d\left(n+k\right)\right]_{\eta_1}$. Similarly, the rows of $\left[M_{p_{rs}}^d\left(n+k+1\right)\right]_{\eta_2}$ are compressions of rows of
$M^d\left(n+k+1\right)$ that are indexed by the sequence $\left\{\bar{Z}^{u+i_q} Z^{t+j_q}\right\}_{q=1}^{\eta_2}$, and the first η_1 of these also index the rows of $\left[M_{p_{rs}}^d\left(n+k\right)\right]_{\eta_1}$. Further, from (4.1) and the above remarks, it is clear that $\left[M_{p_{rs}}^d\left(n+k+1\right)\right]_{\eta_1} = \left[M_{p_{rs}}^d\left(n+k\right)\right]_{\eta_1}$. It now follows from the preceding observations that
$\left[M_{p_{rs}}^d\left(n+k+1\right)\right]_{\eta_2}$ admits a block decomposition of the form

$$M_{p_{rs}}^d\left(n+k+1\right)_{\eta_2} = \begin{pmatrix} \left[M_{p_{rs}}^d\left(n+k\right)\right]_{\eta_1} & B_{p_{rs}}^d\left(n+k+1\right) \\ D_{p_{rs}}^d\left(n+k+1\right) & C_{p_{rs}}^d\left(n+k+1\right) \end{pmatrix}, \tag{4.5}$$

where

$$\left[M_{p_{rs}}^d\left(n+k\right)\right]_{\eta_1} = {}_{\left[\bar{Z}^u Z^t;1,\eta_1\right]} M^d\left(n+k\right)_{\left[\bar{Z}^i Z^j;1,\eta_1\right]}, \tag{4.6}$$

$$B_{p_{rs}}^d\left(n+k+1\right) = {}_{\left[\bar{Z}^u Z^t;1,\eta_1\right]} M^d\left(n+k+1\right)_{\left[\bar{Z}^i Z^j;1+\eta_1,\eta_2\right]}, \tag{4.7}$$

$$D_{p_{rs}}^d\left(n+k+1\right) = {}_{\left[\bar{Z}^u Z^t;\eta_1+1,\eta_2\right]} M^d\left(n+k+1\right)_{\left[\bar{Z}^i Z^j;1,\eta_1\right]}, \tag{4.8}$$

$$C_{p_{rs}}^d\left(n+k+1\right) = {}_{\left[\bar{Z}^u Z^t;\eta_1+1,\eta_2\right]} M^d\left(n+k+1\right)_{\left[\bar{Z}^i Z^j;\eta_1+1,\eta_2\right]}. \tag{4.9}$$

The following lemma is the first step toward proving an analogue of Theorem 4.1 for p_{rs}.

Lemma 4.4. *For each $r,s \in \mathbb{Z}_+^d$ with $|r|+|s| \le \delta$,*

$$\begin{pmatrix} \left[M_{p_{rs}}^d(n+k)\right]_{\eta_1} \\ D_{p_{rs}}^d(n+k+1) \end{pmatrix} W = \begin{pmatrix} B_{p_{rs}}^d(n+k+1) \\ C_{p_{rs}}^d(n+k+1) \end{pmatrix}.$$

Proof. For $1 \le m \le \eta_2 - \eta_1$, the m-th column of $\begin{pmatrix} B^d(n+1) \\ C^d(n+1) \end{pmatrix}$ is the (η_1+m)-th column of $M(n+1)$, and is thus of the form $\bar{Z}^e Z^f \in \mathcal{C}_{M(n+1)}$, with $|e|+|f| = n+1$. If $\left(\alpha_{a,b}^{(m)}\right)_{a,b\in\mathbb{Z}_+^d,\ |a|+|b|\le n}$ denotes the m-th column of W, then we have

$$\bar{Z}^e Z^f = \sum_{|a|+|b|\le n} \alpha_{a,b}^{(m)} \bar{Z}^a Z^b. \tag{4.10}$$

Let $\{V_{a,b}(r,s)\}_{|a|+|b|\le n}$ denote the lexicographic ordering of the columns of $\begin{pmatrix} \left[M_{p_{rs}}^d(n+k)\right]_{\eta_1} \\ D_{p_{rs}}^d(n+k+1) \end{pmatrix}$ and let $U_m(r,s)$ denote the m-th column of $\begin{pmatrix} B_{p_{rs}}^d(n+k+1) \\ C_{p_{rs}}^d(n+k+1) \end{pmatrix}$. It suffices to show that

$$U_m(r,s) = \sum_{|a|+|b|\le n} \alpha_{a,b}^{(m)} V_{a,b}(r,s). \tag{4.11}$$

Since $M^d(n+k+1)$ is a flat, hence positive, extension of $M^d(n+1)$, (4.10) also holds in $\mathcal{C}_{M^d(n+k+1)}$. Now $U_m(r,s)$ is the (η_1+m)-th column of $\left[M_{p_{rs}}^d(n+k+1)\right]_{\eta_2}$, and is thus indexed by the (η_1+m)-th multiple of $\bar{Z}^{i(r,s,k)} Z^{j(r,s,k)}$; thus $U_m(r,s)$ is indexed by $\bar{Z}^{e+i(r,s,k)} Z^{f+j(r,s,k)}$. Since $M^d(n+k+1)$ is recursively generated, (4.10) implies that in $\mathcal{C}_{M^d(n+k+1)}$ we have

$$\bar{Z}^{e+i(r,s,k)} Z^{f+j(r,s,k)} = \sum_{|a|+|b|\le n} \alpha_{a,b}^{(m)} \bar{Z}^{a+i(r,s,k)} Z^{b+j(r,s,k)}; \tag{4.12}$$

thus, via compression of these columns to rows indexed by the first η_2 multiples of $\bar{Z}^u Z^t$, it follows that the relation in (4.12) holds as well in the column space of $\left[M_{p_{rs}}^d(n+k+1)\right]_{\eta_2}$. Since the compression of $\bar{Z}^{e+i(r,s,k)} Z^{f+j(r,s,k)}$ is $U_m(r,s)$ and the compression of $\bar{Z}^{a+i(r,s,k)} Z^{b+j(r,s,k)}$ is $V_{a,b}(r,s)$, we obtain (4.11), so the result follows. □

Lemma 4.5. *For each $r,s \in \mathbb{Z}_+^d$ with $|r|+|s| \le \delta$,*

$$D_{p_{rs}}^d(n+k+1) = B_{\bar{p}_{rs}}^d(n+k+1)^* = W^* \left[M_{p_{rs}}^d(n+k)\right]_{\eta_1}.$$

Proof. Applying Lemma 4.4 to $\bar{p}_{rs}$ ($\equiv \bar{Z}^s Z^r$), we have

$$\begin{aligned} B_{\bar{p}_{rs}}^d(n+k+1) &= \left[M_{\bar{p}_{rs}}^d(n+k)\right]_{\eta_1} W \\ &= \left[M_{p_{rs}}^d(n+k)^*\right]_{\eta_1} W \quad \text{(by (3.2))} \\ &= \left[M_{p_{rs}}^d(n+k)\right]_{\eta_1}^* W, \end{aligned}$$

whence $B^d_{\bar{p}_{rs}}(n+k+1)^* = W^* \left[M^d_{p_{rs}}(n+k)\right]_{\eta_1}$. Now

$$\begin{aligned}
D^d_{p_{rs}}(n+k+1)^* &= {}_{\left[\bar{Z}^i Z^j;1,\eta_1\right]} M^d(n+k+1)^*_{\left[\bar{Z}^u Z^t;\eta_1+1,\eta_2\right]} \\
&\qquad\qquad\qquad\qquad\qquad \text{(Lemma 4.3)} \\
&= {}_{\left[\bar{Z}^i Z^j;1,\eta_1\right]} M^d(n+k+1)_{\left[\bar{Z}^u Z^t;\eta_1+1,\eta_2\right]} \\
&\qquad\qquad\qquad\qquad\qquad \text{(since } M^d(n+k+1) \geq 0\text{)} \\
&= B^d_{\bar{Z}^u Z^t \cdot Z^i \bar{Z}^j}(n+k+1) \quad \text{(by (4.7) applied to } \bar{Z}^s Z^r\text{)} \\
&= B^d_{\bar{p}_{rs}}(n+k+1);
\end{aligned}$$

thus $D^d_{p_{rs}}(n+k+1) = B^d_{\bar{p}_{rs}}(n+k+1)^*$ and the proof is complete. □

Proof of Theorem 4.1. Using the uniqueness of $M^d_p(n+k)$ and of $M^d_p(n+k+1)$, it follows from (4.5)–(4.9) that $M^d_p(n+k+1)$ admits a block decomposition of the form

$$M^d_p(n+k+1) = \begin{pmatrix} M^d_p(n+k) & B^d_p(n+k+1) \\ D^d_p(n+k+1) & C^d_p(n+k+1) \end{pmatrix},$$

where

$$\begin{aligned}
M^d_p(n+k) &= \sum_{|r|+|s|\leq\delta} \alpha_{rs} \left[M^d_{p_{rs}}(n+k)\right]_{\eta_1}, \\
B^d_p(n+k+1) &= \sum_{|r|+|s|\leq\delta} \alpha_{rs} B^d_{p_{rs}}(n+k+1), \\
D^d_p(n+k+1) &= \sum_{|r|+|s|\leq\delta} \alpha_{rs} D^d_{p_{rs}}(n+k+1), \\
C^d_p(n+k+1) &= \sum_{|r|+|s|\leq\delta} \alpha_{rs} C^d_{p_{rs}}(n+k+1).
\end{aligned}$$

Lemma 4.4 implies $B^d_p(n+k+1) = M^d_p(n+k)W$, and Lemma 4.5 implies $D^d_p(n+k+1) = W^* M^d_p(n+k)$. Now Lemmas 4.4 and 4.5 imply $C^d_p(n+k+1) = D^d_p(n+k+1)W = W^* M^d_p(n+k)W$, whence (4.2) holds. Since $M^d_p(n+k)$ is positive, (4.2) implies that $M^d_p(n+k+1)$ is positive and that $\operatorname{rank} M^d_p(n+k+1) = \operatorname{rank} M^d_p(n+k)$ (cf. [CuFi4]). □

5. Existence of minimal representing measures supported in semi-algebraic sets

We begin with the analogue of Theorem 1.1 for the truncated complex multivariable K-moment problem. Recall that if $M^d(n)(\gamma)$ (≥ 0) has a flat extension $M^d(n+1)$, then $M^d n+1)$ admits unique recursive flat (positive) extensions $M^d(n+2), M^d(n+3), \ldots$ (Theorem 2.2).

Theorem 5.1. *Let $\gamma \equiv \gamma^{(2n)} = \{\gamma_i\}_{i\in\mathbb{Z}^d_+,\, |i|\leq 2n}$ be a complex sequence and let $\mathcal{P} \equiv \{p_i\}_{i=1}^m \subseteq \mathbb{C}^d[z,\bar{z}]$ with $\deg p_i = 2k_i$ or $2k_i - 1$ $(1 \leq i \leq m)$. Let $M \equiv M^d(n)(\gamma)$ and let $r := \operatorname{rank} M$. There exists a (minimal) r-atomic representing measure for γ supported in $K_{\mathcal{P}}$ if and only if $M \geq 0$ and M admits a flat extension $M^d(n+1)$ for which $M^d_{p_i}(n+k_i) \geq 0$ $(1 \leq i \leq m)$. In this case, $M^d(n+1)$ admits a unique representing measure ν, which is an r-atomic (minimal) $K_{\mathcal{P}}$-representing measure for γ; moreover, ν has precisely $r - \operatorname{rank} M^d_{p_i}(n+k_i)$ atoms in $\mathcal{Z}(p_i)$ $(1 \leq i \leq m)$.*

Proof. Suppose $M^d n)(\gamma)$ is positive and admits a flat extension $M^d(n+1)$ for which $M^d_{p_i}(n+k_i) \geq 0$ $(1 \leq i \leq m)$. [CuFi1, Corollary 7.9] and [CuFi1, Theorem 7.7] imply that $M^d(n+1)$ admits a unique flat (positive) extension $M^d(\infty)$, and that $M^d(\infty)$ admits an r-atomic representing measure $\nu \equiv \sum_{j=1}^r \rho_j \delta_{z_j}$, with $\rho_j > 0$ and $z_j \in \mathbb{C}^d$ $(1 \leq j \leq r)$. Theorem 1.2 implies that ν is the unique representing measure for $M^d(n+1)$. We will show that $\operatorname{supp}\nu \subseteq K_{\mathcal{P}}$. Fix i, $1 \leq i \leq m$. Since $M^d_{p_i}(n+k_i) \geq 0$, repeated application of Theorem 4.1 shows that $M^d_{p_i}(\infty)$ is a flat, positive extension of $M^d_{p_i}(n+k_i)$; moreover,

$$\left\langle M^d_{p_i}(\infty)\hat{f}, \hat{g}\right\rangle = \int p_i\, f\, \bar{g}\, d\nu, \qquad f, g \in \mathbb{C}^d[z, \bar{z}]. \tag{5.1}$$

Fix j, $1 \leq j \leq r$, and let

$$f_j(z, \bar{z}) = \frac{\|z - z_1\|^2 \cdots \|z - z_{j-1}\|^2 \|z - z_{j+1}\|^2 \cdots \|z - z_r\|^2}{\|z_j - z_1\|^2 \cdots \|z_j - z_{j-1}\|^2 \|z_j - z_{j+1}\|^2 \cdots \|z_j - z_r\|^2}$$

(where, for $z \equiv (z_1, \ldots, z_d)$, $\|z\|^2 := \sum \bar{z}_i z_i \in \mathbb{C}^d[z, \bar{z}]$). Now $f_j \in \mathbb{C}^d[z, \bar{z}]$, so by (5.1),

$$\begin{aligned} 0 \leq \left\langle M^d_{p_i}(\infty)\hat{f}_j, \hat{f}_j\right\rangle &= \int p_i\, |f_j|^2\, d\nu \\ &= \sum_{k=1}^r \rho_k p_i(z_k, \bar{z}_k)\, |f_j(z_k, \bar{z}_k)|^2 \\ &= \rho_j p_i(z_j, \bar{z}_j). \end{aligned}$$

Since $\rho_j > 0$, then $p(z_j, \bar{z}_j) \geq 0$. Repeating the preceding argument for $1 \leq i \leq m$ and $1 \leq j \leq r$, we conclude that $\operatorname{supp}\nu \subseteq K_{\mathcal{P}}$.

We now count the atoms of ν that lie in $\mathcal{Z}(p_i)$. Equations (5.1) and (2.2) show that $M^d_{p_i}(\infty)$ is the moment matrix corresponding to the measure $p_i\, d\nu$, i.e., $M^d_{p_i}(\infty) = M^d(\infty)[p_i\, d\nu]$. Thus, [CuFi1, Proposition 7.6] implies that

$$\operatorname{card}\operatorname{supp}(p_i d\nu) = \operatorname{rank} M_{p_i}(\infty) = \operatorname{rank} M^d_{p_i}(n+k_i).$$

We have

$$\begin{aligned} \Delta_i &:= \operatorname{rank} M^d(n)(\gamma) - \operatorname{rank} M^d_{p_i}(n+k_i) \\ &= \operatorname{card}\operatorname{supp}\nu - \operatorname{card}\operatorname{supp}(p_i d\nu) \\ &= \operatorname{card}(\operatorname{supp}\nu \cap \mathcal{Z}(p_i)), \end{aligned}$$

whence ν has precisely Δ_i atoms in $\mathcal{Z}(p_i)$ $(1 \leq i \leq m)$.

For the converse direction, suppose ν is an r-atomic representing measure for γ with $\operatorname{supp}\nu \subseteq K_{\mathcal{P}}$. Since ν is a representing measure for $M(\infty) \equiv M^d(\infty)[\nu]$, [CuFi1, Proposition 7.6] implies that

$$\begin{aligned} r &= \operatorname{card}\operatorname{supp}\nu = \operatorname{rank} M(\infty) \\ &\geq \operatorname{rank} M(n+1)[\nu] \geq \operatorname{rank} M(n)[\nu] = \operatorname{rank} M(n)(\gamma) = r. \end{aligned}$$

In particular, $M^d(n+1)[\nu]$ is a flat extension of $M^d(n)(\gamma)$, as is
$M^d(n+k_i)[\nu]$ $(1 \leq i \leq m)$; since ν is also a representing measure for $M^d(n+k_i)[\nu]$ and $\operatorname{supp}\nu \subseteq K_{p_i}$, then (3.1) implies that $M^d_{p_i}(n+k_i)[\nu] \geq 0$ $(1 \leq i \leq m)$. $\square$

We next prove Theorem 1.1, the analogue of Theorem 5.1 for moment problems on $\mathbb{R}^N$. We consider a real N-dimensional sequence of degree $2n$, $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z^N_+, |i| \leq 2n}$, and its moment matrix $\mathcal{M} \equiv \mathcal{M}^N(n)(\beta)$. Recall from Theorem 2.19 that β admits a $\operatorname{rank}\mathcal{M}$-atomic (minimal) representing measure if and only if $\mathcal{M} \geq 0$ and $\mathcal{M}$ admits a flat moment matrix extension $\mathcal{M}^N(n+1)$, which in turn admits unique successive flat extensions $\mathcal{M}^N(n+2)$, $\mathcal{M}^N(n+3)$,... . For the reader's convenience, we restate Theorem 1.1, as follows.

Theorem 5.2. *Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i\in\mathbb{Z}_+^N,|i|\leq 2n}$ be an N-dimensional real sequence, and let $\mathcal{Q} \equiv \{q_i\}_{i=1}^m \subseteq \mathbb{C}^N[t]$, with $\deg q_i = 2k_i$ or $2k_i-1$ $(1 \leq i \leq m)$. Let $\mathcal{M} := \mathcal{M}^N(n)(\beta)$ and let $r := \operatorname{rank}\mathcal{M}$. There exists a (minimal) r-atomic representing measure for $\mathcal{M}$ supported in $K_{\mathcal{Q}}$ if and only if $\mathcal{M} \geq 0$ and $\mathcal{M}$ admits a flat extension $\mathcal{M}(n+1)$ such that $\mathcal{M}_{q_i}(n+k_i) \geq 0$ $(1 \leq i \leq m)$. In this case, $\mathcal{M}(n+1)$ admits a unique representing measure μ, which is an r-atomic (minimal) $K_{\mathcal{Q}}$-representing measure for β; moreover, μ has precisely $r - \operatorname{rank}\mathcal{M}_{q_i}(n+k_i)$ atoms in $\mathcal{Z}(q_i) \equiv \left\{t \in \mathbb{R}^N : q_i(t) = 0\right\}$, $1 \leq i \leq m$.*

Proof. Suppose μ is a $\operatorname{rank}\mathcal{M}$-atomic representing measure for β with $\operatorname{supp}\mu \subseteq K_{\mathcal{Q}}$. Exactly as in the proof of Theorem 2.19 (or of Theorem 5.1), $\mathcal{M}(n+1)[\mu]$ is a flat extension of $\mathcal{M}$ $(= \mathcal{M}^N(n)[\mu] \geq 0)$, with unique successive flat extensions $\mathcal{M}^N(n+2)[\mu], \mathcal{M}^N(n+3)[\mu], \ldots$ Since $\operatorname{supp}\mu \subseteq K_{\mathcal{Q}}$, for each $f \in \mathbb{C}_n^N[t]$, we have $\left\langle \mathcal{M}_{q_i}^N(n+k_i)[\mu]\hat{f}, \hat{f} \right\rangle = \int q_i \left|f\right|^2 \ d\mu \geq 0$, whence $\mathcal{M}_{q_i}^N(n+k_i)[\mu] \geq 0$ $(1 \leq i \leq m)$.

For the converse and the location of the atoms, we first consider the case when N is even, say $N = 2d$. Suppose $\mathcal{M} \equiv \mathcal{M}^{2d}(n)(\beta)$ is positive and has a flat extension $\mathcal{M}^{2d}(n+1)(\tilde{\beta})$ for which $\mathcal{M}_{q_i}^{2d}(n+k_i)(\tilde{\beta}) \geq 0$ $(1 \leq i \leq m)$ (cf. Theorem 2.19). Using Proposition 2.18 (and as in the proof of the "even" case of Theorem 2.19), $\mathcal{M}$ corresponds to a complex moment matrix $M^d(n)(\gamma)$ $(= L^{(n)*}\mathcal{M}L^{(n)})$, and the successive flat extensions $\mathcal{M}^{2d}(n+j)(\tilde{\beta})$ of $\mathcal{M}$ correspond to successive flat moment matrix extensions of $M^d(n)(\gamma)$ defined by $M^d(n+j)(\tilde{\gamma}) := L^{(n+j)*}\mathcal{M}^{2d}(n+j)(\tilde{\beta})L^{(n+j)}$ $(j \geq 1)$ (cf. Proposition 2.18).

We will show that $M_{p_i}^d\left(n+k_i\right)(\tilde{\gamma}) \geq 0$, where $p_i := q_i \circ \tau \in \mathbb{C}_{2k_i}^d[z,\bar{z}])$ $(1 \leq i \leq m)$. To this end, recall from Lemma 2.13 that $\tau(z,\bar{z}) = (x,y)$, so that $q_i = p_i \circ \psi$ $(1 \leq i \leq m)$; further Proposition 2.18(vi) implies that

$$\Lambda_{\tilde{\gamma}}(p) = \Lambda_{\tilde{\beta}}(p \circ \psi) \ (p \in \mathbb{C}^d[z,\bar{z}]). \tag{5.2}$$

We assert that

$$M_{p_i}^d\left(n+k_i\right)(\tilde{\gamma}) = L^{(n)*}\mathcal{M}_{q_i}^{2d}(n+k_i)(\tilde{\beta})L^{(n)} \ (1 \leq i \leq m). \tag{5.3}$$

Indeed, for $f, g \in \mathbb{C}_n^d[z,\bar{z}])$ and $1 \leq i \leq m$, we have

$$\begin{aligned}
\left\langle M_{p_i}^d\left(n+k_i\right)(\tilde{\gamma})\hat{f}, \hat{g}\right\rangle &= \Lambda_{\tilde{\gamma}}(p_i f\bar{g}) \\
&= \Lambda_{\tilde{\beta}}((p_i f\bar{g}) \circ \psi) \ \text{ (by (5.2))} \\
&= \left\langle \mathcal{M}_{q_i}^{2d}(n+k_i)(\tilde{\beta})\widetilde{f \circ \psi}, \widetilde{g \circ \psi}\right\rangle \\
&= \left\langle \mathcal{M}_{q_i}^{2d}(n+k_i)(\tilde{\beta})L^{(n)}\hat{f}, L^{(n)}\hat{g}\right\rangle \ \text{(by Lemma 2.13)} \\
&= \left\langle L^{(n)*}\mathcal{M}_{q_i}^{2d}(n+k_i)(\tilde{\beta})L^{(n)}\hat{f}, \hat{g}\right\rangle,
\end{aligned}$$

whence (5.3) follows. Since $\mathcal{M}_{q_i}^{2d}(n+k_i)(\tilde{\beta}) \geq 0$, then (5.3) implies that $M_{p_i}^d\left(n+k_i\right)(\tilde{\gamma}) \geq 0$ $(1 \leq i \leq m)$.

Theorem 5.1 now implies that γ has a $\operatorname{rank} M^d(n)(\gamma)$-atomic representing measure ω, supported in $K_{\mathcal{P}}$, and Proposition 2.17 shows that ω corresponds to a $\operatorname{rank}\mathcal{M}$-atomic representing measure for β supported in $K_{\mathcal{Q}}$. Theorem 5.1 also implies that $M^d\left(n+1\right)(\tilde{\gamma})$ admits a unique representing measure ν, which is a $\operatorname{rank} M^d(n)$-atomic $K_{\mathcal{P}}$-representing measure for γ having $\operatorname{rank} M^d\left(n\right)(\gamma) -$

$\operatorname{rank} M_{p_i}^d(n+k_i)(\tilde{\gamma})$ atoms in $\mathcal{Z}(p_i)$ $(1 \le i \le m)$. From Proposition 2.17, ν corresponds to a unique representing measure $\mu := \nu \circ \psi$ for $\mathcal{M}^{2d}(n+1)(\tilde{\beta})$. Since, from Proposition 2.17, $\operatorname{supp} \nu = \psi(\operatorname{supp} \mu)$, $\mathcal{Z}(p_i) = \psi \circ \mathcal{Z}(q_i)$, and $\operatorname{rank} M^d(n)(\gamma) = \operatorname{rank} \mathcal{M}$, and since (5.3) implies $\operatorname{rank} M_{p_i}^d(n+k_i)(\tilde{\gamma}) = \operatorname{rank} \mathcal{M}_{q_i}^{2d}(n+k_i)(\tilde{\beta})$, it follows that $\operatorname{supp} \mu \subseteq K_{\mathcal{Q}}$ and that μ has precisely $\operatorname{rank} \mathcal{M} - \operatorname{rank} \mathcal{M}_{q_i}^N(n+k_i)(\tilde{\beta})$ atoms in $\mathcal{Z}(q_i)$ $(1 \le i \le m)$. The proof of the "even" case is now complete.

We now consider the case $N = 2d-1$. Suppose $\mathcal{M} \equiv \mathcal{M}^{2d-1}(n)(\beta)$ is positive and has a flat extension $\mathcal{M}^{2d-1}(n+1)(\tilde{\beta})$, with unique successive flat extensions $\mathcal{M}^{2d-1}(n+j)(\tilde{\beta})$ $(j \ge 2)$ (cf. Theorem 2.19); we are assuming $\mathcal{M}_{q_i}^{2d-1}(n+k_i)(\tilde{\beta}) \ge 0$ $(1 \le i \le m)$. As in the proof of the "odd" case of Theorem 2.19, $\mathcal{M}$ corresponds to the positive moment matrix $\mathcal{M}^{\smile} \equiv \mathcal{M}^{2d}(n)(\breve{\beta})$, which has a sequence of successive flat extensions $\mathcal{M}^{2d}(n+j)(\tilde{\lambda})$ satisfying $\mathcal{M}^{2d}(n+j)(\tilde{\lambda}) = \breve{\mathcal{M}}^{2d-1}(n+j)(\tilde{\beta})$ $(j \ge 1)$; the moments of $\tilde{\beta}$ are related to those of $\tilde{\lambda}$ as in (2.15).

Fix ℓ, $1 \le \ell \le m$; for $q_\ell \equiv \sum b_{\ell,s} t^s \in \mathbb{C}^d[t]$ we let $\breve{q}_\ell \in \mathbb{C}^{d+1}[t,u]$ be given by $\breve{q}_\ell(t,u) := q_\ell(t)$ $(t \in \mathbb{R}^{2d-1}, u \in \mathbb{R})$. We claim that $\mathcal{M}_{\breve{q}_\ell}^{2d}(n+k_\ell)(\tilde{\lambda}) \ge 0$. To this end, for $i \in \mathbb{Z}_+^{2d-1}$, $j \in \mathbb{Z}_+$, recall that $\breve{\imath} := (i,j) \in \mathbb{Z}_+^{2d}$, and for $t \in \mathbb{R}^{2d-1}$, $u \in \mathbb{R}$, $\breve{t} := (t,u) \in \mathbb{R}^{2d}$, so $\breve{t}^{\breve{\imath}} = t^i u^j$. We denote $f \in \mathbb{C}_n[t,u]$ by $f(\breve{t}) = \sum_{|\breve{\imath}| \le n} a_{\breve{\imath}} \breve{t}^{\breve{\imath}}$, and we define $[f] \in \mathbb{C}_n[t]$ by $[f](t) := \sum_{|\breve{\imath}| \le n, j=0} a_{\breve{\imath}} t^i$. Now, for $f \in \mathbb{C}_n[t,u]$,

$$
\begin{aligned}
\left\langle \mathcal{M}_{\breve{q}_\ell}^{2d}(n+k_\ell)(\tilde{\lambda})\hat{f}, \hat{f} \right\rangle &= \Lambda_{\tilde{\lambda}}(\breve{q}_\ell\, |f|^2) \\
&= \sum_{|s| \le \deg q_\ell, |\breve{\imath}|, |\breve{\imath}'| \le n} b_{\ell,s} a_{\breve{\imath}} \bar{a}_{\breve{\imath}'} \tilde{\lambda}_{(s+i+i', j+j')} \\
&= \sum_{|s| \le \deg q_\ell, |\breve{\imath}|, |\breve{\imath}'| \le n, j=j'=0} b_{\ell,s} a_{\breve{\imath}} \bar{a}_{\breve{\imath}'} \tilde{\beta}_{s+i+i'} \quad \text{(by Remark 2.20)} \\
&= \Lambda_{\tilde{\beta}}(q_\ell\, |[f]|^2) = \left\langle \mathcal{M}_{q_\ell}^{2d-1}(n+k_\ell)(\tilde{\beta})[\hat{f}], [\hat{f}] \right\rangle \ge 0.
\end{aligned}
$$

Since $\mathcal{M}_{\breve{q}_\ell}^{2d}(n+k_\ell)(\tilde{\lambda}) \ge 0$ $(1 \le \ell \le m)$, by the "even" case (and its proof, above), $\mathcal{M}^{2d}(n+1)(\tilde{\lambda})$ admits a unique representing measure $\tilde{\mu}$, which is a $\operatorname{rank} \mathcal{M}^{\smile}$-atomic $K_{\mathcal{Q}^{\smile}}$-representing measure for $\mathcal{M}^{\smile}$ (where $\mathcal{Q}^{\smile} := \{\breve{q}_1, ..., \breve{q}_m\}$), with precisely $\operatorname{rank} \mathcal{M}^{\smile} - \operatorname{rank} \mathcal{M}_{\breve{q}_i}^{2d}(n+k_i)(\tilde{\lambda})$ atoms in $\mathcal{Z}(\breve{q}_i)$ $(1 \le i \le m)$. Write $\tilde{\mu} \equiv \sum_{s=1}^r \rho_s \delta_{(t_s,u_s)}$; it follows that $\mu := \sum_{s=1}^r \rho_s \delta_{t_s}$ is a representing measure for $\mathcal{M}^{2d-1}(n+1)(\tilde{\beta})$, with $\operatorname{supp} \mu \subseteq K_{\mathcal{Q}}$ and $\operatorname{card}(\operatorname{supp} \mu \bigcap \mathcal{Z}(q_i)) = \operatorname{rank} \mathcal{M} - \operatorname{rank} M_{q_i}^{2d}(n+k_i)(\tilde{\beta})$ $(1 \le i \le m)$. That μ is the unique representing measure for $M^{2d-1}(n+1)(\tilde{\beta})$ follows from Theorem 2.21. $\square$

Proof of Corollary 1.4. Suppose $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+j)$ which in turn has a flat extension $\mathcal{M}(n+j+1)$ satisfying $\mathcal{M}_{q_i}(n+j+k_i) \ge 0$ $(1 \le i \le m)$. We can apply Theorem 1.1 to $\mathcal{M}(n+j)$ to obtain a finitely atomic $K_{\mathcal{Q}}$-representing measure for $\mathcal{M}(n+j)$, and hence for $\mathcal{M}(n)$. For the converse, suppose $\mathcal{M}(n)$ has a finitely atomic representing measure μ with $\operatorname{supp} \mu \subseteq K_{\mathcal{Q}}$. We will estimate the minimum value of j necessary to obtain a positive extension $\mathcal{M}(n+j)$ having a flat extension $\mathcal{M}(n+j+1)$ (with a corresponding $K_{\mathcal{Q}}$-representing measure). Since μ is finitely atomic, it has convergent moments of degree $2n+1$. Thus, [CuFi8, Theorem 1.4] implies that μ has an *inside cubature rule* η of degree $2n$, with $s := \operatorname{card} \operatorname{supp} \eta \le 1 + \dim \mathbb{R}_{2n}^N[t] = 1 + \binom{2n+N}{N}$; in particular, η is a representing measure for $\mathcal{M}(n)$ and $V := \operatorname{supp} \eta \subseteq \operatorname{supp} \mu$ $(\subseteq K_{\mathcal{Q}})$. Since $\operatorname{card} V = s$, Lagrange interpolation implies that every real-valued function on V agrees on V with a polynomial in $\mathbb{R}_{2(s-1)}^N[t]$. (Indeed, if $V \equiv \{v_1, ..., v_s\}$, let $f_\ell(t) := \frac{\prod_{i=1,...,s; i \ne \ell} \|t - v_i\|^2}{\prod_{i=1,...,s; i \ne \ell} \|t_j - v_i\|^2} \in \mathbb{R}_{2(s-1)}^N[t]$ $(1 \le \ell \le s)$.

Then any function $f : V \to \mathbb{R}$ satisfies $f = \sum_{\ell=1}^{s} f(v_\ell) f_\ell$.) In particular, if $i \in \mathbb{Z}_+^N$ with $|i| = 2s-1$, there exists $p_i \in \mathbb{R}^N_{2(s-1)}[t]$ such that $t^i - p_i(t)|_V \equiv 0$. By Proposition 2.1, $T^i = p_i(T)$ in $\mathcal{C}_{\mathcal{M}(2s-1)[\eta]}$, and since $\deg p_i < |i|$, it follows that $\mathcal{M}(2s-1)[\eta]$ is a flat extension of $\mathcal{M}(2s-2)[\eta]$. Theorem 2.19 implies that $\mathcal{M}(2s-1)[\eta]$ has unique successive flat moment matrix extensions, and it is clear from the preceding argument that these extensions are $\mathcal{M}(2s)[\eta], \mathcal{M}(2s+1)[\eta], \ldots$. Since $V \subseteq K_{\mathcal{Q}}$, it follows immediately that $\mathcal{M}_{q_i}(2s-2+k_i)[\eta] \geq 0$ $(1 \leq i \leq m)$. If $n \leq 2(s-1)$, then $j := 2(s-1)-n$ satisfies our requirements, and $j \leq 2\binom{2n+N}{N} - n$. If $n > 2(s-1)$, then $\mathcal{M}(n) = \mathcal{M}(2s-1)[\eta]$ or $\mathcal{M}(n)$ is one of the successive extensions of $\mathcal{M}(2s-1)[\eta]$ listed above, and in this case we can take $j := 0$. $\square$

Remark 5.3. (i) In the case $N = 2, n > 2$, the estimate for j $(j \leq 4n^2 + 3n + 2)$ can be improved to $j \leq 2n^2 + 6n + 6$ (cf. [CuFi3, Theorem 1.5]). We also note that in several examples that we have studied which require $j > 0$, the flat extension $\mathcal{M}(n+j+1)$ can be realized with $j = 1$ [CuFi7], [CuFi9], [Fia2], [FiPe]. In particular, if $K_{\mathcal{Q}}$ is a degenerate hyperbola and $\mathcal{M}(n)$ has a $K_{\mathcal{Q}}$-representing measure, $\mathcal{M}(n)$ might not have a flat extension, but in this case there is always a positive extension $\mathcal{M}(n+1)$ that has a flat extension $\mathcal{M}(n+2)$ [CuFi9].
(ii) Corollary 1.4 implies an exact analogue for complex moment sequences.

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