THE HANKEL MATRIX METHOD FOR GAUSSIAN QUADRATURE IN 1 AND 2 DIMENSIONS

CARLOS SUERO, MAURICIO ALMANZAR

CONTENTS

1. Introduction

The Fundamental Theorem of Calculus allows one to compute the definite integral of a function over an interval $[a, b]$ by using anti-derivatives. Once the anti-derivative of the function is found, then it is evaluated at the end points of the interval. For example, suppose we have a continuous function $f(x)$ on $[a, b]$. If $F(x)$ is an anti-derivative of $f(x)$, i.e. $F'(x) = f(x)$ for all x in $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

[0]This project was carried out during the Spring, 2008 and Summer I, 2008 semesters with Professor Lawrence A. Fialkow as the mentor. During Spring, 2008 the project was sponsored by SUNY/NSF Alliance for Minority Participation at SUNY New Paltz. The New Paltz AMP Director is Professor Stacie Nunes from the Department of Physics. During Summer I, 2008 the project was sponsored under National Science Foundation Grant (number). The authors thank the sponsors for their support.

For many functions $f(x)$ we are not able to compute an anti-derivative in closed form, so we can't use the Fundamental Theorem of Calculus. For example $f(x) = e^{-x^2}$ does not have an anti-derivative in closed form. Instead, we will use numerical methods to estimate $\int_a^b f(x)dx$. Gaussian Quadrature is one such method. The purpose of our research is to find a way of estimating complicated definite integrals using the Gaussian Quadrature rule. There are other numerical integration methods as well, like Simpson's rule and the Trapezoid rule. In some of the experiments, we will compare the efficiency of Gaussian Quadrature to the efficiencies of Simpson's Rule and the Trapezoid Rule.

The idea behind Gaussian Quadrature is that given an integer $M = 2n+1 > 0$, we can find points $x_1 \ldots x_n$ in $[a, b]$, and positive weights $w_1 \ldots w_n$, so that

$$\int_a^b p(x)dx = \sum_{i=1}^{p} w_i p(x_i) \tag{1.1}$$

for every polynomial $p(x)$ with $deg\ p \leq M$. Then, for any continuous function $f(x)$ on the interval $[a, b]$, we can approximate the integral $\int_a^b f(x)dx$ by the Gaussian Quadrature rule $Q(f) := \sum_{i=1}^{n} w_i f(x_i)$, i.e,

$$\int_a^b f(x)dx \approx Q(f) = \sum_{i=1}^{n} w_i f(x_i). \tag{1.2}$$

To explain why this approach is effective, we have to consider the Weierstrass Approximation Theorem (WAT). This theorem says that given a continuous function

$f(x)$ on the interval $[a, b]$ and a small $\epsilon > 0$, we can find a polynomial $p(x)$ so that $|f(x) - p(x)| < \epsilon$ for all x in $[a, b]$. But we also have to take into account that to get the best results from (1.1) and (1.2), the degree of the polynomial $p(x)$ in WAT must be less than the chosen M for the Gaussian Quadrature Rule. If $deg\ p > M$, the approximation in (1.2) may not be satisfactory.

Our definition of Gaussian Quadrature says that $Q(f) = \sum_{i=1}^{p} w_i f(x_i)$. What the Weierstrass Approximation Theorem suggests is that $Q(f) \approx \int_a^b f(x)dx$. To see this, consider the error,

$$E := |Q(f) - \int_a^b f(x)dx|. \tag{1.3}$$

For every polynomial p, we have

$$E = |Q(f) - Q(p) + Q(p) - \int_a^b f(x)dx|$$

$$\leq |Q(f) - Q(p)| + |Q(p) - \int_a^b f(x)dx|.$$

Now suppose that $deg\ p \leq M$. That means that

$$Q(p) = \int_a^b p(x)dx.$$

So,

$$|Q(p) - \int_a^b f(x)dx| \tag{1.4}$$

$$= |\int_a^b p(x)dx - \int_a^b f(x)dx|$$

$$= |\int_a^b (p(x) - f(x))dx|$$

$$\leq \int_a^b |p(x) - f(x)|dx. \tag{1.5}$$

If $|p(x) - f(x)| < \epsilon$ throughout $[a, b]$, then the expression in (1.5) is at most $= \int_a^b \epsilon dx$, which is equal to $\epsilon(b - a)$. So we have

$$|Q(p) - \int_a^b f(x)dx| \leq \epsilon(b - a). \tag{1.6}$$

Now let's consider $|Q(f) - Q(p)|$. This is equal to

$$|\sum w_i f(x_i) - \sum w_i p(x_i)|$$

$$= |\sum w_i (f(x_i) - p(x_i))|$$

$$\leq \sum w_i |(f(x_i) - p(x_i)|$$

$$\leq \sum w_i \epsilon$$

$$= \epsilon \sum w_i$$

$$= \epsilon(b - a) \text{ (since } \textstyle\sum w_i = Q(1) = \int_a^b 1dx = b - a).$$

So we have:

$$|Q(f) - Q(p)| \leq \epsilon(b - a). \tag{1.7}$$

Now we can show that the error (1.3) can be made small, as follows. Suppose we are given an interval $[a, b]$, a continuous function $f(x)$ on $[a, b]$, $\epsilon > 0$, and let $p(x)$ be given by the Weierstrass Approximation Theorem, i.e. $|p(x) - f(x)| < \epsilon$ $(a \leq x \leq b)$. If we choose $M \geq deg\ p$ and apply the Gaussian Quadrature Rule using this M, then from (1.7) and (1.4) we have

$$E = |Q(f) - \int_a^b f(x)dx|$$

is

$$\leq |Q(f) - Q(p)| + |Q(p) - \int_a^b f(x)dx|.$$

$$\leq \epsilon(b-a) + \epsilon(b-a) = 2\epsilon(b-a),$$

and since ϵ is small, E is also small.

Unfortunately, for small ϵ, $deg\ p$ is often very large, so it is not practical to use Gaussian Quadrature with $M \geq deg\ p$. What the experiments will show is that starting with $M = 1$, as M increases we can achieve 6 place accuracy in estimating $\int_a^b f(x)dx$ by using fairly small values of M, though the smallest M that gives 6 place accuracy depends on several factors, such as the interval length $b - a$ and the complexity of $f(x)$.

2. Proof of Gaussian Quadrature

We will derive an implementation of Gaussian Quadrature based on matrix positivity and Lagrange interpolation, as described in [HK, page 115]. Recall the following facts about integrals.

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx. \tag{2.1}$$

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx. \tag{2.2}$$

The same rules apply to Gaussian Quadrature:

$$Q(f(x)+g(x)) = Q(f(x)) + Q(g(x)). \tag{2.3}$$

$$Q(\alpha f(x)) = \alpha Q(f(x)). \tag{2.4}$$

To prove (2.3), we have

$$Q(f(x)+g(x))$$

$$= \sum_{i=0}^{n} w_i(f(x_i)+g(x_i))$$

$$= \sum_{i=0}^{n} (w_i f(x_i) + w_i g(x_i))$$

$$= \sum_{i=0}^{n} w_i(f(x_i)) + \sum_{i=0}^{n} w_i(g(x_i)) = Q(f(x)) + Q(g(x)).$$

The proof of (2.4) is similar.

Equations (2.1) - (2.4) show that it is enough to prove Gaussian Quadrature for $p(x) = x^i$ $(0 \leq i \leq M)$, i.e, we must find points and weights as in (1.1) such that

$$\beta_j := \int_a^b x^j dx = Q(x^j) := \sum_{i=0}^{n} w_i x_i^j, (0 \leq j \leq M). \tag{2.5}$$

Example 2.1. We will illustrate this reduction to monomials with $p(x) = a_0 + a_1 x$. Applying (2.1) and (2.2) we have, $\int (a_o + a_1 x)dx = a_0 \int 1dx + a_1 \int xdx$. If (2.5) holds, then the last expression equals

$$a_0 Q(1) + a_1 Q(x),$$

and by (2.3)-(2.4), this equals

$$Q(a_0 1 + a_1 x),$$

i.e,

$$\int (a_0 + a_1 x)dx = Q(a_0 + a_1 x). \ \square$$

Example 2.2. Throughout this section we shall illustrate the method of Gaussian Quadrature in detail for the case when $m = 3$, $n = 1$, and the interval is $[0, 1]$. We need to find points x_0, x_1 in $[0, 1]$ and positive weights w_0, w_1 such that:

$$\beta_0 := 1 = w_0x_0 + w_1x_1 \tag{2.6}$$

$$\beta_1 := 1/2 = w_0x_0 + w_1x_1 \tag{2.7}$$

$$\beta_2 := 1/3 = w_0x_0^2 + w_1x_1^2 \tag{2.8}$$

$$\beta_3 := 1/4 = w_0x_0^3 + w_1x_1^3 \tag{2.9}$$

In the sequel we will show how to solve (2.6)-(2.9). □

To establish Gaussian Quadrature we need to look at the Hankel matrix H defined as

$$H = \begin{pmatrix} \beta_0 & \beta_1 & \dots & \beta_n \\ \beta_1 & \beta_2 & \dots & \beta_{n+1} \\ \beta_2 & \beta_3 & \dots & \beta_{n+2} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_n & \beta_{n+1} & \dots & \beta_{2n} \end{pmatrix}.$$

We will denote the columns of H by $\mathbf{1}$,$\mathbf{t}$,..., $\mathbf{t^n}$. We also will consider the vector $v :=\mathbf{t^{n+1}}= (\beta_{n+1}, \dots, \beta_{2n+1})^t$.

It is known that H is "positive" and invertible. That means $\langle Ha, a\rangle > 0$ whenever a $\neq 0$. Another way to know that the matrix H is positive and invertible is to show that the "nested" determinants of the "corners" are all positive. If H_i is the $i \times i$ upper left hand corner of H, we must show that det $H_i > 0$ $(1 \leq i \leq n+1)$.

Example 2.3. In our example, we have

$$H = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}.$$

Then det(H_1)= 1 and det(H_2)= $1/3 - 1/4 = 1/12 > 0$. So $H > 0$. □

Since H is invertible, there is an unique vector $a = (a_0, \ldots, a_n)$ such that

$$Ha = v, \tag{2.10}$$

i.e,

$$a = H^{-1}v,$$

or equivalently,

$$
\begin{aligned}
a_0\beta_0 + \ldots + a_n\beta_n &= \beta_{n+1} \\
a_0\beta_1 + \ldots + a_n\beta_{n+1} &= \beta_{n+2} \\
\vdots \quad & \vdots \quad \vdots \\
a_0\beta_n + \ldots + a_n\beta_{2n} &= \beta_{2n+1}.
\end{aligned}
\tag{2.11}
$$

So $\mathbf{t^{n+1}} = a_0\mathbf{1}+a_1\mathbf{t}+ \ldots + a_n\mathbf{t^n}$. Now let $p(t) = t^{n+1} - (a_0 1 + a_1 t + \ldots + a_n t^n)$.

Example 2.4. In our example, we have

$$
\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/4 \end{pmatrix}
$$

so

$$
\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = H^{-1} \begin{pmatrix} 1/3 \\ 1/4 \end{pmatrix}
$$

$$
= 12 \begin{pmatrix} 1/3 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/4 \end{pmatrix}
$$

and we find

$$
a_0 = -1/6, a_1 = 1.
$$

So we have

$$p(t) = t^2 - (-1/6 + t) = t^2 - t + 1/6. \ \square$$

It is known that $p(t)$ has exactly $n+1$ distinct real roots in $[a, b]$ (see the proof of Proposition 3.3 and Theorem 4.1 *(iv)* => *(iii)* in [CF]. Denote these roots by $x_0 \ldots x_n$.

Example 2.5. In our example, the roots of $t^2 - t + 1/6 = 0$ are

$$t = 1 \pm \sqrt{1 - 4(1)(1/6)}/2 = 1 \pm \sqrt{(5/3)}/2,$$

so $x_0 \approx 0.7886751346$ and $x_1 \approx 0.02113248654$. Note that both x_0 and x_1 are in $[0, 1]$. $\square$

Now let's look at the Vandermonde matrix V defined as

$$V = \begin{pmatrix} x_0^0 & \ldots & x_n^0 \\ x_0^1 & \ldots & x_n^1 \\ \vdots & \vdots & \vdots \\ x_0^n & \ldots & x_n^n \end{pmatrix}.$$

It is known that because $x_0, \ldots, x_n$ are distinct, then V is invertible, i.e, $\det(V) \neq 0$ [HK, page 115].

Example 2.6. In our example,

$$V = \begin{pmatrix} 1 & 1 \\ x_0 & x_1 \end{pmatrix}.$$

So $\det(V) = x_1 - x_0 \neq 0$. □

Since V is invertible, there is an unique vector, $\omega := (\omega_0 \dots \omega_n)$, such that

$$V\omega = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \tag{2.12}$$

i.e,

$$\begin{pmatrix} 1 & \dots & 1 \\ x_0 & \dots & x_n \\ {x_0}^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots \\ x_0^n & \dots & x_n^n \end{pmatrix} \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}. \tag{2.13}$$

(2.13) is equivalent to the following system:

$$\begin{aligned}
\beta_0 &= \omega_0 + \ldots + \omega_n \ (= Q(1)) \\
\beta_1 &= \omega_0 x_0 + \ldots + \omega_n x_n \ (= Q(x)) \\
\vdots \ &\ \vdots \ \ \vdots \\
\beta_n &= \omega_0 x_0^n + \ldots + \omega_n x_n^n \ (= Q(x^n)).
\end{aligned} \tag{2.14}$$

We claim that $x_0, \ldots, x_n$ and $\omega_0, \ldots, \omega_n$ solve the Gaussian Quadrature system (2.5).

Example 2.7. In the example we have,

$$\begin{pmatrix} 1 & 1 \\ x_0 & x_1 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix},$$

or equivalently

$$w_0 + w_1 = \beta_0,$$

and

$$w_0 x_0 + w_1 x_1 = \beta_1,$$

where $w_0 \approx 0.623893$ and $w_1 \approx 0.376107$. ▫

The Gaussian Quadrature system (2.5) is equivalent to the system of equations in (2.14) together with the following system:

$$
\begin{aligned}
(2.15) \qquad \beta_{n+1} &= \omega_0 x_0^{n+1} + \ldots + \omega_n x_n^{n+1} \ (= Q(x^{n+1})) \\
\beta_{n+2} &= \omega_0 x_0^{n+2} + \ldots + \omega_n x_n^{n+2} \ (= Q(x^{n+2})) \\
\vdots &= \vdots \\
\beta_{2n+1} &= \omega_0 x_0^{2n+1} + \ldots + \omega_n x_n^{2n+1} \ (= Q(x^{2n+1})).
\end{aligned}
$$

Since (2.14) is satisfied from (2.13), we are going to focus on (2.15). Consider the first equation of (2.15),

$$\beta_{n+1} = \omega_0 x_0^{n+1} + \ldots + \omega_n x_n^{n+1}.$$

We have

$$\beta_{n+1} = \int_a^b x^{n+1} dx$$

and

$$Q(x^{n+1}) \equiv \sum_{i=0}^{n} w_i x_i^{n+1}.$$

Since each x_i is a root of $t^{n+1} = a_0 + a_1 t + \ldots + a_n t^n$, we have

$$Q(x^{n+1}) = \sum_{i=0}^{n} w_i x_i^{n+1}$$

$$= \sum_{i=0}^{n} w_i(a_0 1 + a_1 x_i + \ldots + a_n x_i^n)$$

$$= a_0 \sum w_i + a_1 \sum w_i x_i + \ldots + a_n \sum w_i x_i^n$$

$$= a_0\beta_0 + a_1\beta_1 + \ldots + a_n\beta_n \text{ (by (2.14))}.$$

By the first equation of (2.11), the last expression is equal to β_{n+1}, so we conclude that

$$Q(x^{n+1}) = \beta_{n+1}. \tag{2.16}$$

For the second equation of (2.15), we have

$$Q(x^{n+2}) \equiv \sum w_i x_i^{n+2}.$$

Since each x_i is a root of $t^{n+1} = a_0 + a_1 t + \ldots + a_n t^n$, x_i is also a root of $t^{n+2} = a_0 t + \ldots + a_n t^{n+1}$. So we have

$$Q(x^{n+2}) = \sum w_i x_i^{n+2}$$

$$= \sum w_i(a_0 x_i + \ldots + a_n x_i^{n+1})$$

$$= a_0 \sum w_i x_i + \ldots + a_n \sum w_i x_i^{n+1}$$

$$= a_0\beta_1 + \ldots + a_n\beta_{n+1} \text{ (by (2.14) and (2.16))}.$$

By the second equation of (2.11) the last expression is equal to β_{n+2}, so we conclude that

$$Q(x^{n+2}) = \beta_{n+2}. \tag{2.17}$$

Following the same procedure, we are able to prove the rest of the equations in the set (2.15). This completes the proof that the points $x_i, \ldots, x_n$ and the weights $w_i, \ldots, w_n$ satisfy the Gaussian Quadrature system (1.1).

From the above discussion, we have distinct points $x_0, \ldots, x_n$ in $[a, b]$ and weights $w_0, \ldots, w_n$ such that

$$Q(p) := \sum_{i=0}^{n} w_i p(x_i) = \int_a^b p(x)dx \ (deg\ p \leq 2n+1). \tag{2.18}$$

To complete the proof of Gaussian Quadrature as in (1.1) we must still show that each $w_i > 0$. In what follows we will use Lagrange polynomials to prove that each $w_i > 0$. Lagrange Interpolation says that given distinct points $x_0, \ldots, x_n$ and given numbers $y_0, \ldots, y_n$ there is a unique polynomial $p(x)$ with $deg\ p = n$, such that

$$p(x_i) = y_i \ (0 \leq i \leq n).$$

Example 2.8. Consider Lagrange Interpolation with n=1. The Lagrange polynomial of degree 1 such that $p(x_0) = y_0$ and $p(x_1) = y_1$ is

$$p_1(x) := \frac{y_0(x - x_1)}{(x_0 - x_1)} + \frac{y_1(x - x_0)}{(x_1 - x_0)}. \tag{2.19}$$

The curve $y = p(x)$ is the line connecting (x_0, y_0) and (x_1, y_1). $\square$

The Lagrange polynomial $p_1(x)$ in (2.19) is the basis for the Trapezoidal Rule. In this rule, we subdivide $[a, b]$ into n equal subintervals $[x_i, x_{i+1}]$ $(0 \leq i \leq n + 1, x_0 = a, x_n = b)$. On interval $[x_i, x_{i+1}]$, we approximate $f(x)$ by the line connecting $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$; this line is the graph of the Lagrange polynomial $p_1(x)$ such that

$$p_1(x_i) = y_i := f(x_i)$$

and

$$p_1(x_{i+1}) = y_{i+1} := f(x_{i+1}).$$

We then approximate $\int_{x_i}^{x_{i+1}} f(x)dx$ by the area of the trapezoid determined by the graph of p_1 between x_i and x_{i+1}:

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{1}{2}(x_{i+1} - x_i)(f(x_i) + f(x_{i+1})).$$

Letting $h = x_{i+1} - x_i$ $(0 \leq i \leq n - 1)$, we may express the Trapezoidal Rule by

$$(2.20) \qquad \int_a^b f(x)dx = h\sum_{i=1}^{n-1} f(a+ih) + h(\frac{f(a)+f(b)}{2}).$$

Example 2.9. For $n = 2$, we have distinct points x_0, x_1, x_2, and values y_0, y_1, y_2. The Lagrange polynomial $p(x)$ looks like the following:

$$(2.21) \qquad p_2(x) := \frac{y_0(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{y_1(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + \frac{y_2(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

The curve $y = p_2(x)$ is the unique parabola passing through (x_0, y_0), (x_1, y_1), (x_2, y_2). $\square$

The Lagrange polynomial $p_2(x)$ in (2.21) is the basis for Simpson's Rule. In this rule, we approximate $y = f(x)$ by a piecewise-parabolic curve $y = g(x)$ on $[a, b]$, using $p_2(x)$ for each piece of $g(x)$. We can then approximate $\int_a^b f(x)dx$ by $\int_a^b g(x)dx$. If we let the function f be tabulated at points x_0, x_1 and x_2, equally spaced by distance h, and we let $y_i = f_i := f(x_i)$ $(0 \le i \le 2)$, then Simpson's rule says that

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_0+2h} f(x)dx$$

$$\approx \int_{x_0}^{x_0+2h} p_2(x)dx = \frac{1}{3}h(f_0 + 4f_1 + f_2).$$

If we use n double-intervals with equally spaced points $x_0, x_1, \ldots, x_{2n}$, then by using the above method on each double-interval and adding up, we get

$$\int_a^b f(x)dx \approx \frac{h}{3}(\sum_{i=0}^{n-1}(2f(a+2ih)+4f(a+(2i+1)h))+f(b)-f(a)). \tag{2.22}$$

The general formula for the Lagrange polynomial $p(x)$ such that $p(x_i) = y_i$ $(p \leq i \leq n)$ is clear from (2.19) and (2.21). If $q(x)$ is another polynomial of *deg* n such that $q(x_i) = y_i$ $(0 \leq i \leq n)$, then $(p-q)(x_i) = 0$. Since *deg* $p - q \leq n$ and $p - q$ has $n + 1$ distinct roots, it follows that $p - q = 0$, i.e. $p = q$. So there is a unique Lagrange polynomial of *deg* n.

Now we return to Gaussian Quadrature, and the claim that $\omega_i > 0$ $(0 \leq i \leq n)$. We fix j and we let $p(x) \equiv p_j(x)$ be the Lagrange polynomial of degree n such that $p(x_i) = 0$ for $i \neq j$ and $p(x_j) = 1$. Now consider $q(x) = p(x)^2$. Since *deg* $q = 2n$ $(< 2n + 1)$, then from (2.18),

$$\int_a^b q(x)dx = \sum_{i=0}^{n} w_i q(x_i). \tag{2.23}$$

Note that $q(x) \geq 0$ and $q(x_j) = 1$, so

$$\int_a^b q(x)dx > 0.$$

Therefore,

$$0 < \int_a^b q(x)dx = \sum_{i=0}^{n} w_i q(x_i) = w_j,$$

so we conclude that $w_j > 0$ $(0 \le j \le n)$. This completes the proof of the Gaussian Quadrature rule (1.1).

3. Iterated 2-Dimensional Gaussian Quadrature

In 1-dimensional Gaussian Quadrature we showed that given $n > 0$ and an interval $[a, b]$, there exist points $x_0, \ldots, x_n$ in [a,b] and positive weights $\omega_0, \ldots, \omega_n$, such that $\int_a^b p(x)dx = \sum_{i=0}^n w_i p(x_i)$ $(deg\ p \le 2n + 1)$.

Now we are going to use 1-dimensional Gaussian Quadrature to find a rule that works in two dimensions. We are given a rectangle in the plane, $R = [a, b] \times [c, d]$, and a continuous function $f(x, y)$ on R. We want to approximate $\int\int_R f(x, y)dxdy$. Given n, let $x_0, \ldots, x_n$, $\omega_0, \ldots, \omega_n$ be the Gaussian Quadrature points and weights for $[a, b]$, so (1.1) holds. Let $y_0, \ldots, y_n$, $s_0, \ldots, s_n$ be the Gaussian Quadrature points and weights for $[c, d]$, so that

$$\int_c^d q(y)dy = \sum_{j=0}^{n} s_j q(y_j) \ (deg\ q \le 2n + 1).$$

Now for a function $f(x, y)$, defined on rectangle R, let

$$Q(f) := \sum_{i=0}^{n} \sum_{j=0}^{n} s_j w_i f(x_i, y_j).$$

For a polynomial $p(x,y)$, let $p_y(x) = p(x,y)$(a polynomial in x with y fixed), and let $p_x(y) = p(x,y)$(a polynomial in y with x fixed).

We are claming that if $p(x,y)$ is a polynomial, with $deg\ p_y(x) \leq 2n+1$ and $deg\ p_x(y) \leq 2n+1$, then

$$Q(p) = \int_c^d \int_a^b p(x,y)dxdy (\equiv \int \int_R p(x,y)dxdy).$$

If we fix x and consider

$$g(y) := p_x(y) = p(x,y),$$

then

$$deg\ g \leq 2n+1,$$

so

$$\int_c^d g(y)dy = \sum_{j=0}^{n} s_j g(y_j),$$

which is the same as

$$H(x) := \int_c^d p(x,y)dy = \sum_{j=0}^{n} s_j p(x,y_j)\ (a \leq x \leq b).$$

So we have

$$\int_a^b \int_c^d p(x,y)dydx = \int_a^b \int_c^d p_x(y)dydx = \int_a^b H(x)dx$$

$$= \int_a^b (\sum_{j=0}^{n} s_j p(x,y_j))dx$$

$$= \sum_{j=0}^{n} s_j \int_a^b p(x,y_j)dx\ (deg\ p(x,y_j) \leq 2n+1)$$

$$= \sum_{j=0}^{n} s_j \sum_{i=0}^{n} w_i p(x_i, y_j) = \sum_{j=0}^{n} \sum_{i=0}^{n} s_j w_i p(x_i, y_j)$$

$$= Q(p(x, y)).$$

So $Q(p) = \int \int_R p(x, y) dx dy$ whenever $deg\ p_x(y) \leq 2n + 1$ and $deg\ p_y(x) \leq 2n + 1$. For a general function $f(x, y)$, that is defined and continuous on R, we may approximate $\int \int_R f(x, y) dx dy$ by $Q(f)$. In appendix-5 we will illustrate this approximation with numerical examples.

References

[CF] R.E. Curto, L.A. Fialkow,"Recursiveness, positivity, and truncated moment problems", Houston Journal of Mathematics 17(1991), 603-635.

[HK] Hoffman, R.Kunze, Linear Algebra, Prentice-Hill, 1961.

Department of Computer Science, State University of New York, New Paltz, NY 12561, USA