TRUNCATED MULTIVARIABLE MOMENT PROBLEMS WITH FINITE VARIETY

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Abstract. Let $\beta \equiv \{\beta_i\}_{i\in\mathbb{Z}_+^d,|i|\le 2n}$ denote a real d-dimensional multisequence of degree $2n$, with moment matrix $\mathcal{M}(n)$, and let $\mathcal{V} \equiv V(\mathcal{M}(n))$ denote the associated algebraic variety. For the case $v \equiv card\ \mathcal{V} < +\infty$, we prove that β has a representing measure if and only if $r \equiv rank\ \mathcal{M}(n) \le v$ and there exists a positive moment matrix extension $\mathcal{M} \equiv \mathcal{M}(n+v-r+1)$ satisfying $rank\ \mathcal{M} \le card\ V(\mathcal{M})$. For the class of *recursively determinate* moment matrices $\mathcal{M}(n)$, we present a computational algorithm for establishing the existence (or nonexistence) of an extension $\mathcal{M}$ as above and, in the positive case, for computing a minimal representing measure for β. We also show that for the case $r < v < +\infty$, it is possible for β to admit a representing measure μ with $card\ supp\ \mu < v$; equivalently, in this case $supp\ \mu$ may be a proper subset of $V(\mathcal{M}(n))$.

1. Introduction

Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i\in\mathbb{Z}_+^d,|i|\le 2n}$ denote a real d-dimensional multisequence of degree $2n$. The *truncated moment problem* for β concerns necessary and sufficient conditions for the existence of a positive Borel measure μ on $\mathbb{R}^d$ such that $\beta_i = \int_{\mathbb{R}^d} x^i\, d\mu$, $|i| \le 2n$ (here, for $x \equiv (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $i \equiv (i_1,\ldots,i_d) \in \mathbb{Z}_+^d$, we set $x^i := x_1^{i_1}\cdots x_d^{i_d}$ and $|i| = i_1+\cdots+i_d$). Conditions for such a *representing measure* μ are usually expressed in terms of positivity and extension properties of the moment matrix $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ corresponding to β, or conditions on the algebraic variety $\mathcal{V} \equiv V(\mathcal{M}(n))$ associated to β (cf. Theorems 1.1 and 3.1). Positivity, extension, and variety are also prominent themes in the classical full moment problem (cf. [Akh] [PV] [ST] [Sch] [Sto1] [SZ]); a result of Stochel [Sto2] shows that a full moment sequence $\beta^{(\infty)}$ admits a representing measure supported in a closed set $K \subset \mathbb{R}^d$ if and only if each truncation $\beta^{(2n)}$ admits such a measure. In the present note, we solve certain cases of the truncated moment problem *algorithmically.* For these cases, we do not have a set of necessary and sufficient conditions in the traditional sense, but we can nevertheless test an individual sequence β to determine whether or not it admits a representing measure and, if so, we can explicitly compute a finitely atomic representing measure having the fewest atoms possible.

As we discuss in Section 2, a sequence $\beta \equiv \beta^{(2n)}$ has a representing measure if and only if there is some integer $k \ge 1$ such that $\mathcal{M}(n)$ admits a positive moment matrix extension $\mathcal{M}(n+k)$ satisfying $rank\ \mathcal{M}(n+k) = rank\ \mathcal{M}(n+k-1)$. The crux of the truncated moment problem is to predict the existence and estimate the minimal value of such an integer k. Linear dependence relations in the columns of $\mathcal{M}(n)$ determine both its rank and its variety $\mathcal{V} \equiv V(\mathcal{M}(n))$, and in the sequel we study interrelationships between the column structure of $\mathcal{M}(n)$ and the variety. For the case when $v \equiv card\ \mathcal{V} < +\infty$, in Theorem 2.1 we show that β has a representing measure if and only if $r \equiv rank\ \mathcal{M}(n) \le v$ and $\mathcal{M}(n)$ admits successive positive moment matrix extensions $\mathcal{M}(n+1),\ldots,\mathcal{M}(n+v-r+1)$ such that $rank\ \mathcal{M}(n+v-r+1) \le card\ V(\mathcal{M}(n+v-r+1))$; in this case, we can take k (as above) with $k \le v-r+1$. For the class of *recursively determinate* moment matrices with finite variety, we show that the existence (or nonexistence) of a convergent extension sequence and a representing measure (as above) can be completely determined from the

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dependence relations in the columns of $\mathcal{M}(n)$ in at most $v-r+1$ extension steps (Theorem 4.3 and Algorithm 4.10). In these results, it is not necessary to explicitly compute the points of $\mathcal{V}$; it is sufficient to know that the variety is finite. Further, a version of the algorithm applies to an arbitrary recursively determinate $\mathcal{M}(n)$, and the algorithm can also be used to decide whether an arbitrary $\mathcal{M}(n)$ is recursively determinate (cf. Remark 4.11).

Representing measures μ always satisfy *card supp* $\mu \geq$ *rank* $\mathcal{M}(n)$ (cf. (1.2) below). Theorem 3.2 provides a test for the existence of *rank* $\mathcal{M}(n)$-atomic (minimal) representing measures in cases where β has finite variety. This allows us to exhibit for the first time a sequence β, with finite variety $\mathcal{V}$, having a representing measure whose support is a proper subset of $\mathcal{V}$ (Example 3.3); equivalently, this example shows that it is possible to have a *rank* $\mathcal{M}(n)$-atomic representing measure in a case where $r < v < +\infty$. More generally, Theorem 3.10 implies the existence of *rank* $\mathcal{M}(n)$-atomic representing measures in cases in which $v < +\infty$ and $v-r$ is arbitrarily large, and in all of these cases we can take k (as above) with $k=1$. The truncated moment problem has applications in multivariable cubature [EFP] [FP] and polynomial optimization [Las1] [Las2] [Lau1] [Lau2] [Lau3]. With a view toward such applications, we provide several numerical examples which illustrate how to implement our methods with concrete moment data (cf. Examples 3.3, 3.9, 3.11 4.15, 4.17, 4.18).

Let $\mathcal{P} \equiv \mathbb{R}[x_1,\ldots,x_d]$ denote the algebra of real valued d-variable polynomials, and for $m \geq 1$, let $\mathcal{P}_m$ denote the subspace of polynomials p with *deg* $p \leq m$; we note for future reference that *dim* $\mathcal{P}_m = \begin{pmatrix} d+m \\ m \end{pmatrix}$. For $p(x) \equiv \sum_{|i|\leq m} a_i x^i \in \mathcal{P}_m$, let $\hat{p} \equiv (a_i)$ denote the coefficient (column) vector of p relative to the basis for $\mathcal{P}_m$ consisting of the monomials in $\mathcal{P}_m$ in degree-lexicographic order. Corresponding to β we have the *Riesz functional* $\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \longmapsto \mathbb{R}$, which assigns to $p(x) \equiv \sum_{|i|\leq 2n} a_i x^i$ the value $\Lambda(p) := \sum_{|i|\leq 2n} a_i \beta_i$; if μ is a representing measure for β, then clearly $\Lambda(p) = \int p \; d\mu$. Following [CF1] [CF5], we associate to β the *moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$, with rows and columns X^i indexed by the monomials of $\mathcal{P}_n$ in degree-lexicographic order; for example, with $d=3$, $n=2$, the columns of $\mathcal{M}(2)$ are 1, X_1, X_2, X_3, X_1^2, X_1X_2, X_1X_3, X_2^2, X_2X_3, X_3^2. The entry in row X^i, column X^j of $\mathcal{M}(n)$ is β_{i+j}, so $\mathcal{M}(n)$ is a real symmetric matrix characterized by $\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = \Lambda(pq)$ $(p, q \in \mathcal{P}_n)$. If μ is a representing measure for β, then $\langle \mathcal{M}(n)\hat{p}, \hat{p}\rangle = \Lambda(p^2) = \int p^2 \; d\mu \geq 0$, and since $\mathcal{M}(n)$ is real symmetric, it follows that $\mathcal{M}(n)$ is positive semidefinite ($\mathcal{M}(n) \succeq 0$).

Let $\mathcal{C}_{\mathcal{M}(n)}$ denote the column space of $\mathcal{M}(n)$. Corresponding to $p(x) \equiv \sum_{|i|\leq n} a_i x^i \in \mathcal{P}_n$ is the element $p(X)$ of $\mathcal{C}_{\mathcal{M}(n)}$ defined by $p(X) := \sum_{|i|\leq n} a_i X^i$; thus, $p(X) = \mathcal{M}(n)\hat{p}$. If β admits a representing measure μ, then

$$(1.1) \qquad for \; p \in \mathcal{P}_n, \; p|supp \; \mu \equiv 0 \Longleftrightarrow p(X) = 0 \; [CF1, Prop.3.1] \; [CF6, Prop.2.10]$$

and

$$(1.2) \qquad r \equiv rank \; \mathcal{M}(n) \leq card \; supp \; \mu \; [CF1, Cor.3.7] \; [CF6, Cor.2.12].$$

We say that a representing measure μ is *minimal* if *card supp* $\mu \leq$ *card supp* ν for every representing measure ν; (1.2) shows that a *rank* $\mathcal{M}(n)$-atomic representing measure is minimal. The following result of [CF1] is our basic tool for constructing *rank* $\mathcal{M}(n)$-atomic minimal representing measures.

Theorem 1.1. *(Flat Extension Theorem, Part 1 [CF1, Thm. 7.10] [CF6, Thm. 2.19]) $\beta \equiv \beta^{(2n)}$ admits a rank $\mathcal{M}(n)$-atomic representing measure if and only if $\mathcal{M}(n) \succeq 0$ and $\mathcal{M}(n)$ admits an extension to a moment matrix $\mathcal{M}(n+1)$ satisfying rank $\mathcal{M}(n+1) =$ rank $\mathcal{M}(n)$.*

In the sequel, we refer to a rank-preserving extension $\mathcal{M}(n+1)$ as a *flat* extension of $\mathcal{M}(n)$ (cf. [CF1] [CF6])).

Following [CF1], we define the *algebraic variety* of β (or of $\mathcal{M}(n)$) by $V(\mathcal{M}(n)) := \bigcap_{p\in\mathcal{P}_n, p(X)=0} \; \mathcal{Z}(p)$, where $\mathcal{Z}(p) := \{w \in \mathbb{R}^d : p(w) = 0\}$. It is straightforward to check that if $\{q_i\}_{i=1}^s$ is a linear basis for $\mathcal{N}_n \equiv \{p \in \mathcal{P}_n : p(X) = 0\}$, then $V(\mathcal{M}(n)) = \bigcap_{1\leq i\leq s} \; \mathcal{Z}(q_i)$. In view of (1.1), it is clear that if

μ is a representing measure for β, then

$$supp\ \mu \subset V(\mathcal{M}(n)), \tag{1.3}$$

whence

$$r \leq card\ supp\ \mu \leq v \equiv card\ V(\mathcal{M}(n)). \tag{1.4}$$

The *extremal* case of the truncated moment problem, when $r = v$, has been solved in [CFM]; in this case, $\beta^{(2n)}$ has a representing measure if and only if $\mathcal{M}(n)$ is positive and β is *consistent* (cf. Section 3). In the present note we focus primarily on the more general case when $\mathcal{M}(n)$ has *finite variety*, i.e., $v < +\infty$. The following result provides the main tool for explicitly computing a representing measure associated with a flat extension.

Theorem 1.2. *(Flat Extension Theorem, Part 2 [CF6, Thm. 1.2]) Suppose $\mathcal{M}(n)$ is positive semidefinite and admits a flat extension $\mathcal{M}(n+1)$. Then card $V(\mathcal{M}(n+1)) = r \equiv rank\ \mathcal{M}(n)$, and suppose $V(\mathcal{M}(n+1)) \equiv \{w_j\}_{j=1}^{r} \subset \mathbb{R}^d$. Suppose $\mathcal{B} \equiv \{X^{i_1}, \ldots, X^{i_r}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$ and let $U_{\mathcal{B}}$ denote the $r \times r$ matrix whose element in row k, column j is $w_j^{i_k}$. Then $U_{\mathcal{B}}$ is invertible, and the unique representing measure for $\mathcal{M}(n+1)$ is of the form $\mu \equiv \sum_{j=1}^{r} \rho_j \delta_{w_j}$, where $\rho \equiv (\rho_1, \ldots, \rho_r)$ is uniquely determined from $U_{\mathcal{B}}\rho^T = (\beta_{i_1}, \ldots, \beta_{i_r})^T$.*

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2. Representing measures in the finite variety case.

In this section we characterize the existence of representing measures for $\beta \equiv \beta^{(2n)}$ in terms of moment matrix extensions. Theorem 1.1 shows that if $\mathcal{M}(n)$ ($\succeq 0$) admits a flat extension $\mathcal{M}(n+1)$, then β certainly has a representing measure. In cases where $\mathcal{M}(n)$ does not have a flat extension, β may nevertheless admit a representing measure, and in the sequel we study this phenomenon. In [CF2, Thm. 1.5]] we proved that β has a *finitely atomic* representing measure if and only if $\mathcal{M}(n)$ has a positive extension $\mathcal{M}(n+k-1)$ (for some $k \geq 1$) such that $\mathcal{M}(n+k-1)$ admits a flat extension $\mathcal{M}(n+k)$. A recent result of Bayer and Teichmann concerning multivariable cubature [BT] readily implies that if β admits a representing measure, then β actually has a finitely atomic representing measure (cf. [Lau3]). Thus, the above extension criterion is both necessary and sufficient for the existence of representing measures for $\beta^{(2n)}$.

Let $k \geq 1$ and suppose that $\mathcal{M}(n+1), \ldots, \mathcal{M}(n+k)$ is a sequence of successive positive extensions of $\mathcal{M}(n)$ such that $\mathcal{M}(n+i)$ is a rank-increasing extension of $\mathcal{M}(n+i-1)$ if $k > 1$ and $1 \leq i \leq k-1$, and $\mathcal{M}(n+k)$ is a flat extension of $\mathcal{M}(n+k-1)$. We refer to such a sequence as a *convergent extension sequence of length* k, which we denote by $\mathcal{M}(n) \longrightarrow \cdots \longrightarrow \mathcal{M}(n+k)$. [CF6] implies that in this case $\mathcal{M}(n+k)$ admits unique successive positive extensions $\mathcal{M}(n+k+1)$, $\mathcal{M}(n+k+2), \ldots$, all of which are flat extensions. Further, Theorem 1.2 implies that $\mathcal{M}(n+k)$ admits a unique representing measure ν, which is clearly a representing measure for β, and which is characterized by $supp\ \nu = V(\mathcal{M}(n+k))$ and $card\ supp\ \nu = rank\ \mathcal{M}(n+k)$).

We begin with the following characterization of the existence of representing measures in the finite variety case.

Theorem 2.1. *Suppose $v < +\infty$. $\beta \equiv \beta^{(2n)}$ admits a representing measure if and only if $r \leq v$ and $\mathcal{M}(n)(\beta)$ has a positive moment matrix extension $\mathcal{M}(n+v-r+1)$ satisfying rank $\mathcal{M}(n+v-r+1) \leq$ card $V(\mathcal{M}(n+v-r+1))$. In this case, $\mathcal{M}(n)$ has a convergent extension sequence with length at most $v-r+1$.*

As a practical matter, the extension $\mathcal{M}(n+v-r+1)$ is determined via successive intermediate extensions $\mathcal{M}(n+1), \ldots, \mathcal{M}(n+v-r)$. Thus, $v-r$, which we refer to as the *gap* in $\mathcal{M}(n)(\beta)$, provides a measure of the complexity of determining the existence of a representing measure in the finite variety case. Let β be an arbitrary sequence, possibly with infinite variety. A modification of the proof of Theorem 2.1 shows that if β has a representing measure, then it has a convergent extension sequence with length at most $dim\ \mathcal{P}_{2n} - r + 1$ (cf. Proposition 2.3). Part of our motivation for focusing on the finite variety case can be seen from examples. In Example 4.15 we have a planar $\mathcal{M}(5)$ with $v - r = 6$ and $dim\ \mathcal{P}_{10} - r = 47$, and in Example 4.17 we have a 3-dimensional $\mathcal{M}(4)$ with $v - r = 7$ and $dim\ \mathcal{P}_8 - r = 138$.

The upper estimate $v - r + 1$ for the length of some convergent extension sequence for $\mathcal{M}(n)$ is sharp in the following sense. If $\mathcal{M}(n)(\beta)$ is extremal, i.e., $v = r$, and β admits a representing measure (cf. Theorem 3.4), then (1.3) and Theorem 1.1 imply that $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$, so in this case the minimal length of a convergent extension sequence is precisely $v - r + 1$ ($= 1$). Further, [CF7] illustrates cases of the truncated moment problem for measures supported in an hyperbola, with $v - r = 1$ and where the minimal length convergent extension sequence has length 2 ($= v - r + 1$). On the other hand, in Example 4.15 we have $v - r + 1 = 7$, with a convergent extension sequence of length 3. The most dramatic divergence of $v - r + 1$ from the minimal length of a convergent extension sequence is described by Theorem 3.10 and the remarks immediately preceding it.

For the proof of Theorem 2.1 we require certain facts about positive extensions of moment matrices. Consider a moment matrix extension

$$\mathcal{M}(n+1) \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^T & C(n+1) \end{pmatrix}.$$

A result of Smul'jan [Smu] implies that $\mathcal{M}(n+1) \succeq 0$ if and only if $\mathcal{M}(n) \succeq 0$, there exists a matrix W such that $B(n+1) = \mathcal{M}(n)W$ (equivalently, $Ran\ B(n+1) \subset Ran\ \mathcal{M}(n)$ [D]), and $C(n+1) \succeq W^T\mathcal{M}(n)W$. In this case, $\mathcal{M}(n+1)$ is a flat extension, i.e., $rank\ \mathcal{M}(n+1) = rank\ \mathcal{M}(n)$, if and only if $C(n+1) = W^T\mathcal{M}(n)W$. Suppose $\mathcal{M}(n+1) \geq 0$ and let $p \in \mathcal{P}_n$; the Extension Principle of [F1] shows that if $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, then $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$, i.e., column dependence relations in $\mathcal{M}(n)$ extend to $\mathcal{M}(n+1)$. It follows that if $\mathcal{M}(n+1) \geq 0$, then $V(\mathcal{M}(n+1)) \subset V(\mathcal{M}(n))$. In this case, we note for future reference that in computing $V(\mathcal{M}(n+1))$, we may ignore any dependence relation in $\mathcal{C}_{\mathcal{M}(n+1)}$ of the form $(pq)(X) = 0$, where $p \in \mathcal{P}_n$, $pq \in \mathcal{P}_{n+1}$, and $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. Indeed, it is clear that $V(\mathcal{M}(n)) \subset \mathcal{Z}(pq)$, so the relation $(pq)(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$ cannot subtract from $V(\mathcal{M}(n))$ in computing $V(\mathcal{M}(n+1))$.

Proof of Theorem 2.1. Suppose $r \leq v$ and $\mathcal{M}(n)(\beta)$ has a positive extension $\mathcal{M}(n+v-r+1)$ satisfying $rank\ \mathcal{M}(n+v-r+1) \leq card\ V(\mathcal{M}(n+v-r+1))$. We have $r \equiv rank\ \mathcal{M}(n) \leq rank\ \mathcal{M}(n+1) \leq \cdots \leq rank\ \mathcal{M}(n+v-r+1)$. Since $\mathcal{M}(n+v-r+1) \geq 0$, the Extension Principle implies that $card\ V(\mathcal{M}(n+v-r+1)) \leq \cdots \leq card\ V(\mathcal{M}(n+1)) \leq card\ V(\mathcal{M}(n)) \equiv v$. Since $rank\ \mathcal{M}(n+v-r+1) \leq card\ V(\mathcal{M}(n+v-r+1))$, if each extension $\mathcal{M}(n+i)$ $(1 \leq i \leq v-r+1)$ is strictly rank increasing, then it follows that $v \geq card\ V(\mathcal{M}(n+v-r+1)) \geq rank\ \mathcal{M}(n+v-r+1) \geq r + (v - r + 1) = v + 1$, a contradiction. Thus, there exists i, $1 \leq i \leq v - r + 1$, such that $rank\ \mathcal{M}(n+i-1) = rank\ \mathcal{M}(n+i)$, whence the existence of a $rank\ \mathcal{M}(n+i)$-atomic representing measure follows from Theorem 1.1.

Conversely, suppose β admits a representing measure ν. (1.4) shows that $r \leq v$. Since $supp\ \nu \subset V(\mathcal{M}(n))$ (cf. (1.3)) and $v < +\infty$, then ν is finitely atomic, and thus has finite moments of all orders. Now $\mathcal{M}(n) \equiv \mathcal{M}(n)[\nu]$ (computed using $\beta_i := \int x^i\ d\nu$) admits positive extensions $\mathcal{M}(n+i)[\nu]$ for every $i \geq 1$. In particular, since $\mathcal{M}(n+v-r+1)[\nu]$ admits a representing measure (namely, ν), (1.4) implies that $rank\ \mathcal{M}(n+v-r+1)[\nu] \leq card\ V(\mathcal{M}(n+v-r+1)[\nu])$. $\square$

In Section 4 we show how to verify the existence of the extensions in Theorem 2.1 for $\mathcal{M}(n)$ recursively determinate. Theorem 2.1 shows that in the finite variety case, $v - r$ is an upper bound on the minimal length of a convergent extension sequence. For this reason, it would be desirable if $v - r$ were bounded, but the following result shows that this is not the case.

Proposition 2.2. *Let $d = 2$. For $n > 0$, there exists $\mathcal{M}(n)$ (having a representing measure) with $v < +\infty$ and $v - r \geq (n-1)(n-2)/2$.*

Proof. Recall from Bezout's Theorem [CLO1] that if $\mathcal{M}(n)$ has finite variety, then $v \leq n^2$. Let $p(x,y)$ and $q(x,y)$ denote polynomials of degree n having exactly n^2 common zeros in the plane. For example, let $p(x,y) = y - (x - x_1)\cdots(x - x_n)$ for distinct $x_1, \ldots, x_n$ and let $q(x,y) = (y - y_1)\cdots(y - y_n)$ for distinct $y_1, \ldots, y_n$ sufficiently close to 0. Let $\mathcal{V} = \mathcal{Z}(p) \bigcap \mathcal{Z}(q)$ and let μ be a positive measure with *supp* $\mu = \mathcal{V}$. Consider $M = \mathcal{M}(n)[\mu]$. Since $n^2 = card\ \mathcal{Z}(p) \bigcap \mathcal{Z}(q) \geq v \geq card\ supp\ \mu = n^2$, we have $v = n^2$. Further, (1.1) implies that $p(X,Y) = 0$ and $q(X,Y) = 0$ in $\mathcal{C}_M$, so $r = rank\ M \leq dim\ \mathcal{P}_n - 2 = (n+1)(n+2)/2 - 2$, whence the result follows. □

We conclude this section with a partial analogue of Theorem 2.1 for an arbitrary sequence $\beta^{(2n)}$, which may have infinite variety.

Proposition 2.3. *If $\beta \equiv \beta^{(2n)}$ has a representing measure, then there is a convergent extension sequence $\mathcal{M}(n)(\beta) \longrightarrow \ldots \longrightarrow \mathcal{M}(n+k)$ with $k \leq dim\ \mathcal{P}_{2n} - r + 1$.*

Proof. Since β has a representing measure, [BT] implies that there is a representing measure ν with *card supp* $\nu \leq dim\ \mathcal{P}_{2n}$. If, for some j, we have a strictly rank increasing extension sequence $\mathcal{M}(n)(\beta) \equiv \mathcal{M}(n)[\nu], \ldots, \mathcal{M}(n+j)[\nu]$, then, with $r = rank\ \mathcal{M}(n)$, we have $r + j \leq rank\ \mathcal{M}(n+j)[\nu] \leq card\ supp\ \nu \leq dim\ \mathcal{P}_{2n}$. Thus, $j \leq dim\ \mathcal{P}_{2n} - r$, and if $j = dim\ \mathcal{P}_{2n} - r$, then $\mathcal{M}(n+j+1)[\nu]$ is a flat extension of $\mathcal{M}(n+j)[\nu]$ □

Remark 2.4. (i) For the case of the plane ($d = 2$), Proposition 2.3 implies a convergent extension sequence with length at most $dim\ \mathcal{P}_{2n} - r + 1$, where $dim\ \mathcal{P}_{2n} = (2n+1)(n+1)$. For planar moment matrices with finite variety, Theorem 2.1 gives the improved estimate $v - r + 1$, since in this case $v \leq n^2$ (by Bezout's Theorem).
(ii) In [CF2, Theorem 1.5], for the truncated complex moment problem in the plane, the existence of a finitely atomic representing measure for a complex sequence $\gamma^{(2n)}$ is shown to be equivalent to the existence of a convergent extension sequence of complex moment matrices of length at most $2n^2 + 7n + 7$.

3. Flat extensions in the case $r \leq v < +\infty$.

In this section we study the existence of flat extensions in the finite variety case. Our motivation comes from the following solution to the truncated moment problem on planar curves of degree 1 or 2.

Theorem 3.1. *([F3] [CF4] [CF5] [CF7]) Let $d = 2$ and suppose $deg\ p(x,y) \leq 2$. $\beta^{(2n)}$ has a representing measure supported in the curve $p(x,y) = 0$ if and only if $\mathcal{M}(n)$ has a column dependence relation $p(X,Y) = 0$ and $\mathcal{M}(n)$ is positive semidefinite, recursively generated, and satisfies $r \leq v$.*

Under the conditions of Theorem 3.1, whenever $v = +\infty$, then $\mathcal{M}(n)$ admits a flat extension. By contrast, in every case in which $r < v < +\infty$, then it transpires that $v = r + 1$ and a minimal representing measure corresponds to a positive, rank-increasing extension $\mathcal{M}(n+1)$ followed by a flat extension $\mathcal{M}(n+2)$. Thus, in the finite variety case of Theorem 3.1, each representing measure μ satisfies *supp* $\mu = V(\mathcal{M}(n))$. In the sequel we examine whether such rigidity is a general feature of the truncated moment problem in the finite variety case.

We begin with a computational test for the existence of flat extensions $\mathcal{M}(n+1)$ in the case when $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ has finite variety and the elements of $V(\mathcal{M}(n))$ are known exactly. Let $\mathcal{V} \equiv \{w_1, \ldots, w_s\}$ be a finite subset of $\mathbb{R}^d$. Following [CFM], we define the matrix $W_m[\mathcal{V}]$ with s rows and with columns $\mathcal{X}^j$ indexed by the monomials in $\mathcal{P}_m$ in degree-lexicographic order. The entry of $W_m[\mathcal{V}]$ in row i ($1 \leq i \leq s$), column $\mathcal{X}^j$ ($j \in \mathbb{Z}_+^d,\ |j| \leq m$) is w_i^j; we further set $U_m[\mathcal{V}] = W_m[\mathcal{V}]^T$ (the transpose). In the sequel we set $\tau(m) := dim\ \mathcal{P}_m = \begin{pmatrix} m+d \\ m \end{pmatrix}$ and we let $p_1, \ldots, p_\tau$ denote the list of monomials in $\mathcal{P}_m$ in degree-lexicographic order. Given $\mathcal{M}(n)(\beta)$, let $\tau \equiv \tau(2n)$, $r = rank\ \mathcal{M}(n)$, $v = card\ V(\mathcal{M}(n))$, and set $L_\beta := (\Lambda_\beta(p_1), \ldots, \Lambda_\beta(p_\tau))^T \in \mathbb{R}^\tau$. Let $\mathcal{B} \equiv \{X^{i_1}, \ldots, X^{i_r}\}$ denote a basis for $\mathcal{C}_{\mathcal{M}(n)}$, the column space of $\mathcal{M}(n)$. For the case when $\mathcal{V}$ (as above) is a subset of $V(\mathcal{M}(n))$, let $W_\mathcal{B}[\mathcal{V}]$ denote the compression of $W_n[\mathcal{V}]$ to columns $\mathcal{X}^{i_1}, \ldots, \mathcal{X}^{i_r}$ and let $U_\mathcal{B}[\mathcal{V}] = W_\mathcal{B}[\mathcal{V}]^T$.

Theorem 3.2. *For $\beta \equiv \beta^{(2n)}$, suppose $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta) \geq 0$ and let $r = rank\ \mathcal{M}(n)$. β admits an r-atomic representing measure μ (equivalently, $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$) if and only if there exists an r-element subset $\mathcal{V}$ of $V(\mathcal{M}(n))$ for which $L_\beta \in Ran\ U_{2n}[\mathcal{V}]$. In this case, if $\mathcal{V} \equiv \{w_1, \ldots, w_r\}$ and if $\mathcal{B} \equiv \{X^{i_1}, \ldots, X^{i_r}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$, then we can take $\mu := \sum_{i=1}^r \rho_i \delta_{w_i}$, where the densities $\rho \equiv (\rho_1, \ldots, \rho_r)$ are uniquely determined by $U_\mathcal{B}[\mathcal{V}]\rho^T = (\beta_{i_1}, \ldots, \beta_{i_r})^T$; a flat extension of $\mathcal{M}(n)$ is then $\mathcal{M}(n+1)[\mu]$.*

There is no requirement in Theorem 3.2 that $V(\mathcal{M}(n))$ be finite, and examples of [CF5] [CF7] illustrate flat extensions in cases where the variety is infinite. However, since Theorem 3.2 entails testing the r-element subsets of $V(\mathcal{M}(n))$, it is of practical interest primarily in the case when $v < +\infty$ and v is close to r. We illustrate Theorem 3.2 with an example in which $v = r + 1$. This appears to be the first example in the literature of a flat extension in a case with $r < v < +\infty$. It also provides the first example in the finite variety case of a representing measure whose support is a proper subset of $V(\mathcal{M}(n))$.

Example 3.3. Consider $\mathcal{M}(3)(\beta)$ defined as $\mathcal{M}(3) := \begin{pmatrix} \mathcal{M}(2) & B(3) \\ B(3)^T & C(3) \end{pmatrix}$, where

$$\mathcal{M}(2) := \begin{pmatrix} 8 & 0 & 0 & 78 & 1446 & 32838 \\ 0 & 78 & 1446 & 0 & 0 & 0 \\ 0 & 1446 & 32838 & 0 & 0 & 0 \\ 78 & 0 & 0 & 1446 & 32838 & 794886 \\ 1446 & 0 & 0 & 32838 & 794886 & 19651398 \\ 32838 & 0 & 0 & 794886 & 19651398 & 489352326 \end{pmatrix},$$

$$B(3) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1446 & 32838 & 794886 & 19651398 \\ 32838 & 794886 & 19651398 & 489352326 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$C(3) := \begin{pmatrix} 32838 & 794886 & 19651398 & 489352326 \\ 794886 & 19651398 & 489352326 & 12216629958 \\ 19651398 & 489352326 & 12216629958 & 305262005766 \\ 489352326 & 12216629958 & 305262005766 & 7630169896518 \end{pmatrix}.$$

A calculation using nested determinants reveals that $\mathcal{M}(3)$ is positive semidefinite, with rank 8 and column dependence relations $f(X,Y) = 0$ and $g(X,Y) = 0$, where $f(x,y) := y - x^3$ and $g(x,y) := 900x - 361x^3 - 900y + 399x^2y - 39xy^2 + y^3$. It follows that $V(\mathcal{M}(3)) = \mathcal{Z}(f) \bigcap \mathcal{Z}(g) \equiv \{w_i\}_{i=1}^9$, where $w_1 = (0,0)$, $w_2 = (-3,-27)$, $w_3 = (-2,-8)$, $w_4 = (-1,-1)$, $w_5 = (-5,-125)$, $w_6 = (1,1)$, $w_7 = (2,8)$, $w_8 = (3,27)$, $w_9 = (5,125)$. Let $\mathcal{V}_i := V(\mathcal{M}(3)) \setminus \{w_i\}$ $(1 \leq i \leq 9)$. For $2 \leq i \leq 9$, calculations show that $L_\beta \notin Ran\ U_{2n}[\mathcal{V}_i]$, so there is no 8-atomic representing measure for β with support $\mathcal{V}_i$. However, for $\mathcal{V}_1 (= \{w_i\}_{i=2}^9)$, we see that $L_\beta \in Ran\ U_{2n}[\mathcal{V}_1]$. With the column basis $\mathcal{B} := \{1,\ X, Y,\ X^2,\ XY,\ Y^2,\ X^2Y,\ XY^2\}$, Theorem 3.2 implies that β has a minimal representing measure of the form $\mu := \sum_{i=1}^8 \rho_i \delta_{w_{i+1}}$, where the densities $\rho \equiv (\rho_1, \ldots, \rho_8)$ are uniquely determined by $U_\mathcal{B}[\mathcal{V}_1]\rho^T = (8,\ 0,\ 0, 78\ , 1446,\ 32838, 0, 0)^T$. A calculation yields $\rho_i = 1$

$(1 \le i \le 8)$. Theorem 3.2 now shows that $\mathcal{M}(4)[\mu]$ is a flat extension of $\mathcal{M}(3)$, where

$$B(4)[\mu] = \begin{pmatrix} 1446 & 32838 & 794886 & 19651398 & 489352326 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 32838 & 794886 & 19651398 & 489352326 & 12216629958 \\ 794886 & 19651398 & 489352326 & 12216629958 & 305262005766 \\ 19651398 & 489352326 & 12216629958 & 305262005766 & 7630169896518 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $C(4)[\mu] =$

$$\begin{pmatrix} 794886 & 19651398 & 489352326 & 12216629958 & 305262005766 \\ 19651398 & 489352326 & 12216629958 & 305262005766 & 7630169896518 \\ 489352326 & 12216629958 & 305262005766 & 7630169896518 & 190741838947206 \\ 12216629958 & 305262005766 & 7630169896518 & 190741838947206 & 4768434352539078 \\ 305262005766 & 7630169896518 & 190741838947206 & 4768434352539078 & 119209854443408646 \end{pmatrix}.$$

Note also that the column relations $f(X,Y)=0$ and $g(X,Y)=0$ show that $\mathcal{M}(3)$ is recursively determinate (cf. Section 4), so the existence of a representing measure can also be approached through Algorithm 4.10. Indeed, in the language of Section 4, one can construct a *recursively determined* extension $\mathcal{M}(4)$ by imposing $(xf)(X,Y)=0$ and $(yg)(X,Y)=0$. A calculation shows that *rank* $\mathcal{M}(4)=8$, so $\mathcal{M}(4)$ is a flat (hence positive) extension, with a corresponding representing measure which coincides with μ. $\square$

For the proof of Theorem 3.2, we require some preliminary results and notation. Recall from [CFM] that $\beta \equiv \beta^{(2n)}$ is *consistent* if

$$p \in \mathcal{P}_{2n},\ p|V(\mathcal{M}(n)) \equiv 0 \Longrightarrow \Lambda_\beta(p)=0. \tag{3.1}$$

Consistency is clearly a necessary condition for representing measures. Consistency plays an essential role in the *extremal* truncated moment problem, the case when $r=v$. In this case, if $\mathcal{B} \equiv \{X^{i_1},\dots,X^{i_r}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$ and $\mathcal{V} \equiv V(\mathcal{M}(n)) = \{w_1,\dots,w_r\}$, let $U_{\mathcal{B}}[\mathcal{V}]$ denote the $r \times r$ matrix whose element in row k, column j is $w_j^{i_k}$.

Theorem 3.4. *([CFM, Theorem 4.2) Suppose $r=v$. $\beta \equiv \beta^{(2n)}$ has a representing measure if and only if $\mathcal{M}(n)$ is positive semidefinite and β is consistent. In this case, $U_{\mathcal{B}}[\mathcal{V}]$ is invertible, and the unique representing measure for β is of the form $\mu \equiv \sum_{j=1}^r \rho_j \delta_{w_j}$, where $\rho \equiv (\rho_1,\dots,\rho_r)$ is determined from $U_{\mathcal{B}}[\mathcal{V}]\rho^T = (\beta_{i_1},\dots,\beta_{i_r})^T$.*

The following basic problem of [CFM] remains unsolved:

Question 3.5. *If $\mathcal{M}(n) \succeq 0$, $r \le v$, and β is consistent, does β have a representing measure?*

Let us compare Theorem 3.4 to the case of Theorem 2.1 when $r=v$. For this case, Theorem 2.1 shows that β has a representing measure if and only if $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+1)$ satisfying *rank* $\mathcal{M}(n+1) \le$ *card* $V(\mathcal{M}(n+1))$. Since $\mathcal{M}(n+1) \succeq 0$, we have $r =$ *rank* $\mathcal{M}(n) \le$ *rank* $\mathcal{M}(n+1) \le$ *card* $V(\mathcal{M}(n+1)) \le$ *card* $V(\mathcal{M}(n)) = v = r$, so $\mathcal{M}(n+1)$ must be a flat extension. Thus, Theorem 2.1 entails a flat extension, whereas Theorem 3.4 entails consistency. (Of course, if β in Theorem 3.4 is consistent, then the unique representing measure for β also corresponds to a flat extension $\mathcal{M}(n+1)$.) Depending on the specific column dependence relations in a given problem, either consistency or the existence of a flat extension may be easier to check. In the case when $v < +\infty$ and the points of $V(\mathcal{M}(n))$ are known exactly, our next result provides a simple computational test for consistency.

Let $\mathcal{V} \subset V(\mathcal{M}(n))$. We say that β is *$\mathcal{V}$-consistent* if $p \in \mathcal{P}_{2n}$, $p|\mathcal{V} \equiv 0 \Longrightarrow \Lambda_\beta(p) = 0$. In particular, β is consistent if and only if β is $\mathcal{V}$-consistent for $\mathcal{V} = V(\mathcal{M}(n))$. Clearly, if β admits a representing measure μ with *supp* $\mu \subset \mathcal{V} \subset V(\mathcal{M}(n))$, then β is $\mathcal{V}$-consistent. We note that when β is $\mathcal{V}$-consistent, we can always take $\mathcal{V}$ to be finite. Indeed, let $\tau \equiv \tau(2n)$ and for $w \in \mathcal{V}$, let $\pi(w) = (p_1(w), \ldots, p_\tau(w)) \in \mathbb{R}^\tau$ (where, as above, $p_1, ..., p_\tau$ is a listing of the monomials in $\mathcal{P}_{2n}$ in degree-lexicographic order). Let $\{\pi(w_1), \ldots, \pi(w_s)\}$ $(1 \leq s \leq \tau)$ denote a maximal independent subset of $\{\pi(w) : \ w \in \mathcal{V}\}$, and let $\mathcal{V}' = \{w_1, \ldots, w_s\}$. If $p \in \mathcal{P}_{2n}$ and $p|\mathcal{V}' \equiv 0$, then $\langle \hat{p}, \pi(w_i)\rangle = 0$ $(1 \leq i \leq s)$, whence $\langle \hat{p}, \pi(w)\rangle = 0$ for $w \in \mathcal{V}$, i.e., $p|\mathcal{V} \equiv 0$. Now, $p|\mathcal{V} \equiv 0 \Longrightarrow \Lambda_\beta(p) = 0$, so β is $\mathcal{V}'$-consistent. The following reformulation of $\mathcal{V}$-consistency shows that in the case when $\mathcal{V}$ is finite, to establish $\mathcal{V}$-consistency, we need not verify the defining property, but can instead rely on a simpler test.

Proposition 3.6. *Let $\mathcal{V} \equiv \{w_1, \ldots, w_s\} \subset V(\mathcal{M}(n))(\beta)$. β is $\mathcal{V}$-consistent if and only if $L_\beta \in$ Ran $U_{2n}[\mathcal{V}]$. In particular, if $v < +\infty$, β is consistent if and only if $L_\beta \in$ Ran $U_{2n}[V(\mathcal{M}(n))]$.*

Proof. Let $\tau \equiv \tau(2n)$; for $\mathcal{S} \subset \mathbb{R}^\tau$, let $\mathcal{S}^\perp = \{t \in \mathbb{R}^\tau : \ \langle t, s\rangle = 0 \ \forall s \in \mathcal{S}\}$. β is $\mathcal{V}$-consistent if and only if $p \in \mathcal{P}_{2n}$, $p|\mathcal{V} \equiv 0 \Longrightarrow \Lambda_\beta(p) = 0$. Now $p|\mathcal{V} \equiv 0 \Longleftrightarrow W_{2n}[\mathcal{V}]\hat{p} = 0$, and $\Lambda_\beta(p) = 0 \Longleftrightarrow \langle \hat{p}, L_\beta\rangle = 0$. Thus, β is $\mathcal{V}$-consistent if and only if *ker* $W_{2n}[\mathcal{V}] \subset \{L_\beta\}^\perp$ (relative to $\mathbb{R}^\tau$). Since *ker* $W_{2n}[\mathcal{V}] = (Ran\ U_{2n}[\mathcal{V}])^\perp$, it follows that β is $\mathcal{V}$-consistent if and only if $(Ran\ U_{2n}[\mathcal{V}])^\perp \subset \{L_\beta\}^\perp$, or, equivalently (since the underlying spaces are finite dimensional), $L_\beta \in$ *Ran* $U_{2n}[\mathcal{V}]$. □

Remark 3.7. For the case $r = v$ and $\mathcal{V} = V(\mathcal{M}(n))$, the condition of Theorem 3.2 that $L_\beta \in$ *Ran* $U_{2n}[\mathcal{V}]$ is equivalent to the condition that β be consistent. Thus, in this case, Theorem 3.2 is equivalent to [CFM, Thm. 4.2], although the condition $L_\beta \in$ *Ran* $U_{2n}[\mathcal{V}]$ in Theorem 3.2 is apparently easier to verify than is the original consistency condition in [CFM]. The idea of using duality to reformulate consistency is due to H. M. Möller and is used in the proofs of [CFM, Lemmas 2.2-2.3].

Proof of Theorem 3.2. Suppose first that $\mathcal{V} \equiv \{w_1, \ldots, w_r\} \subset V(\mathcal{M}(n)(\beta))$ satisfies $L_\beta \in$ *Ran* $U_{2n}[\mathcal{V}]$. Proposition 3.6 shows that β is $\mathcal{V}$-consistent. Let $\rho \equiv (\rho_1, \ldots, \rho_r)$ satisfy $U_{2n}[\mathcal{V}]\rho^T = L_\beta$, so that $\Lambda_\beta(p_i) = \sum_{j=1}^r \rho_j p_i(w_j)$ (where, as above, $p_1, \ldots, p_\tau$ is a listing of the monomials in $\mathcal{P}_{2n}$). If we define the r-atomic measure μ by $\mu := \sum_{j=1}^r \rho_j \delta_{w_j}$, then clearly μ interpolates all of the moments of β, i.e., $\beta_i = \int x^i d\mu(x)$ $(|i| \leq 2n)$. To complete the proof it suffices to show that $\mu \geq 0$, i.e., $\rho_j > 0$ $(1 \leq j \leq r)$. To this end, let $\mathcal{B} \equiv \{X^{i_1}, \ldots, X^{i_r}\}$ denote a maximal independent set of columns of $\mathcal{M}(n)$. Let $W_\mathcal{B}[\mathcal{V}]$ denote the compression of $W_n[\mathcal{V}]$ to the columns, $\mathcal{X}^{i_1}, \ldots, \mathcal{X}^{i_r}$, indexed by the same monomials which index $\mathcal{B}$. We claim that $W_\mathcal{B}[\mathcal{V}]$ is invertible. Indeed, for $p \equiv a_1 x^{i_1} + \cdots + a_r x^{i_r} (\in \mathcal{P}_n)$, $W_\mathcal{B}[\mathcal{V}]\hat{p} = 0 \Longleftrightarrow p|\mathcal{V} \equiv 0$. In this case, for each $q \in \mathcal{P}_n$, we have $pq \in \mathcal{P}_{2n}$ and $pq|\mathcal{V} \equiv 0$, so $\mathcal{V}$-consistency implies that $\Lambda_\beta(pq) = 0$. Now, $\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = \Lambda_\beta(pq) = 0$ $(q \in \mathcal{P}_n)$, so $\sum_{j=1}^r a_j X^{i_j} = \mathcal{M}(n)\hat{p} = 0$, whence $\hat{p} = 0$ (since $\mathcal{B}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$).

Consider $U \equiv U_\mathcal{B}[\mathcal{V}] = W_\mathcal{B}[\mathcal{V}]^T$. Since $U_{2n}[\mathcal{V}]\rho^T = L_\beta$, then $U\rho^T = (\beta_{i_1}, \ldots, \beta_{i_r})^T$, and since U is invertible, this relation uniquely determines ρ. For $1 \leq k \leq r$, let $U_k \equiv U_k(x)$ denote the matrix obtained from U by replacing w_k (in column k) by the variable $x \in \mathbb{R}^d$, and define $g_k \in \mathcal{P}_n$ by $g_k(x) := det\ U_k(x)$. Now, $g_k(w_j) = \delta_{kj} det\ U$ $(1 \leq k, j \leq r)$. Since $g_k^2 \in \mathcal{P}_{2n}$ and μ interpolates all of the moments of β, we have $0 \leq \langle \mathcal{M}(n)\hat{g_k}, \hat{g_k}\rangle = \Lambda_\beta(g_k^2) = \int g_k^2 d\mu = \sum_{j=1}^r \rho_j g_k^2(w_j) = \rho_k (det\ U)^2$. Since *det* $U \neq 0$, it follows that $\rho_k \geq 0$, and since *card supp* $\mu \geq r$ (by (1.2)), then $\rho_k > 0$. Now μ is a representing measure for β with *supp* $\mu = \mathcal{V}$, so (1.2) implies that $r \equiv$ *rank* $\mathcal{M}(n) \leq$ *rank* $\mathcal{M}(n+1)[\mu] \leq$ *card supp* $\mu =$ *card* $\mathcal{V} = r$, whence $\mathcal{M}(n+1)[\mu]$ is a flat extension of $\mathcal{M}(n)$.

Conversely, let μ denote an r-atomic representing measure for β. Let $\mathcal{V} =$ *supp* μ $(\subset V(\mathcal{M}(n)))$. For $p \in \mathcal{P}_{2n}$, if $p|\mathcal{V} \equiv 0$, then $\Lambda_\beta(p) = \int p d\mu = 0$. Thus β is $\mathcal{V}$-consistent, so the conclusion that $L_\beta \in$ *Ran* $U_{2n}[\mathcal{V}]$ now follows from Proposition 3.6. □

In the sequel we will establish a general framework for constructing examples of $\mathcal{M}(n)$ satisfying $r \leq v < +\infty$ and having flat extensions $\mathcal{M}(n+1)$. To motivate this, we recall from [CFM] that β is *weakly consistent* if $p \in \mathcal{P}_n$, $p|V(\mathcal{M}(n)) \equiv 0 \Longrightarrow p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$; (1.1) shows that weak consistency is a necessary condition for representing measures, and [CFM, Prop. 2.1] shows that β consistent $\Longrightarrow \beta$ weakly consistent $\Longrightarrow \mathcal{M}(n)$ *recursively generated* (cf. Section 4). In [CFM, Theorem 5.2] we presented the first example of a positive, weakly consistent moment matrix satisfying $r \leq v$ but having no representing measure (cf. Theorem 3.1). In this example, $\mathcal{M}(3)$ is weakly consistent and extremal, and the choice of data is motivated by considerations from algebraic geometry. The proofs of weak consistency and of the nonexistence of a representing measure are also established using techniques from algebraic geometry. In Example 3.9 (below) we provide a numerical instance of this example, with a new proof based entirely on moment matrix methods. For this, we require the following reformulation of weak consistency in the finite variety case.

Proposition 3.8. *Suppose $\mathcal{V} \equiv V(\mathcal{M}(n))$ is finite. $\beta \equiv \beta^{(2n)}$ is weakly consistent if and only if Ran $\mathcal{M}(n) \subset$ Ran $U_n[\mathcal{V}]$; equivalently, there exists a matrix Z such that $\mathcal{M}(n) = U_n[\mathcal{V}]Z$.*

Proof. Recall that β is weakly consistent if and only if $p \in \mathcal{P}_n$, $p|\mathcal{V} \equiv 0 \Longrightarrow p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. Now for $p \in \mathcal{P}_n$, $p|\mathcal{V} \equiv 0 \Longleftrightarrow W_n[\mathcal{V}]\hat{p} = 0$, and $p(X) = 0 \Longleftrightarrow \mathcal{M}(n)\hat{p} = 0$. Thus, β is weakly consistent if and only if *ker* $W_n[\mathcal{V}] \subset$ *ker* $\mathcal{M}(n)$, or equivalently (since the underlying spaces are finite dimensional), $[\mathit{ker}\ \mathcal{M}(n)]^\perp \subset [\mathit{ker}\ W_n[\mathcal{V}]]^\perp$. Since the underlying spaces are finite dimensional and $\mathcal{M}(n)$ is real-symmetric, the latter inclusion is equivalent to *Ran* $\mathcal{M}(n) \subset$ *Ran* $U_n[\mathcal{V}]$, which in turn is equivalent to the existence of a factorization $\mathcal{M}(n) = U_n[\mathcal{V}]Z$ [D]. $\square$

Example 3.9. We use moment matrix methods to discuss a numerical instance of the example of [CFM, Section 5]. Consider $\mathcal{M}(3)(\beta)$ defined as $\mathcal{M}(3) := \begin{pmatrix} \mathcal{M}(2) & B(3) \\ B(3)^T & C(3) \end{pmatrix}$, where

$$\mathcal{M}(2) := \begin{pmatrix} 14 & \frac{7}{2} & \frac{-67}{8} & \frac{79}{4} & \frac{1055}{16} & \frac{18195}{64} \\ \frac{7}{2} & \frac{79}{4} & \frac{1055}{16} & \frac{-67}{8} & \frac{-1935}{32} & \frac{-43115}{128} \\ \frac{-67}{8} & \frac{1055}{16} & \frac{18195}{64} & \frac{-1935}{32} & \frac{-43115}{128} & \frac{-926695}{512} \\ \frac{79}{4} & \frac{-67}{8} & \frac{-1935}{32} & \frac{1055}{16} & \frac{18195}{64} & \frac{336151}{256} \\ \frac{1055}{16} & \frac{-1935}{32} & \frac{-43115}{128} & \frac{18195}{64} & \frac{336151}{256} & \frac{6407195}{1024} \\ \frac{18195}{64} & \frac{-43115}{128} & \frac{-926695}{512} & \frac{336151}{256} & \frac{6407195}{1024} & \frac{124731423}{4096} \end{pmatrix},$$

$$\mathcal{B}(3) := \begin{pmatrix} \frac{-67}{8} & \frac{-1935}{32} & \frac{-43115}{128} & \frac{-926695}{512} \\ \frac{1055}{16} & \frac{1895}{64} & \frac{336151}{256} & \frac{6407195}{1024} \\ \frac{18195}{64} & \frac{336151}{256} & \frac{6407195}{1024} & \frac{124731423}{4096} \\ \frac{-1935}{32} & \frac{-43115}{128} & \frac{-926695}{512} & \frac{-19736547}{2048} \\ \frac{-43115}{128} & \frac{-926695}{512} & \frac{-19736547}{2048} & \frac{-419176415}{8192} \\ \frac{-926695}{512} & \frac{-19736547}{2048} & \frac{-419176415}{8192} & \frac{-8894873563}{32768} \end{pmatrix},$$

$$\mathcal{C}(3) := \begin{pmatrix} \frac{18195}{64} & \frac{336151}{256} & \frac{6407195}{1024} & \frac{124731423}{4096} \\ \frac{336151}{256} & \frac{6407195}{1024} & \frac{124731423}{4096} & \frac{2469281827}{16384} \\ \frac{6407195}{1024} & \frac{124731423}{4096} & \frac{2469281827}{16384} & \frac{49568350247}{65536} \\ \frac{124731423}{4096} & \frac{2469281827}{16384} & \frac{49568350247}{65536} & \frac{1006568996907}{262144} \end{pmatrix}.$$

Calculations with nested determinants show that $\mathcal{M}(3)$ is positive semidefinite, with a column basis $\mathcal{B} \equiv \{1,\ X,\ Y,\ X^2,\ XY,\ Y^2,\ X^2Y,\ XY^2\}$ and column dependence relations

$$X^3 = Y \tag{3.2}$$

and

$$Y^3 = 3X + \frac{45}{4}Y - 13X^2 + \frac{65}{4}XY - \frac{13}{4}Y^2 - 22X^2Y + \frac{35}{4}XY^2. \tag{3.3}$$

Thus $\mathcal{V} \equiv V(\mathcal{M}(3)) = \mathcal{Z}(f) \bigcap \mathcal{Z}(g)$, where $f(x,y) = y - x^3$ and $g(x,y) = y^3 - (3x + \frac{45}{4}y - 13x^2 + \frac{65}{4}xy - \frac{13}{4}y^2 - 22x^2y + \frac{35}{4}xy^2)$. A calculation shows that $\mathcal{V}$ consists of the following 8 points $w_i \equiv (x_i, y_i)$ $(1 \le i \le 8)$: $w_1 = (0,0)$, $w_2 = (-1,-1)$, $w_3 = (-2,-8)$, $w_4 = (\frac{1}{2}(-1+\sqrt{13}), -5+2\sqrt{13})$, $w_5 = (1,1)$, $w_6 = (2,8)$, $w_7 = (\frac{1}{2}(-1-\sqrt{13}), -5-2\sqrt{13})$, $w_8 = (\frac{1}{2}, \frac{1}{8})$. Thus, $\mathcal{M}(3)$ is positive and extremal, with $r = v = 8$.

A calculation (for example, in *Mathematica*, using $Z = \mathit{LinearSolve}[U_n[\mathcal{V}],\ \mathcal{M}(3)]$) shows that there is a factorization $\mathcal{M}(3) = U_n[\mathcal{V}]Z$, so Proposition 3.8 implies that β is weakly consistent. A futher calculation shows that $L_\beta \notin \mathit{Ran}\ U_{2n}[\mathcal{V}]$, so Proposition 3.6 implies that β is not consistent, and thus admits no representing measure. In particular, $\mathcal{M}(3)$ admits no flat extension $\mathcal{M}(4)$. $\square$

Note that (3.2) and (3.3) imply that $\mathcal{M}(3)$ is recursively determinate (cf. Prop. 4.2). In Section 4 we will use Algorithm 4.10 to provide an alternate proof that β admits no representing measure, by showing that $\mathcal{M}(3)$ admits no positive, recursively generated extension $\mathcal{M}(4)$ and, in particular, no flat moment matrix extension (cf. Example 4.18). We next provide a general result which implies that in this example, if μ is a positive measure with *supp* $\mu = \mathcal{V}$, then $\mathcal{M}(3)[\mu]$ has the same rank and variety as $\mathcal{M}(3)$, but does admit a flat extension $\mathcal{M}(4)$. More generally, the next result, when combined with Proposition 2.2, shows that flat extensions can occur with moment matrices having arbitrarily large gaps $v - r$.

Theorem 3.10. *Let $\tilde{\mathcal{M}} = \mathcal{M}(n)(\tilde{\beta})$ and suppose that $r \le v < +\infty$ and $\tilde{\beta}$ is weakly consistent. There exists an r-element subset $\mathcal{V}$ of $\tilde{\mathcal{V}} \equiv V(\tilde{\mathcal{M}})$, such that if μ is a positive measure with supp $\mu = \mathcal{V}$, then $\mathcal{M} \equiv \mathcal{M}(n)[\mu]$ satisfies rank $\mathcal{M} = r$ and $V(\mathcal{M}) = \tilde{\mathcal{V}}$, so $\mathcal{M}$ and $\tilde{\mathcal{M}}$ have the same gap, and $\mathcal{M}$ has a flat extension $\mathcal{M}(n+1)$.*

Proof. Let $w_1, \ldots, w_v$ denote the distinct points of $\tilde{\mathcal{V}}$. Let $\mathcal{B} \equiv \{X^{i_1}, \ldots, X^{i_r}\}$ be a maximal independent set of columns of $\tilde{\mathcal{M}}$. Let $W_{\mathcal{B}} \equiv W_{\mathcal{B}}[\tilde{\mathcal{V}}]$ denote the compression of $W_n[\tilde{\mathcal{V}}]$ to the columns $\mathcal{X}^{i_1}, \ldots, \mathcal{X}^{i_r}$ (indexed by the same monomials which index $\mathcal{B}$). We assert that the columns of $W_{\mathcal{B}}$ are independent. For otherwise, there is a nonzero polynomial $p(x) = \sum_{j=1}^{r} a_j x^{i_j} \in \mathcal{P}_n$ such that $W_{\mathcal{B}}\hat{p} = 0$, i.e., $p|\tilde{\mathcal{V}} \equiv 0$. Since $\tilde{\beta}$ is weakly consistent, it follows that $p(X) = 0$ in $\mathcal{C}_{\tilde{\mathcal{M}}}$, whence $\sum_{j=1}^{r} a_j X^{i_j} = 0$, contradicting the independence of the elements of $\mathcal{B}$. Now, *row rank* $W_{\mathcal{B}} =$ *column rank* $W_{\mathcal{B}} = r$; it follows that there exists an r-element subset of $\tilde{\mathcal{V}}$, say $\mathcal{V} \equiv \{w_{j_1}, \ldots, w_{j_r}\}$, such that $W_{\mathcal{B}}[\mathcal{V}]$, the compression of $W_{\mathcal{B}}$ to rows indexed by $w_{j_1}, \ldots, w_{j_r}$, is invertible.

Let μ be a positive measure with *supp* $\mu = \mathcal{V}$, i.e., μ is of the form $\mu = \sum_{i=1}^{r} a_i \delta_{w_{j_i}}$ with each $a_i > 0$. Let $\beta \equiv \beta^{(2n)}[\mu]$ be the corresponding moment sequence of degree $2n$, and consider $\mathcal{M} \equiv \mathcal{M}(n)(\beta)$ $(= \mathcal{M}(n)[\mu] \succeq 0)$. We will show that *rank* $\mathcal{M} = r$ and $V(\mathcal{M}) = \tilde{\mathcal{V}}$. Let

$D = diag(a_1, \ldots, a_r)$. Since $supp\ \mu = \mathcal{V}$, a calculation (cf. [Lau1, Lemma 2.4] [CFM, Lemma 2.5]) shows that

$$\mathcal{M} = W_n[\mathcal{V}]^T D W_n[\mathcal{V}]. \tag{3.4}$$

Now $\mathcal{M} = \mathcal{W}^T\mathcal{W}$, where $\mathcal{W} = D^{1/2}W_n[\mathcal{V}]$, so it follows that $rank\ \mathcal{M} = rank\ \mathcal{W} = rank\ W_n[\mathcal{V}] = rank\ W_{\mathcal{B}}[\mathcal{V}] = r$. Note also that if $p \in \mathcal{P}_n$ and $\tilde{\mathcal{M}}\hat{p} = 0$, then $p|\tilde{\mathcal{V}} \equiv 0$, so it follows from (3.4) that $p|\mathcal{V} \equiv 0 \Longrightarrow W_n[\mathcal{V}]\hat{p} = 0 \Longrightarrow \mathcal{M}\hat{p} = 0$. Thus, $ker\ \tilde{\mathcal{M}} \subset ker\ \mathcal{M}$. Since $rank\ \mathcal{M} = rank\ \tilde{\mathcal{M}}$, we have $ker\ \mathcal{M} = ker\ \tilde{\mathcal{M}}$, whence $V(\mathcal{M}) = V(\tilde{\mathcal{M}})$. Finally, $\mathcal{M}(n+1)[\mu]$ is a flat extension of $\mathcal{M}(n)[\mu]$, since, using (1.2), we have $r = rank\ \mathcal{M}(n)[\mu] \leq rank\ \mathcal{M}(n+1)[\mu] \leq card\ supp\ \mu = r$.

□

Example 3.11. We illustrate Theorem 3.10 with a continuation of Example 3.9. Recall that $\mathcal{M}(3)(\beta)$ is positive and extremal, with $r = v = 8$, and β is weakly consistent, but β has no representing measure. Since $r = v$, the content of Theorem 3.10 in this case is that if μ is a positive measure with $supp\ \mu = \mathcal{V} \equiv V(\mathcal{M}(3))$, then $\mathcal{M} \equiv \mathcal{M}(3)[\mu]$ satisfies $rank\ \mathcal{M} = 8$, $V(\mathcal{M}) = \mathcal{V}$, and $\mathcal{M}$ has a flat extension $\mathcal{M}(4)$. With $\{w_i\}_{i=1}^8$ as in Example 3.9, let $\mu = \sum_{i=1}^8 \rho_i \delta_{w_i}$, where $\rho_i = i$ except that $\rho_4 = \rho_7 = 1$. We have $\mathcal{M}(3)[\mu] := \begin{pmatrix} \mathcal{M}(2) & B(3) \\ B(3)^T & C(3) \end{pmatrix}$, where

$$\mathcal{M}(2) := \begin{pmatrix} 36 & 12 & 18 & 52 & \frac{365}{2} & \frac{5897}{8} \\ 12 & 52 & \frac{365}{2} & 18 & \frac{153}{4} & \frac{801}{16} \\ 18 & \frac{365}{2} & \frac{5897}{8} & \frac{153}{4} & \frac{801}{16} & \frac{-17343}{64} \\ 52 & 18 & \frac{153}{4} & \frac{365}{2} & \frac{5897}{8} & \frac{99521}{32} \\ \frac{365}{2} & \frac{153}{4} & \frac{801}{16} & \frac{5897}{8} & \frac{99521}{32} & \frac{1719041}{128} \\ \frac{5897}{8} & \frac{801}{16} & \frac{-17343}{64} & \frac{99521}{32} & \frac{1719041}{128} & \frac{30274049}{512} \end{pmatrix},$$

$$\mathcal{B}(3) := \begin{pmatrix} 18 & \frac{153}{4} & \frac{801}{16} & \frac{-17343}{64} \\ \frac{365}{2} & \frac{5897}{8} & \frac{99521}{32} & \frac{1719041}{128} \\ \frac{5897}{8} & \frac{99521}{32} & \frac{1719041}{128} & \frac{30274049}{512} \\ \frac{153}{4} & \frac{801}{16} & \frac{-17343}{64} & \frac{-893439}{256} \\ \frac{801}{16} & \frac{-17343}{64} & \frac{-893439}{256} & \frac{-27228159}{1024} \\ \frac{-17343}{64} & \frac{-893439}{256} & \frac{-27228159}{1024} & \frac{-709193727}{4096} \end{pmatrix},$$

$$\mathcal{C}(3) := \begin{pmatrix} \frac{5897}{8} & \frac{99521}{32} & \frac{1719041}{128} & \frac{30274049}{512} \\ \frac{99521}{32} & \frac{1719041}{128} & \frac{30274049}{512} & \frac{543551489}{2048} \\ \frac{1719041}{128} & \frac{30274049}{512} & \frac{543551489}{2048} & \frac{9954181121}{8192} \\ \frac{30274049}{512} & \frac{543551489}{2048} & \frac{9954181121}{8192} & \frac{185950830593}{32768} \end{pmatrix}.$$

Calculations with nested determinants show that $\mathcal{M}$ is positive semidefinite, with a column basis $\mathcal{B} \equiv \{1,\ X,\ Y,\ X^2,\ XY,\ Y^2,\ X^2Y,\ XY^2\}$, and column dependence relations $f(X,Y) = 0$

and $g(X,Y)=0$ (with f and g as in Example 3.9). It follows readily that $r=8$ and that $V(\mathcal{M}) = \mathcal{Z}(f)\bigcap\mathcal{Z}(g)=\mathcal{V}$. To show that $\mathcal{M}(4)[\mu]$ is a flat extension of $\mathcal{M}(3)[\mu]$, note that $V(\mathcal{M}(4)[\mu]) \subset V(\mathcal{M}(3)[\mu])$, whence (via (1.2)) $8 = \textit{rank}\ \mathcal{M}(3)[\mu] \le \textit{rank}\ \mathcal{M}(4)[\mu] \le \textit{card}\ V(\mathcal{M}(4)[\mu]) \le \textit{card}\ V(\mathcal{M}(3)[\mu]) = \textit{card}\ \mathcal{V} = 8$. $\square$

4. An algorithmic solution to the recursively determinate truncated moment problem

For a general sequence β, it may be very difficult to verify the condition of Theorem 2.1. In this section we introduce the class of *recursively determinate* moment matrices; for this class, it is possible to determine the existence of extensions $\mathcal{M}(n+1),\dots,\mathcal{M}(n+v-r+1)$ algorithmically, in a purely mechanical fashion (cf. Algorithm 4.10). To this end, we recall from [CF2] that $\mathcal{M}(n)$ is *recursively generated* if the following property holds:

$$p,q,pq \in \mathcal{P}_n,\ p(X)=0 \Longrightarrow (pq)(X)=0. \tag{4.1}$$

It follows from (1.1) that recursiveness is a necessary condition for representing measures. It is straightforward to verify recursiveness: it suffices to check that whenever the column relation $X^i = p(X)$ expresses column X^i as a linear combination of columns to its left in $\mathcal{M}(n)$ (for some $p \in \mathcal{P}_n$), then $X^{i+j} = (x^j p)(X)$ whenever $|j| \le n-|i|$. Note, in particular, that if $\mathcal{M}(n-1)$ is nonsingular, then (vacuously) $\mathcal{M}(n)$ is recursively generated. Roughly speaking, the recursively determinate matrices comprise the largest class for which the existence of a convergent extension sequence can be detected solely by imposing recursiveness. We next motivate the definition of recursive determinancy with a basic example.

Example 4.1. Let $d=2$ (the plane). Suppose $\mathcal{M}(n)$ has column dependence relations of the form

$$X^n = p(X,Y)\ (p \in \mathcal{P}_{n-1}) \tag{4.2}$$

and

$$Y^n = q(X,Y)\ (q \in \mathcal{P}_n \textit{ is free of the term } y^n). \tag{4.3}$$

We will show that the preceding relations determine any possible positive, recursively generated moment matrix extensions $\mathcal{M}(n+1)$, $\mathcal{M}(n+2),\dots$. Indeed, any extension $\mathcal{M}(n+1)$ is of the form

$$\mathcal{M}(n+1) \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^T & C(n+1) \end{pmatrix},$$

and since $d=2$, columns X^{n+1} and Y^{n+1} of $\begin{pmatrix} B(n+1) \\ C(n+1) \end{pmatrix}$ contain all of the "new moments" of degrees $2n+1$ and $2n+2$ (see below). Since $\mathcal{M}(n+1)$ is to be positive, the Extension Principle [F1] (cf. Section 2) implies that relations (4.2) and (4.3) must hold for the columns of $\mathcal{M}(n+1)$. The requirement of recursiveness in $\mathcal{M}(n+1)$ then implies that in $\mathcal{C}_{\mathcal{M}(n+1)}$ we must have

$$X^{n+1} = (xp)(X,Y) \tag{4.4}$$

and

$$Y^{n+1} = (yq)(X,Y). \tag{4.5}$$

These relations, when applied to the columns of $\begin{pmatrix} \mathcal{M}(n) & B(n+1) \end{pmatrix}$, completely determine block $B(n+1)$. To see this, consider the moments of degree $2n+1$ in $B(n+1)$, which form the block

$$M_{n,n+1} \equiv \begin{pmatrix} \beta_{2n+1,0} & \beta_{2n,1} & \dots & \beta_{n+2,n-1} & \beta_{n+1,n} & \beta_{n,n+1} \\ \beta_{2n,1} & \beta_{2n-1,2} & \dots & \beta_{n+1,n} & \beta_{n,n+1} & \beta_{n-1,n+2} \\ \beta_{2n-1,2} & \beta_{2n-2,3} & \dots & \beta_{n,n+1} & \beta_{n-1,n+2} & \beta_{n-2,n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n+1,n} & \beta_{n,n+1} & \dots & \beta_{2,2n-1} & \beta_{1,2n} & \beta_{0,2n+1} \end{pmatrix}$$

(with columns $X^{n+1},\ X^nY,\ldots,XY^n,\ Y^{n+1}$). Since $deg\ p \leq n-1$, relation (4.4) uniquely determines column X^{n+1} in block $B(n+1)$ as a linear combination of columns of $\mathcal{M}(n)$. Thus the moments $\beta_{2n+1,0},\ \beta_{2n,1},\ldots,\ \beta_{n+1,n}$, which define column X^{n+1} in $M_{n,n+1}$, are determined, and these moments propagate through the upper left triangle of $M_{n,n+1}$. Now assume $deg\ q = n$. In the first row of $M_{n,n+1}$, the moments transposed from column X^{n+1} by moment matrix structure are used with (4.5) to determine $\beta_{n,n+1}$ in column Y^{n+1}. Then, in the second row, these moments and $\beta_{n,n+1}$ are used with (4.5) to determine $\beta_{n-1,n+2}$. In row 3, $\beta_{n,n+1}$ and $\beta_{n-1,n+2}$ are used with the earlier moments and (4.5) to determine $\beta_{n-2,n+3}$, etc. In this manner we successively determine all of the moments of column Y^{n+1} in block $M_{n,n+1}$. In case $deg\ q < n$, column Y^{n+1} in block $B(n+1)$ is determined from (4.5) simply as a linear combination of columns of $\mathcal{M}(n)$.

Having determined $B(n+1)$, when relations (4.4) and (4.5) are then applied to the columns of $\begin{pmatrix} B(n+1)^T & C(n+1) \end{pmatrix}$, the above method yields all of the moments of degree $2n+2$ needed to determine block $C(n+1)$ (which is a Hankel matrix, since $d=2$). In this way we complete the construction of an extension $\mathcal{M}(n+1)$. If $\mathcal{M}(n+1)$ is positive and recursively generated, then (4.4), (4.5), and recursiveness can be used (as above) to uniquely determine an extension $\mathcal{M}(n+2)$, and recursiveness in $\mathcal{M}(n+2)$ uniquely determines $\mathcal{M}(n+3)$, and so on. We illustrate this procedure in Example 4.15 (below). $\square$

Example 4.1 encompasses any extremal planar $\mathcal{M}(3)$ satisfying $Y = X^3$. Indeed, it is shown in [CFM, Sections 4-5] that extremality implies that $Y^3 = q(X,Y)$ where $q \in \mathcal{P}_3$ is free of the term y^3, so conditions (4.2) and (4.3) are satisfied. Theorem 3.1 describes the solution of the truncated moment problem for measures supported in a planar curve $p(x,y)=0$ with $deg\ p \leq 2$. [CFM] is related to the truncated moment problem for $y = x^3$, which remains unsolved. More generally, little is known about the truncated moment problem for $y = x^n$ with $n > 2$, but Example 4.1 (and Proposition 4.2 and Theorem 4.3 below) provide a framework for generating examples and testing hypotheses (cf. Example 4.18).

We now procede to define recursive determinancy formally. Consider a proposed extension $\mathcal{M}(n+1)$ in which $B(n+1)$ and $C(n+1)$ are as yet undetermined, and consider also a finite family of column dependence relations in $\mathcal{M}(n)$,

$$p_i(X) = 0 \ (deg\ p_i = n,\ 1 \leq i \leq s). \tag{4.6}$$

Suppose further that there are corresponding monomials of degree 1, $x^{j_1},\ldots,x^{j_s}$, such that in the column space of $\begin{pmatrix} \mathcal{M}(n) & B(n+1) \end{pmatrix}$, the formal system

$$(x^{j_i}p_i)(X) = 0 \ (1 \leq i \leq s) \tag{4.7}$$

determines all new moments of degree $2n+1$, leaving no "free" choices. Since (4.7) is a system of linear equations in variables representing the new moments of degree $2n+1$, the requirement that there are no free choices implies that the moments of degree $2n+1$ are uniquely determined by (4.7). Using these moments, we may form block $B(n+1)$. Consider now the formal system (4.7) in the column space of $\begin{pmatrix} B(n+1)^T & C(n+1) \end{pmatrix}$, and suppose that this system determines all moments of degree $2n+2$ for $C(n+1)$, with no free choices. As above, since there are no free choices and the system is linear in the moments of degree $2n+2$, these moments are uniquely determined. Defining $C(n+1)$ in this way, we say that the resulting extension $\mathcal{M}(n+1)$ is *recursively determined*. Note that there can be only one recursively determined extension $\mathcal{M}(n+1)$ that is recursively generated, because in such an extension, all systems (4.7) must be valid simultaneously.

$\mathcal{M}(n)$ is *recursively determinate* if it enjoys the following property: for each $i \geq 0$, if $\mathcal{M}(n+i)$ is a positive, recursively generated extension of $\mathcal{M}(n)$, then $\mathcal{M}(n+i)$ admits a recursively determined extension $\mathcal{M}(n+i+1)$. In this definition, we do not presuppose that $\mathcal{M}(n+i+1)$ is itself recursively generated or positive semidefinite. For $i=0$, we consider $\mathcal{M}(n)$ to be a (trivial) extension of itself, so if $\mathcal{M}(n)$ is recursively determinate, positive, and recursively generated, then the definition entails a recursively determined extension $\mathcal{M}(n+1)$. Note that if $\mathcal{M}(n)$ is recursively determinate and $\mathcal{M}(n+i)$ $(i>0)$ is a positive and recursively generated extension, then $\mathcal{M}(n+i)$ is also recursively determined. Indeed, the Extension Principle [F1] implies that $\mathcal{M}(n+i-1)$ is positive and recursively generated, so $\mathcal{M}(n+i-1)$ has a recursively determined extension, which must then coincide with $\mathcal{M}(n+i)$.

The fundamental example of a recursively determinate moment matrix concerns the case of *flat data*, where $\mathcal{M}(n) \succeq 0$ satisfies $rank\ \mathcal{M}(n) = rank\ \mathcal{M}(n-1)$. In this case, for each $i \in \mathbb{Z}_+^d$ with $|i| = n$, there exists $p_i \in \mathcal{P}_{n-1}$ such that $X^i = p_i(X)$ in $\mathcal{C}_{\mathcal{M}(n)}$. The main result of [CF2] (closely related to Theorem 1.1) is that in this case, the formal system $X^{i+j} = (x^j p_i)(X)$ $(|i| = n,\ |j| = 1)$ admits a unique solution, leading to a unique flat, positive, recursively generated extension $\mathcal{M}(n+1)$. Further, Theorem 1.2 implies that in this case, $v = r$, so the existence of a flat extension $\mathcal{M}(n+1)$ is consistent with Theorem 4.3 (below). We next identify a family of recursively determinate planar moment matrices; instances of this family appear in Examples 3.3, 3.9, 4.15, 4.18.

Proposition 4.2. *Let $d = 2$. $\mathcal{M}(n)$ is recursively determinate if it has column relations of the form $X^n = p(X, Y)$, with $deg\ p < n$, and $Y^n = q(X, Y)$, where $deg\ q \le n$ and q is free of the term y^n.*

The proof of Proposition 4.2 is essentially contained in Example 4.1; observe that if $\mathcal{M}(n+i)$ is a positive, recursively generated extension of $\mathcal{M}(n)$, then in $\mathcal{C}_{\mathcal{M}(n+i)}$ we have $X^{n+i} = (x^i p)(X, Y)$ and $Y^{n+i} = (y^i q)(X, Y)$, so (as in Example 4.1) we may define a recursively determined extension $\mathcal{M}(n+i+1)$ via the relations $X^{n+i+1} = (x^{i+1}p)(X, Y)$ and $Y^{n+i+1} = (y^{i+1}q)(X, Y)$. A modification of the argument in Example 4.1 shows that Proposition 4.2 also holds if the roles of p and q are reversed.

The main result of this section, which follows, shows that for the class of recursively determinate moment matrices with finite variety, we can refine Theorem 2.1 so as to detect minimal representing measures.

Theorem 4.3. *Suppose $\mathcal{M}(n)(\beta)$ is recursively determinate, with $r \le v < +\infty$. The following are equivalent:*
i) β has a representing measure;
ii) There exists i, $0 \le i \le v - r$, such that $\mathcal{M}(n)$ admits successive positive, recursively determined extensions $\mathcal{M}(n+1), \ldots, \mathcal{M}(n+i+1)$, and $\mathcal{M}(n+i+1)$ is a flat extension of $\mathcal{M}(n+i)$.
If the preceding conditions hold, and if i is minimal with respect to the flat extension property for a particular extension sequence satisfying ii), then the unique representing measure for $\mathcal{M}(n+i+1)$ (cf. Thm. 1.2) is a minimal representing measure for β.

As with Theorem 2.1, a version of Theorem 4.3 holds for the case when the variety is not necessarily finite (cf. Remark 4.11). Theorem 4.3 shows that if $\mathcal{M}(n)$ (with finite variety) is recursively determinate and has a representing measure, then a minimal representing measure always corresponds to minimum-length convergent extension sequence. Whether this behavior holds for arbitrary $\mathcal{M}(n)$ is a question that we discuss further below (cf. Question 4.13 and Example 4.14). In Algorithm 4.10 we describe a computational procedure for verifying whether the conditions of Theorem 4.3(ii) are satisfied.

For the proof of Theorem 4.3 we require several preliminary results. We begin by deriving another necessary condition for the existence of a positive moment matrix extension $\mathcal{M}(n+1) \equiv \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^T & C(n+1) \end{pmatrix}$. In the sequel, for a vector h with components indexed by the monomials in $\mathcal{P}_{n+1}$ in degree-lexicographic order, let $[h]_n$ denote the projection of h onto the components indexed by the monomials in $\mathcal{P}_n$.

Proposition 4.4. *Suppose $\mathcal{M}(n+1) \succeq 0$ and let $p \in \mathcal{P}_n$ with $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. For each polynomial q with $deg\ q \le n+1-deg\ p$,*
(i) $[(pq)(X)]_n = 0$ in the column space of $\begin{pmatrix} \mathcal{M}(n) & B(n+1) \end{pmatrix}$, and
(ii) if $\mathcal{M}(n+1)$ is recursively generated, then $(pq)(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$.

We require the following lemma.

Lemma 4.5. *Suppose $\mathcal{M}(n+1) \geq 0$ and let $p \in \mathcal{P}_{n-1}$ with $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. If $|j| = 1$, then $(x^j p)(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$.*

Proof. Since $\mathcal{M}(n+1) \succeq 0$ and $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, the Extension Principle implies that $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$, i.e., $\mathcal{M}(n+1)\hat{p} = 0$. Since $deg\ p < n$ and $|j| = 1$, then for $s \in \mathcal{P}_n$, $\langle \mathcal{M}(n)\widehat{px^j}, \hat{s}\rangle = \Lambda(px^j s)$ $= \langle \mathcal{M}(n+1)\hat{p}, \widehat{x^j s}\rangle = 0$, whence $(px^j)(X) = \mathcal{M}(n)\widehat{px^j} = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. $\square$

Proof of Proposition 4.4. (i) Let $k = deg\ q$ $(\leq n+1-deg\ p)$, so that $q(x) = \sum_{|j|\leq k} a_j x^j$. Since $[(pq)(X)]_n = \sum_{|j|\leq k}[a_j(px^j)(X)]_n$, we may assume that q is a monomial, of the form $q(x) = x^j$ for some j with $|j| = k$. Consider first the case when $deg\ p = n$ and $k = 1$. Since $p(X) = 0$ in $\mathcal{M}(n)$ and $\mathcal{M}(n+1) \geq 0$, the Extension Principle implies that $p(X) = 0$ in $\mathcal{M}(n+1)$, and thus $\langle \mathcal{M}(n+1)\hat{p}, \hat{u}\rangle = 0$ for every $u \in \mathcal{P}_{n+1}$. In particular, for each $s \in \mathcal{P}_n$, if $u = x^j s$, then $0 = \langle \mathcal{M}(n+1)\hat{p}, \widehat{x^j s}\rangle = \Lambda(px^j s) = \langle \mathcal{M}(n+1)\widehat{px^j}, \hat{s}\rangle$, whence $[(px^j)(X)]_n = 0$.

Now suppose $k > 1$ and write $x^j = x^{j_1}\cdots x^{j_k}$, where $|j_i| = 1$, $1 \leq i \leq k$. If $k + deg\ p \leq n$, we may apply Lemma 4.5 successively to conclude that in $\mathcal{C}_{\mathcal{M}(n)}$, $(x^{j_1}p)(X) = 0$, $(x^{j_1}x^{j_2}p)(X) = 0$,..., $(x^j p)(X) = (x^{j_1}\cdots x^{j_k}p)(X) = 0$. In the remaining case, $k + deg\ p = n+1$. We may apply Lemma 4.5 successively (as above) to derive $(x^{j_2}\cdots x^{j_k}p)(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, and then apply the first case (when $deg\ p = n$ and $k = 1$) to conclude that $[(x^j p)(X)]_n = [(x^{j_1}(x^{j_2}\cdots x^{j_k}p))(X)]_n = 0$. This completes the proof of (i).

(ii) Suppose $\mathcal{M}(n+1)$ is both positive and recursively generated. Since $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, the Extension Principle implies that $p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$, whence recursiveness implies that $(pq)(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$. $\square$

Corollary 4.6. *Suppose $\mathcal{M}(n+1)$ is positive and that in $\mathcal{C}_{\mathcal{M}(n)}$, $X^{j_1} = p_1(X)$ $(|j_1| \leq n$, $deg\ p_1 \leq n$, $p_1(x)$ is free of $x^{j_1})$ and $X^{j_2} = p_2(X)$ $(|j_2| \leq n$, $deg\ p_2 \leq n$, $p_2(x)$ is free of $x^{j_2})$. If $i_1, i_2 \in \mathbb{Z}_+^d$ satisfy $i_1 + j_1 = i_2 + j_2$ and $|i_1| + |j_1| = |i_2| + |j_2| = n+1$, then $(x^{i_1}p_1)(X) = (x^{i_2}p_2)(X)$ in $\mathcal{C}_{\left(\begin{smallmatrix}\mathcal{M}(n) & B(n+1)\end{smallmatrix}\right)}$. Moreover, if $\mathcal{M}(n+1)$ is also recursively generated, then $(x^{i_1}p_1)(X) = (x^{i_2}p_2)(X)$ in $\mathcal{C}_{\mathcal{M}(n+1)}$.*

For the truncated complex moment problem, an analogue of Proposition 4.4 appears in [CF4, Thm. 1.6]. We illustrate Proposition 4.4 and Corollary 4.6 with an example of a positive, recursively generated $\mathcal{M}(n)$ with no positive extension $\mathcal{M}(n+1)$.

Example 4.7. Let $d = 2$ and consider $\mathcal{M}(3)$ of the form

$$\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \alpha & 0 & b & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & b & 0 & 2-b \\
1 & 0 & 0 & \alpha & 0 & b & 0 & \gamma & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 & \gamma & 0 & 0 & 0 \\
1 & 0 & 0 & b & 0 & 2-b & 0 & 0 & 0 & \delta \\
0 & \alpha & 0 & 0 & \gamma & 0 & e & 0 & b\alpha & 0 \\
0 & 0 & b & \gamma & 0 & 0 & 0 & b\alpha & 0 & b^2 \\
0 & b & 0 & 0 & 0 & 0 & b\alpha & 0 & b^2 & 0 \\
0 & 0 & 2-b & 0 & 0 & \delta & 0 & b^2 & 0 & f
\end{pmatrix},$$

where $\alpha = \frac{-2+b+4b^2}{-1+4b}$, $\delta = -4b(b-1)$, and $\gamma = \frac{-4b(b-1)}{-1+4b}$. A calculation with nested determinants shows that for $0 < b < 1/4$ and e and f sufficiently large, $\mathcal{M}(3)$ is positive and recursively generated, with $\mathcal{M}(2) \succ 0$, $rank\ \mathcal{M}(3) = 8$, and column relations

$$X^2Y = 1 + bY - \frac{1}{2}X^2 - \frac{1}{2}Y^2 \tag{4.8}$$

and

$$XY^2 = bX. \tag{4.9}$$

Proposition 4.4(i) implies that in any positive extension $\mathcal{M}(4)$ we must have $Y+bY^2-\frac{1}{2}X^2Y-\frac{1}{2}Y^3 = X^2Y^2 = bX^2$ in the columns of $\begin{pmatrix} \mathcal{M}(3) & B(4) \end{pmatrix}$. Now the last entry in bX^2 is 0, but if f is sufficiently large, we can insure that the last entry in $Y+bY^2-\frac{1}{2}X^2Y-\frac{1}{2}Y^3$ is nonzero. Thus $\mathcal{M}(3)$ admits no positive extension $\mathcal{M}(4)$. $\square$

In Example 4.7, the system (4.8)-(4.9) recursively determined two different choices for column X^2Y^2. It is clear from Corollary 4.6 that if a system of dependence relations in $\mathcal{M}(n)$ leads to such conflicting choices for moments of degree $2n+1$, then $\mathcal{M}(n)$ admits no positive extension $\mathcal{M}(n+1)$ (and thus admits no representing measure). This observation is a basic ingredient in Algorithm 4.10 (below).

Corollary 4.8. *Suppose $\mathcal{M}(n)$ is recursively determinate. If β has a representing measure, then $\mathcal{M}(n)$ has unique successive recursively determined extensions $\mathcal{M}(n+1)$, $\mathcal{M}(n+2), \ldots$, which are also the unique successive positive, recursively generated extensions of $\mathcal{M}(n)$.*

Proof. If β has a representing measure, then [BT] implies that β has a finitely atomic representing measure ν (cf. Section 2), and thus $\mathcal{M}(n+1)[\nu]$, $\mathcal{M}(n+2)[\nu], \ldots$ are positive, recursively generated extensions. Suppose $\mathcal{M}(n+1)$ is a recursively determined extension, as determined from some system (4.6)-(4.7). Proposition 4.4 implies that (4.6)-(4.7) also hold in $\mathcal{M}(n+1)[\nu]$, so $\mathcal{M}(n+1)$ must coincide with $\mathcal{M}(n+1)[\nu]$ and, in particular, $\mathcal{M}(n+1)$ is also positive and recursively generated. Assume by induction that $\mathcal{M}(n+i)[\mu]$ is the unique recursively determined extension of $\mathcal{M}(n)$ of degree $2(n+i)$. Since $\mathcal{M}(n)$ is recursively determinate and $\mathcal{M}(n+i)[\mu]$ is positive and recursively generated, some system $\mathcal{S}$ in the column space of $\mathcal{M}(n+i)[\mu]$, together with recursiveness, determines an extension $\mathcal{M}(n+i+1)$. As above, Proposition 4.4 and $\mathcal{S}$ imply that $\mathcal{M}(n+i+1)$ coincides with $\mathcal{M}(n+i+1)[\mu]$. Thus $\mathcal{M}(n+i+1)[\mu]$ is the unique recursively determined extension of $\mathcal{M}(n)$ of degree $2(n+i+1)$ and, clearly, $\mathcal{M}(n+i+1)[\mu]$ is positive and recursively generated. $\square$

Proof of Theorem 4.3. Suppose β has a representing measure. If follows as in the proof of Theorem 2.1 that there exists a positive, recursively generated extension $\mathcal{M}(n+v-r+1)$ satisfying $v \equiv$ *card* $V(\mathcal{M}(n)) \geq \cdots \geq$ *card* $V(\mathcal{M}(n+v-r+1)) \geq$ *rank* $\mathcal{M}(n+v-r+1) \geq \ldots$ *rank* $\mathcal{M}(n) \equiv r$. [F1] implies that each extension $\mathcal{M}(n+i)$ $(1 \leq i \leq v-r)$ is positive and recursively generated, and is thus recursively determined (see the remarks following the definition of recursive determinancy). As in the proof of Theorem 2.1, it now follows that for some i, $0 \leq i \leq v-r$, $\mathcal{M}(n+i+1)$ is a flat extension of $\mathcal{M}(n+i)$. Conversely, the existence of such a flat extension, together with Theorem 1.1, immediately implies the existence of a *rank* $\mathcal{M}(n+i)$-atomic representing measure for β.

Now suppose (ii) holds and that i is minimal with respect to the flat extension property. Let μ be the unique representing measure corresponding to the flat extension $\mathcal{M}(n+i+1)$ (cf. Theorem 1.2), which is *rank* $\mathcal{M}(n+i)$-atomic. If ν is a minimal representing measure for β, then ν is finitely atomic, and the matrices $\mathcal{M}(n+1)[\nu]$, $\mathcal{M}(n+2)[\nu], \ldots$ are positive and recursively generated. Since $\mathcal{M}(n)$ is recursively determinate, Corollary 4.8 implies that these extensions must coincide with the positive, recursively generated extensions $\mathcal{M}(n+1)$ $(= \mathcal{M}(n+1)[\mu])$, $\mathcal{M}(n+2)$ $(= \mathcal{M}(n+2)[\mu])$, $\ldots$. It follows from [CF6, Cor. 2.6] that $\mu = \nu$, so *rank* $\mathcal{M}(n+i)$ is the size of the support of a minimal representing measure for β. $\square$

Remark 4.9. An example of recursive determinancy in the truncated complex moment problem is implicit in [F2]. In the moment problem for a complex sequence $\gamma \equiv \gamma^{(2n)}$, suppose the complex moment matrix $M(n)(\gamma)$ enjoys a column dependence relation of the form $Z^n = p(Z, \bar{Z})$, $deg\ p < n$. Since column Z^{n+1} (for a proposed extension $M(n+1)$) contains all moments of degrees $2n+1$ and $2n+2$ (up to conjugation), we may define an *"analytic" extension* by imposing $Z^{n+1} = (zp)(Z, \bar{Z})$. In [F2, Thm. 2.2] we proved that such a sequence γ has a finitely atomic representing measure if and only if $M(n)(\gamma)$ admits successive positive, recursively generated analytic extensions $M(n+1), \ldots, M(2n-2)$ (where each $M(n+j)$ is determined by $Z^{n+j} = (z^jp)(Z, \bar{Z})$).

Suppose $\mathcal{M}(n)(\beta)$ is recursively determinate, with $r \leq v < +\infty$. Theorem 4.3 leads immediately to the following procedure for determining whether or not β has a representing measure and, in the positive case, for determining a minimal representing measure. For the case when the variety is not necessarily finite, see Remark 4.11. Remark 4.11 also describes how to use this procedure to decide whether or not a given moment matrix is recursively determinate.

Algorithm 4.10. *Test for representing measures for $\mathcal{M}(n)$ recursively determinate with $r \leq v < +\infty$.*

Algorithm. We may assume that $\mathcal{M}(n)$ is positive and recursively generated, for otherwise there is no representing measure. Since $\mathcal{M}(n)$ is recursively determinate, we may use the column dependence relations in $\mathcal{M}(n)$ to determine data for block $B(n+1)$ of some recursively determined extension $\mathcal{M}(n+1)$. Proposition 4.4 and Corollary 4.6 show that if the various dependence relations in $\mathcal{M}(n)$ lead to conflicting definitions for the data in $B(n+1)$, then β has no representing measure. If $B(n+1)$ is well-defined, we test whether $Ran\ B(n+1) \subset Ran\ \mathcal{M}(n)$, or, equivalently, whether there is a matrix W satisfying $B(n+1) = \mathcal{M}(n)W$. If such a factorization is impossible, β has no representing measure, since there cannot be a positive extension in this case (cf. Section 2). If W does exist, it can be computed using elementary linear algebra. In this case, since $\mathcal{M}(n)$ is recursively determinate, we may use recursive relations in the columns of $B(n+1)^T$ to determine data for a block $C(n+1)$. Indeed, since $B(n+1)^T = W^T\mathcal{M}(n)$, each dependence relation in $\mathcal{M}(n)$ extends to the columns of $B(n+1)^T$. By our assumption that $\mathcal{M}(n)$ is recursively determinate, these dependence relations propagate so as to determine data for some moment matrix block $C(n+1)$. If this cannot be done unambiguously, then (as above, using Proposition 4.4 and Corollary 4.6), there is no representing measure for β.

If $\mathcal{M}(n+1)$ is well-defined, we check whether $\mathcal{M}(n+1) \succeq 0$, or, equivalently, whether $C(n+1) \succeq C \equiv B(n+1)^T W$. If such is not the case, β has no representing measure. Suppose $\mathcal{M}(n+1) \succeq 0$. If $rank\ \mathcal{M}(n+1) = rank\ \mathcal{M}(n)$, then the unique representing measure for $\mathcal{M}(n+1)$ is a minimal representing measure for β and may be computed as in Theorem 1.2. Suppose $\mathcal{M}(n+1)$ is not a flat extension. If $\mathcal{M}(n+1)$ is not recursively generated, then Corollary 4.6 implies that there is no representing measure. If $\mathcal{M}(n+1)$ is recursively generated, then, since $\mathcal{M}(n)$ is recursively determinate, we may repeat the above procedure starting with $\mathcal{M}(n+1)$, so as to construct a recursively determined extension $\mathcal{M}(n+2)$. In general, having constructed a positive, recursively generated extension $\mathcal{M}(n+i)$ (as above), if the recursively determined extension $\mathcal{M}(n+i+1)$ fails to be positive and recursively generated, then β has no representing measure. On the other hand, if a recursively determined extension $\mathcal{M}(n+i+1)$ is a flat extension of $\mathcal{M}(n+i)$, then the unique representing measure for $\mathcal{M}(n+i+1)$ is a representing measure for β, and a minimal representing measure for β corresponds to a minimal i with the flat extension property (Theorem 4.3). In the case when a measure exists, Theorem 4.3 guarantees that there is a flat extension $\mathcal{M}(n+i+1)$ for some $i \leq v - r$. Thus, after at most $v - r$ extension steps we are able to conclude whether or not β has a representing measure, and if a measure exists, we are able to compute a minimal representing measure using Theorem 4.3 and Theorem 1.2. □

Remark 4.11. (i) In many cases we can expect the preceding algorithm to terminate in far fewer than $v-r$ steps. To see this, let $d = 2$ (the plane) and consider a recursively determinate $\mathcal{M}(n)$ whose only column dependence relations are of the form $X^n = p(X,Y)$ ($deg\ p < n$) and $Y^n = q(X,Y)$ ($deg\ q < n$). In this case, $r = \frac{(n+1)(n+2)}{2} - 2$. Let us assume that f and g have n^2 common zeros in the plane, so that the gap is $v - r = \frac{n^2-3n+2}{2}$. When we examine how the above dependence relations propagate in successive recursively determined extensions $\mathcal{M}(n+1)$, $\mathcal{M}(n+2)\ldots$, we see that in the case where there is a measure, we must achieve a flat extension after at most $n-1$ steps, for in the extension $\mathcal{M}(2n-1)$ $(= \mathcal{M}(n+(n-1)))$, every column of degree $2n-1$ will be recursively determined. Thus, the actual number of extensions leading to a first flat extension is of order at most n, as compared to the order n^2 estimate from the gap.

(ii) As noted in Section 2, [BT] implies that if $\beta^{(2n)}$ has a representing measure, then it has a representing measure ν with *card supp* $\nu \leq$ *dim* $\mathcal{P}_{2n}$. Using this result, it is not difficult to derive versions of Theorem 4.3 and Algorithm 4.10 that apply to the case when the variety of β is not necessarily finite; in such results, we replace the estimate $v - r$ by *dim* $\mathcal{P}_{2n} - r$ (cf. Proposition 2.3 and Remark 2.4).

(iii) When Algorithm 4.10 is applied to a moment matrix $\mathcal{M}(n)$ that we know is recursively determinate (e.g., from Proposition 4.2 or Proposition 4.16), then we can be certain that the procedure will determine whether or not $\beta^{(2n)}$ has a representing measure. But the procedure of Algorithm 4.10 can actually be applied to an arbitrary moment matrix $\mathcal{M}(n)$ to decide whether or not $\mathcal{M}(n)$ is recursively determinate. We explain how this can be done by considering several cases.

First, consider the case when we can follow the steps of Algorithm 4.10 to construct unique positive recursively determined extensions $\mathcal{M}(n+1), \ldots, \mathcal{M}(k)$ (for some $k > 0$), such that $\mathcal{M}(k)$ is a flat extension of $\mathcal{M}(k-1)$. (In establishing each extension, we are not assuming that $\mathcal{M}(n)$ is recursively determinate; instead, at each stage we recursively propagate the column dependence relations, and we observe that this leads to a well-defined recursively determined extension.) Using [CF1], it then follows that there exist unique successive positive, recursively determined (flat) extensions $\mathcal{M}(k+1)$, $\mathcal{M}(k+2), \ldots$. Now, if $\mathcal{M}$ is a positive, recursively generated extension of $\mathcal{M}(n)$ of degree $2n+2i$, then we must have $\mathcal{M} = \mathcal{M}(n+i)$, so $\mathcal{M}$ has the recursively determined extension $\mathcal{M}(n+i+1)$; thus $\mathcal{M}(n)$ is recursively determinate (and has a representing measure).

Next, consider the case when we can follow the steps of Algorithm 4.10 to determine unique recursively determined extensions $\mathcal{M}(n+1), \ldots,\ \mathcal{M}(k)$ (for some $k > 0$), such that $\mathcal{M}(k-1)$ is positive and recursively generated, but $\mathcal{M}(k)$ fails to be positive and recursively generated. In this case, $\mathcal{M}(n)$ is recursively determinate, but there is no representing measure.

Consider the case when either $k = 0$, or we have determined unique positive recursively determined extensions extensions $\mathcal{M}(n+1), \ldots, \mathcal{M}(k)$ (for some $k > 0$). When we try to construct an extension $\mathcal{M}(k+1)$ by recursively propagating all of the column relations in $\mathcal{M}(k)$, it may happen that there are some "free" (undetermined) moments of degree $2k+1$ or $2k+2$. In this case, $\mathcal{M}(n)$ is not recursively determinate and Algorithm 4.10 simply does not apply to $\mathcal{M}(n)$. Sometimes it is possible to choose free moments which lead to a representing measure, but this is not done by Algorithm 4.10. For example, consider the truncated moment problem for planar measures supported in the parabola $y = x^2$. Theorem 3.1 shows that $\beta^{(2n)}$ has a representing measure supported in this curve if and only if $\mathcal{M}(n)$ is positive and recursively generated, has a column relation $Y = X^2$, and satisfies $r \leq v$. There are cases where $\mathcal{M}(n)$ satisfies these conditions but is not recursively determinate. In these cases, it is always possible to construct a positive, rank-increasing extension $\mathcal{M}(n+1)$ which has a flat extension $\mathcal{M}(n+2)$ (yielding a representing measure supported in the parabola), but this construction is beyond the scope of Algorithm 4.10 and uses special features of a moment matrix $\mathcal{M}(n)$ with a column relation $Y = X^2$.

If we attempt to apply Algorithm 4.10 in a case where $\mathcal{M}(n)$ is not recursively determinate, the results may be inconclusive. For example, if $\mathcal{M}(n)$ is invertible, then all of the moments in block $B(n+1)$ are free choices. Once we select a definite choice for the new moments, this may lead (via Algorithm 4.10) to successive positive extensions $\mathcal{M}(n+1), \ldots, \mathcal{M}(k)$ (for some $k > 0$) before finally leading to the conclusion that $\mathcal{M}(k)$ has no representing measure. But this says nothing about whether $\mathcal{M}(n)$ has a measure. One would have to start with another choice for $B(n+1)$ and try again; since there are infinitely many choices for $B(n+1)$, it is impossible to implement this case algorithmically. Another possibility is that free choices at each extension step lead to a sequence of rank-increasing positive extensions $\mathcal{M}(n+1),\ \mathcal{M}(n+2), \ldots$ with no conclusion at each stage. The ultimate $\mathcal{M}(\infty)$ may or may not have a measure, and the present theory of the full moment problem may not be able to decide. Further, if it is possible to determine that $\mathcal{M}(\infty)$ does not have a measure, this conclusion does not necessarily apply to $\mathcal{M}(n)$. (It is known that a planar $\mathcal{M}(1) \succ 0$ always admits a representing measure, but the analogous problem for $\mathcal{M}(2) \succ 0$ is open. For an example of a planar $\mathcal{M}(3) \succ 0$ having no representing measure, see [CF4].)

We may summarize the preceding discussion as follows. $\mathcal{M}(n)$ is recursively determinate if and only if the following holds: when the steps of Algorithm 4.10 are applied to $\mathcal{M}(n)$, we arrive at a recursively determined extension $\mathcal{M}(n+k+1)$ of some positive, recursively generated extension $\mathcal{M}(n+k)$, and either i) *rank* $\mathcal{M}(n+k+1) =$ *rank* $\mathcal{M}(n+k)$ (a measure exists), or ii) $\mathcal{M}(n+k+1)$ is not recursively generated (there is no measure). In every other case, within $1 + \textit{dim}\ \mathcal{P}_{2n} - r$ extension steps, some positive, recursively generated extension $\mathcal{M}(n+k)$ fails to have a recursively determined extension, so $\mathcal{M}(n)$ is not recursively determinate. □

Whether the general result of Remark 4.11(ii) is actually needed is unclear, since the following question is open.

Question 4.12. *If $\mathcal{M}(n)$ is recursively determinate, is the variety of $\mathcal{M}(n)$ finite?*

Theorem 4.3 shows that for $\mathcal{M}(n+1)$ recursively determinate, minimal representing measures correspond to minimal-length convergent extension sequences. Recall from Proposition 2.3 that if a general $\beta^{(2n)}$ has a representing measure, then it has a convergent extension sequence. It is unclear how the length of such a sequence is related to the size of the corresponding measure.

Question 4.13. *Suppose $\beta \equiv \beta^{(2n)}$ has a representing measure. Does a minimal representing measure always correspond to a minimal-length convergent extension sequence?*

In the positive direction (apart from Theorem 4.3), we note that if a minimal-length convergent extension sequence has length 0 or 1, then the measure corresponding to the terminating flat extension is *rank* $\mathcal{M}(n)$-atomic, and is thus a minimal representing measure for β (by (1.2)). On the other hand, we next show that convergent extension sequences of equal length may lead to representing measures of differing sizes, which perhaps suggests a negative answer to Question 4.13.

Example 4.14. Let $\mu_{\mathbb{D}}$ denote planar Lebesgue measure restricted to the closed unit disk $\mathbb{D}$. It is known that $\mu_{\mathbb{D}}$ has a cubature rule ν of degree 8 with 16 nodes [CR] [Co2] [CK]. Consider the extension sequence $\mathcal{M}(4)[\nu]$ ($= \mathcal{M}(4)[\mu_{\mathbb{D}}]$), $\mathcal{M}(5)[\nu]$, $\mathcal{M}(6)[\nu]$. Since the disk has nonempty interior, *rank* $\mathcal{M}(4)[\mu_{\mathbb{D}}] = 15$. If *rank* $\mathcal{M}(5)[\nu] = 15$, then $\mathcal{M}(5)[\nu]$ is a flat extension of $\mathcal{M}(4)[\nu]$, whence Theorem 1.2 implies that the unique representing measure for $\mathcal{M}(5)[\nu]$ is 15-atomic. Since ν is a 16-atomic representing measure for $\mathcal{M}(5)[\nu]$, we conclude that $16 \leq$ *rank* $\mathcal{M}(5)[\nu] \leq$ *card supp* $\nu = 16$, whence *rank* $\mathcal{M}(5)[\nu] = 16$. Similarly, we have $16 =$ *rank* $\mathcal{M}(5)[\nu] \leq$ *rank* $\mathcal{M}(6)[\nu] \leq$ *card supp* $\nu = 16$, so $\mathcal{M}(6)[\nu]$ is a flat extension of $\mathcal{M}(5)[\nu]$. Thus $\mathcal{M}(4)[\mu_{\mathbb{D}}] \longrightarrow \mathcal{M}(5)[\nu] \longrightarrow \mathcal{M}(6)[\nu]$ is a convergent extension sequence of length 2 leading to a 16-atomic representing measure for $\beta \equiv \beta^{(8)}[\mu_{\mathbb{D}}]$.

We will now show that there also exists a length-2 convergent extension sequence leading to an 18-atomic representing measure for β. Indeed, it is known that a minimal cubature rule for $\mu_{\mathbb{D}}$ of degree 9 has 18 nodes (see [Co1] [Co2] [CR] and the discussion and references in [FP, Section 5]). It is proved in [FP, Cor. 5.11] that each such rule arises by first completing $\begin{pmatrix} \mathcal{M}(4)[\mu_{\mathbb{D}}] & B(5)[\mu_{\mathbb{D}}] \\ B(5)[\mu_{\mathbb{D}}]^T & \end{pmatrix}$ to a rank 18 $\mathcal{M}(5)$ and by then constructing a flat extension $\mathcal{M}(6)$. (The result in [FP] is stated in terms of complex moment matrices, but the corresponding result for real moment matrices follows from the equivalence of the real and complex truncated moment problems (cf. [CF6, Section 2]).) The resulting length-2 convergent extension sequence $\mathcal{M}(4)[\mu_{\mathbb{D}}] \longrightarrow \mathcal{M}(5) \longrightarrow \mathcal{M}(6)$ thus leads to an 18-atomic representing measure for β. This example would provide a negative answer to Question 4.13 were it known that the 16-node cubature rule ν cited above is a minimal degree-8 rule for $[\mu_{\mathbb{D}}]$; however, it remains an open question as to whether $\mathcal{M}(4)[\mu_{\mathbb{D}}]$ admits a flat extension $\mathcal{M}(5)$ and a corresponding 15-node minimal rule of degree 8. □

We conclude this section with several examples illustrating Theorem 4.3 and Algorithm 4.10. For these examples, we explicitly compute the variety of each extension, but this is not really necessary. Indeed, once we know that v is finite (say, from Bezout's Theorem), then we know in advance that Algorithm 4.10 will determine the existence of a measure within $v - r + 1$ extension steps. As noted in Remark 4.11(iii), we can also apply the method of Algorithm 4.10 without knowing that v is finite or even that $\mathcal{M}(n)$ is recursively determinate. In this case, if it is possible to achieve a recursively determined, positive and recursively generated extension $\mathcal{M}(n+k)$, then we can use the condition $rank\ \mathcal{M}(n+k) = rank\ \mathcal{M}(n+k-1)$ as an exit test which guarantees the existence of a finitely atomic representing measure. Only if we wish to know the atoms and densities of this measure is it necessary to compute the variety of $\mathcal{M}(n+k)$.

In the sequel, when we consider a planar moment matrix ($d = 2$), we may describe $\mathcal{M}(n)$ via the block decomposition $\mathcal{M}(n) = (M_{ij})_{0 \le i,j \le n}$, where $M_{i,j}$ is the $(i+1) \times (j+1)$ matrix of the form

$$\begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \cdots & \beta_{i+1,j-1} & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i,j} & \beta_{i-1,j+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \cdots & \beta_{1,i+j-1} & \beta_{0,i+j} \end{pmatrix}.$$

Example 4.15. Let $d = 2$. We begin by constructing a positive, recursively generated, recursively determinate $\mathcal{M}(5)$ with a gap of 6. Consider $\beta \equiv \beta^{(10)}$ defined as follows:
moment of degree 0: $\beta_{00} = 61$;
moments of degree 1: $\beta_{10} = 84,\ \beta_{01} = 168$;
moments of degree 2: $\beta_{20} = 726,\ \beta_{11} = 228,\ \beta_{02} = 570$;
moments of degree 3: $\beta_{30} = 5046,\ \beta_{21} = 2232,\ \beta_{12} = 774, \beta_{03} = 2184$;
moments of degree 4: $\beta_{40} = 45378,\ \beta_{31} = 16356,\ \beta_{22} = 8274,\ \beta_{13} = 3000,\ \beta_{04} = 8994$;
moments of degree 5: $\beta_{50} = 421734,\ \beta_{41} = 157944,\ \beta_{32} = 63138,\ \beta_{23} = 34032,\ \beta_{14} = 12558,\ \beta_{05} = 38808$;
moments of degree 6: $\beta_{60} = 4071426,\ \beta_{51} = 1554648,\ \beta_{42} = 636882,\ \beta_{33} = 267708,\ \beta_{24} = 148530,\ \beta_{15} = 55128,\ \beta_{06} = 173010$;
moments of degree 7: $\beta_{70} = 40134066,\ \beta_{61} = 15765360,\ \beta_{52} = 6478974,\ \beta_{43} = 2773752,\ \beta_{34} = 1195890,\ \beta_{25} = 672432,\ \beta_{16} = 249774,\ \beta_{07} = 790104$;
moments of degree 8: $\beta_{80} = 402564066,\ \beta_{71} = 161892924,\ \beta_{62} = 67511586,\ \beta_{53} = 28774680,\ \beta_{44} = 12609762,\ \beta_{35} = 5513196,\ \beta_{26} = 3119394,\ \beta_{17} = 1157160,\ \beta_{08} = 3675234$;
moments of degree 9: $\beta_{90} = 4094720430,\ \beta_{81} = 1679825112,\ \beta_{72} = 708902562,\ \beta_{63} = 304637616,\ \beta_{54} = 132458718,\ \beta_{45} = 58844664,\ \beta_{36} = 25944498,\ \beta_{27} = 14721552,\ \beta_{18} = 5449518,\ \beta_{09} = 17342808$;
moments of degree 10: $\beta_{10,0} = 42137314386,\ \beta_{91} = 17570281368,\ \beta_{82} = 7493058354,\ \beta_{73} = 3241156524,\ \beta_{64} = 1416559890,\ \beta_{55} = 623436648,\ \beta_{46} = 279382002,\ \beta_{37} = 123845628,\ \beta_{28} = 70358610,\ \beta_{19} = 25985208,\ \beta_{0,10} = 82772850$.

A basis $\mathcal{B}$ for the columns of $\mathcal{M}(5)$ consists of the columns indexed by $\mathcal{P}_4$, together with X^4Y, X^3Y^2, X^2Y^3, and XY^4. Calculations with nested determinants show that $\mathcal{M}_{\mathcal{B}}$, the compression of $\mathcal{M}(5)$ to rows and columns indexed by $\mathcal{B}$, is positive definite, whence $\mathcal{M}(5)$ is positive semidefinite. Further, the column dependence relations

$$X^5 = -12X^3 + 8X^4 + 12XY - 14X^2Y + X^4Y + 6Y^2 + XY^2 - 2X^2Y^2 + Y^3 \tag{4.10}$$

and

$$Y^5 = 120 \cdot 1 - 274Y + 225Y^2 - 85Y^3 + 15Y^4, \tag{4.11}$$

imply that $rank\ \mathcal{M}(5) = 19$ and that $\mathcal{V}(\mathcal{M}(5)) = \mathcal{Z}(u) \bigcap \mathcal{Z}(s)$, where

$$u(x,y) = x^5 - p(x,y),\ with\ p(x,y) =$$
$$-12x^3 + 8x^4 + 12xy - 14x^2y + x^4y + 6y^2 + xy^2 - 2x^2y^2 + y^3 \tag{4.12}$$

and

$$s(x,y) = y^5 - q(x,y),\ with\ q(x,y) = 120 - 274y + 225y^2 - 85y^3 + 15y^4. \tag{4.13}$$

A calculation shows that $\mathcal{V}(\mathcal{M}(5))$ consists of the following 25 points $w_i \equiv (x_i, y_i)$ $(1 \le i \le 25)$ (so the gap is 6): $w_1 = (7,1)$, $w_2 = (8,2)$, $w_3 = (9,3)$, $w_4 = (10,4)$, $w_5 = (11,5)$, $w_6 = (-1,1)$,

$w_7 = (1,1)$, $w_8 = (1-\sqrt{2},1)$, $w_9 = (1+\sqrt{2},1)$, $w_{10} = (-\sqrt{2},2)$, $w_{11} = (\sqrt{2},2)$, $w_{12} = (1-\sqrt{3},2)$, $w_{13} = (1+\sqrt{3},2)$, $w_{14} = (-1,3)$, $w_{15} = (3,3)$, $w_{16} = (-\sqrt{3},3)$, $w_{17} = (\sqrt{3},3)$, $w_{18} = (-2,4)$, $w_{19} = (2,4)$, $w_{20} = (1-\sqrt{5},4)$, $w_{21} = (1+\sqrt{5},4)$, $w_{22} = (-\sqrt{5},5)$, $w_{23} = (\sqrt{5},5)$, $w_{24} = (1-\sqrt{6},5)$, $w_{25} = (1+\sqrt{6},5)$.

From Proposition 4.2 (but reversing the roles of p and q), we see that $\mathcal{M}(5)$ is recursively determinate. We will use Theorem 4.3 and Algorithm 4.10 to determine whether β admits a representing measure; since $v - r = 6$, at most 7 extension steps will be required, and it will be possible to revise this estimate downward as we procede. Following Algorithm 4.10, we first construct a recursively determined extension $\mathcal{M}(6)$, with moments of degrees 11 and 12 determined from the relations

$$X^6 = (xp)(X,Y) \tag{4.14}$$

and

$$Y^6 = (yq)(X,Y). \tag{4.15}$$

$\mathcal{M}(6)$ is positive semidefinite and recursively generated, with a column basis $\mathcal{B}'$ consisting of the 19 columns in $\mathcal{M}(6)$ corresponding to $\mathcal{B}$, together with columns X^6, X^3Y^3, X^2Y^4; thus $r = 22$. In $\mathcal{M}(6)$, the column relations $X^5Y = (yp)(X,Y)$ and $Y^5X = (xq)(X,Y)$, together with (4.10)-(4.11) and (4.14)-(4.15), readily imply that $ker\ \mathcal{M}(6) = \langle \hat{u},\ \hat{s}, \widehat{xu},\ \widehat{yu}, \widehat{xs},\ \widehat{ys} \rangle$, whence $\mathcal{V}' \equiv \mathcal{V}(\mathcal{M}(6)) = \mathcal{Z}(u) \bigcap \mathcal{Z}(s) = \mathcal{V}(\mathcal{M}(5))$, so we again have $v = 25$. Since in $\mathcal{M}(6)$ we have $v - r = 3$ and $\mathcal{M}(6)$ is recursively determinate (by Proposition 4.2), we see that at most 4 additional extension steps are needed to resolve the existence of a representing measure for β.

We next construct a recursively determined extension $\mathcal{M}(7)$ from the relations $X^7 = (x^2p)(X,Y)$ and $Y^7 = (y^2q)(X,Y)$, and it results that $\mathcal{M}(7)$ is positive semidefinite and recursively generated, with $r = 23$. Indeed, a column basis $\mathcal{B}''$ consists of the 22 columns in $\mathcal{M}(7)$ corresponding to $\mathcal{B}'$, together with column X^7. To analyze $\mathcal{V}(\mathcal{M}(7))$, note that by positivity, the column dependence relations in $\mathcal{M}(6)$ extend to $\mathcal{M}(7)$. Next, consider the dependence relations $X^7 = (x^2p)(X,Y)$, $X^6Y = (xyp)(X,Y)$, $X^5Y^2 = (y^2p)(X,Y)$, $Y^7 = (y^2q)(X,Y)$, $Y^6X = (yxq)(X,Y)$, and $Y^5X^2 = (x^2q)(X,Y)$. The common zeros of x^2u ($= x^7 - x^2p$), xyu ($= x^6y - xyp$), y^2u ($= x^5y^2 - y^2p$), y^2s ($= y^7 - y^2q$), xys ($= y^6x - yxq$), y^2s ($= y^5x^2 - x^2q$), and of the polynomials corresponding to $ker\ \mathcal{M}(6)$, clearly coincide with $\mathcal{Z}(p) \bigcap \mathcal{Z}(q) = \mathcal{V}(\mathcal{M}(5))$. However, there is also a dependence relation $X^3Y^4 = h(X,Y)$, where $h(x,y) = -1320 + 120x + 2750y + 264x^2 - 250xy - 1925y^2 - 24x^3 - 550x^2y + 175xy^2 + 550y^3 + 50x^3y + 385x^2y^2 - 50xy^3 - 55y^4 - 35x^3y^2 - 110x^2y^3 + 5xy^4 + 10x^3y^3 + 11x^2y^4$. Thus, a basis for $ker\ \mathcal{M}(7)$ consists of $\{\widehat{x^iy^jp}\}_{i,j\geq 0, i+j\leq 2}$, $\{\widehat{x^iy^jq}\}_{i,j\geq 0, i+j\leq 2}$, and $\hat{g}$ (where $g(x,y) = x^3y^4 - h(x,y)$), so $\mathcal{V}'' \equiv \mathcal{V}(\mathcal{M}(7)) = \mathcal{Z}(u) \bigcap \mathcal{Z}(s) \bigcap \mathcal{Z}(g) = \{w_i\}_{1\leq i\leq 23}$, whence $\mathcal{M}(7)$ is extremal (with $r = v = 23$).

At this point, we can procede either by continuing with Theorem 4.3 and Algorithm 4.10 to compute a recursively determined extension $\mathcal{M}(8)$, or by applying Theorem 3.4 (concerning the extremal case). In the first approach, since the gap in $\mathcal{M}(7)$ is 0, the next extension step is decisive. Indeed, we find that $\mathcal{M}(8)$, determined using $X^8 = (x^3p)(X,Y)$ and $Y^8 = (y^3q)(X,Y)$, is a flat extension of $\mathcal{M}(7)$, with a corresponding representing measure $\mu \equiv \sum_{i=1}^{23} \rho_i \delta_{w_i}$ that can be computed as in Theorem 1.2. Alternately, using Theorem 3.2 (and the notation from Section 3), we find that $L_{\beta^{(14)}} \in Ran\ U_{14}[\mathcal{V}'']$, so that $\beta^{(14)}$ is consistent. The unique representing measure for $\beta^{(14)}$ is thus of the above form, where $\rho \equiv (\rho_1, \ldots, \rho_{23})$ is determined by $U_{\mathcal{B}''}[\mathcal{V}'']\rho^T = v_{\mathcal{B}''}^T$ ($v_{\mathcal{B}''}$ is the vector of moments corresponding to the monomials in basis $\mathcal{B}''$ in degree-lexicographic order; cf. Theorem 3.2). Using either approach, we find that $\rho_1 = \rho_2 = 2$, $\rho_3 = \rho_4 = \rho_5 = 1$, and $\rho_i = 3$ $(6 \leq i \leq 23)$. Thus, Theorem 4.3 implies that μ is a minimal representing measure for β. $\square$

The next example concerns $d = 3$ and requires the following preliminary result, which identifies, from among the $(n+1)(n+2)/2$ columns of degree n in $\mathcal{M}(n)$, a determinate set of $n+2$ columns.

Proposition 4.16. *Let $d = 3$. $\mathcal{M}(n)$ is recursively determinate if each of the following columns of degree n,*

$$\mathcal{F}_n : \; X^n, \; X^{n-i}YZ^{i-1} \; (1 \le i \le n-1), \; Y^n, \; Z^n,$$

can be expressed as linear combinations of columns of strictly smaller degree.

Proof. We observe first that the columns of $\mathcal{F}_n$ contain all moments of degrees $2n-1$ and $2n$. Indeed, in block $B(n)$, column X^n contains β_{jrs} with $n \le j \le 2n-1$, $r, s \ge 0$ and $r + s + j = 2n - 1$. Further, for $1 \le i \le n-1$, columns Y^n, Z^n, and $X^{n-i}YZ^{i-1}$ contain all moments $\beta_{n-i,r,s}$ with $r, \; s \ge 0$ and $r + s + n - i = 2n - 1$. Columns Y^n and Z^n together contain the moments $\beta_{0,r,s}$ $(r + s = 2n - 1, \; r, s \ge 0)$. In block $C(n)$, column X^n contains all moments β_{jrs} with $n \le j \le 2n$, $r, s \ge 0$, and $r+s+j = 2n$. For $1 \le i \le n-1$, the moments $\beta_{n-i,r,s}$ with $r, s \ge 0$ and $n-i+r+s = 2n$ are located in columns $X^{n-i}YZ^{i-1}$, Y^n, and Z^n. Finally, columns Y^n and Z^n contain the moments $\beta_{0,r,s}$ with $r, s \ge 0$ and $r + s = 2n$.

Now suppose that $\mathcal{M}(n+i)$ is a positive, recursively generated extension of $\mathcal{M}(n)$. By applying recursiveness to the columns in $\mathcal{F}_n$ (extended into $\mathcal{M}(n+i)$ via positivity), we see that in the column space of $\mathcal{M}(n+i)$, each column in $\mathcal{F}_{n+i}$ can be expressed as some linear combination of columns of strictly smaller degree. It follows that by propagating these relations into degree $n+i+1$ (which can be done in various ways), we can determine columns for $\mathcal{F}_{n+i+1}$ (first in block $B(n+i+1)$, then in block $C(n+i+1)$), and thereby define a recursively determined extension $\mathcal{M}(n+i+1)$. □

Example 4.17. We consider a 3-dimensional moment matrix $\mathcal{M}(4)(\beta)$ with a gap of 7. Since there are 165 distinct monomials $x^i y^j z^k$ in $\mathcal{P}_8$, we record the moments of $\beta^{(8)}$ in an appendix (Section 5). The size of $\mathcal{M}(4)$ is 35×35; by using nested determinants, we see that $\mathcal{M}(2)$ is positive definite, i.e., $\mathcal{M}(2) \succ 0$. In $\mathcal{M}(3)$ we have the single column dependence relation

$$X^2Y = 5XY - 4Y, \tag{4.16}$$

and if we delete from $\mathcal{M}(3)$ the row and column corresponding to X^2Y, the resulting compression is positive definite; thus $\mathcal{M}(3) \succeq 0$. In $\mathcal{M}(4)$ we have the following degree-4 column relations:

$$X^4 = 12X^3 - 49X^2 + 78X - 40, \tag{4.17}$$

$$X^3Y = 5X^2Y - 4XY, \tag{4.18}$$

$$X^2Y^2 = 5XY^2 - 4Y^2, \tag{4.19}$$

$$X^2YZ = 5XYZ - 4YZ, \tag{4.20}$$

$$XYZ^2 = YZ^2 + 4XY - 4Y, \tag{4.21}$$

$$Y^4 = 2Y^3 + Y^2 - 2Y, \tag{4.22}$$

$$Z^4 = 2XZ^2 - X^2 + 2XZ - 2Z^3. \tag{4.23}$$

In view of (4.17), (4.18), (4.20)-(4.23) and Proposition 4.16, $\mathcal{M}(4)$ is recursively determinate.

Since relations (4.18)-(4.20) correspond to (4.16) via recursiveness, we see that $V(\mathcal{M}(4))$ depends only on (4.16)-(4.17) and (4.21)-(4.23), and consists of the following 34 points: letting $\alpha = -1, \; 0, \; 2, \; 1$, we have

$$\begin{gathered}(1, \alpha, \pm 1), \; (1, \alpha, -1 \pm \sqrt{2}), \; (4, \alpha, \pm 2), \; (4, 0, -1 \pm \sqrt{5}), \; (2, 0, \pm\sqrt{2}), \\ (2, 0, -1 \pm \sqrt{3}), \; (5, 0, \pm\sqrt{5}), \; (5, 0, -1 \pm \sqrt{6}).\end{gathered} \tag{4.24}$$

If we delete from $\mathcal{M}(4)$ the eight pairs of rows and columns corresponding to (4.16)-(4.23), nested determinants implies that the resulting compression is positive definite. Thus $\mathcal{M}(4) \succeq 0$, with $r = 27$ and $v = 34$, leaving a gap of 7.

Since $\mathcal{M}(4)$ is recursively determinate, Algorithm 4.10 permits us to resolve the existence of a representing measure for $\beta^{(8)}$ with at most 8 extensions. We see that the recursively determined extension $\mathcal{M}(5)$ is positive and recursively generated. In $\mathcal{M}(5)$ all of the columns X^5, Y^5, Z^5, X^4Y, X^3YZ, X^2YZ^2, XYZ^3, X^4Z, X^3Y^2, X^2Y^3, X^2Y^2Z, XY^2Z^2, XY^4, Y^4Z, XZ^4, YZ^4 are recursively determined and we also find the following "nonrecursive" column relation:

$$\begin{aligned} Y^2Z^3 = -YZ^3 + (-1+\sqrt{2})Y^2Z^2 + (-1+\sqrt{2})YZ^2 + XY^2Z \\ +XYZ - (-1+\sqrt{2})XY^2 + (1-\sqrt{2})XY. \end{aligned} \tag{4.25}$$

The 31 columns of $\mathcal{M}(5)$ complementary to the columns in the preceding 25 dependence relations define a column basis $\mathcal{B}$, and the compression $\mathcal{M}_{\mathcal{B}}$ of $\mathcal{M}(5)$ to the rows and columns corresponding to $\mathcal{B}$ satisfies $\mathcal{M}_{\mathcal{B}} \succ 0$. Thus $\mathcal{M}(5) \succeq 0$, *rank* $\mathcal{M}(5) = 31$, and $V(\mathcal{M}(5))$ consists of the points of $V(\mathcal{M}(4))$ that are also zeros of

$$\begin{aligned} v(x,y,z) := y^2z^3 + yz^3 - (-1+\sqrt{2})y^2z^2 - (-1+\sqrt{2})yz^2 - xy^2z - xyz \\ +(-1+\sqrt{2})xy^2 - (1-\sqrt{2})xy = y(1+y)(-1+\sqrt{2}-z)(x-z^2). \end{aligned} \tag{4.26}$$

Comparison of (4.26) with (4.24) shows that $V(\mathcal{M}(5)) = V(\mathcal{M}(4)) \backslash \{(1,1,-1-\sqrt{2}),(1,2,-1-\sqrt{2})\}$, so in $\mathcal{M}(5)$ we have $r = 31$, $v = 32$, and the gap has been reduced to 1, implying that at most 2 additional extensions are needed.

We next use the degree-5 column relations of $\mathcal{M}(5)$ to compute the following determinate set of columns for a positive, recursively generated extension $\mathcal{M}(6)$: X^6, Y^6, Z^6, X^5Y, X^4YZ, X^3YZ^2, X^2YZ^3, XYZ^4. There are 28 new columns (of degree 6) in the resulting $\mathcal{M}(6)$. Of these, 27 are recursively determined from dependence relations in $\mathcal{M}(5)$. However, a calculation shows that column X^3Y^3 is linearly independent of the columns in $\begin{pmatrix} \mathcal{M}(5) \\ \mathcal{B}(6)^T \end{pmatrix}$, so *rank* $\mathcal{M}(6) = 1+\mathcal{M}(5) = 32$. A calculation with compressions and nested determinants now shows that $\mathcal{M}(6) \geq 0$. Further, since the only new column dependence relations in $\mathcal{M}(6)$ are recursively determined, we have $V(\mathcal{M}(6)) = V(\mathcal{M}(5))$ (see the remarks immediately preceding the proof of Theorem 2.1), whence $\mathcal{M}(6)$ is extremal, with $r = v = 32$. When we next compute a recursively determined extension $\mathcal{M}(7)$, we see that $\mathcal{M}(7)$ is well-defined and that every column is recursively determined from $\mathcal{M}(6)$. Thus $\mathcal{M}(7)$ is a flat extension of $\mathcal{M}(6)$. Theorem 4.3 now implies that $\beta^{(8)}$ has a representing measure and that the unique minimal representing measure μ is 32-atomic. It follows that *supp* $\mu = V(\mathcal{M}(7)) = V(\mathcal{M}(6)) \equiv \{\omega_i\}_{i=1}^{32}$, and a calculation as in Theorem 1.2 shows that $\mu = \sum_{i=1}^{32} \delta_{\omega_i}$. $\square$

In our final example we show how to use Algorithm 4.10 to establish the nonexistence of a representing measure.

Example 4.18. We consider again $\mathcal{M}(3)$ defined as in Example 3.9. Recall that $\mathcal{M}(3)$ is positive and extremal, with $r = v = 8$ and column dependence relations

$$X^3 = Y \tag{4.27}$$

and

$$Y^3 = 3X + \frac{45}{4}Y - 13X^2 + \frac{65}{4}XY - \frac{13}{4}Y^2 - 22X^2Y + \frac{35}{4}XY^2. \tag{4.28}$$

Proposition 4.2 implies that $\mathcal{M}(3)$ is recursively determinate, and we can determine column X^4 for block $B(4)$ of any positive, recursively generated $\mathcal{M}(4)$ by $X^4 = XY$; we find $\beta_{70} = \frac{-43115}{128}$, $\beta_{61} = \frac{-926695}{512}$, $\beta_{52} = \frac{-19736547}{2048}$, $\beta_{43} = \frac{-419176415}{8192}$. Using these values and the column relation $Y^4 = 3XY + \frac{45}{4}Y^2 - 13X^2Y + \frac{65}{4}XY^2 - \frac{13}{4}Y^3 - 22X^2Y^Y + \frac{35}{4}XY^3$, we then successively compute $\beta_{34} = \frac{-88948735563}{32768}$, $\beta_{25} = \frac{-188695052247}{131072}$, $\beta_{16} = \frac{-4002599665619}{524288}$, $\beta_{07} = \frac{-84900703109071}{2097152}$. We will impose the Smul'jan criteria for positivity of a moment matrix extension (cf. Section 2). A calculation using *Mathematica*'s $W = LinearSolve[\mathcal{M}(3), B(4)]$ shows that there exists W satisfying $B(4) = \mathcal{M}(3)W$.

As in Example 4.1, we next use the above relations for X^4 and Y^4 in the columns of $\begin{pmatrix} B(4)^T & C(4) \end{pmatrix}$ to determine the moments for block $C := C(4)$. The resulting recursively determined extension $\mathcal{M}(4)$ is positive semidefinite if and only if $C \succeq C' := W^T \mathcal{M}(3) W$. Now $C_{33} := C_{51} = \frac{49568350247}{65536} \approx 756353$ and $C'_{33} = \frac{1264843863151430003}{1671544766464} \approx 756692$. Since $C_{33} < C'_{33}$, there is no positive, recursively generated extension $\mathcal{M}(4)$, whence Theorem 4.3 implies that $\beta^{(6)}$ has no representing measure. $\square$

In the examples of [CF5] [CF7], as in Example 4.18, when a positive, recursively determinate $\mathcal{M}(n)$ fails to have a representing measure, it transpires that $\mathcal{M}(n)$ does not admit a positive, recursively generated extension $\mathcal{M}(n+1)$. Moreover, if a positive moment matrix $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$, then it admits unique successive flat, positive, recursively generated extensions $\mathcal{M}(n+2)$, $\mathcal{M}(n+3), \ldots$ (cf. [CF2] [CF10]). These observations and Theorem 4.3 suggest the following question.

Question 4.19. *If $\mathcal{M}(n)$ is recursively determinate and admits a positive, recursively generated extension $\mathcal{M}(n+1)$, does it admit successive positive, recursively generated extensions $\mathcal{M}(n+2)$, $\mathcal{M}(n+3), \ldots$ (and a corresponding representing measure (cf. Theorem 4.3))?*

5. Appendix

We present below the data for Example 4.17. We define a moment matrix $\mathcal{M}(4)(\beta)$ in three real variables x, y, z. The moment data for each degree k $(0 \le k \le 8)$ follows the degree-lexicographic ordering of the monomials of degree k. Thus, for degree 3, the ordering is x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3, and the corresponding moments are presented in the order β_{300}, β_{210}, β_{201}, β_{120}, β_{111}, β_{102}, β_{030}, β_{021}, β_{012}, β_{003}. In the sequel, s denotes $\sqrt{2}$.
Degree 0 moment: 32. Degree 1 moments: 82, 9, $-12+2s$. Degree 2 moments: 290, 21, $-28+2s$, 31, $-1+3s$, $106-4s$. Degree 3 moments: 1186, 69, $-96+2s$, 67, $-1+3s$, $346-4s$, 39, $-7+5s$, $23-6s$, $-132+10s$. Degree 4 monents: 5138, 261, $-400+2s$, 211, $-1+3s$, $1378-4s$, 87, $-7+5s$, $71-6s$, $-400+10s$, 91, $-7+9s$, $81-10s$, $-7+15s$, $610-24s$. Degree 5 moments: 22882, 1029, $-1800+2s$, 787, $-1+3s$, $5938-4s$, 279, $-7+5s$, $263-6s$, $-1584+10s$, 199, $-7+9s$, $225-10s$, $-7+15s$, $2178-24s$, 159, $19+17s$, $101-18s$, $-49+25s$, $85-36s$, $-1232+58s$. Degree 6 moments: 103730, 4101, $-8368+2s$, 3091, $-1+3s$, $26482-4s$, 1047, $-7+5s$, $1031-6s$, $-7000+10s$, 631, $-7+9s$, $801-10s$, $-7+15s$, $9106-24s$, 351, $-19+17s$, $293-18s$, $-49+25s$, $277-36s$, $-4368+58s$, 331, $-31+33s$, $237-34s$, $-49+45s$, $323-60s$, $-41+87s$, $4642-140s$. Degree 7 moments: 476866, 16389, $-39576+2s$, 12307, $-1+3s$, $120466-4s$, 4119, $-7+5s$, $4103-6s$, $-32304+10s$, 2359, $-7+9s$, $3105-10s$, $-7+15s$, $40482-24s$, 1119, $-19+17s$, $1061-18s$, $-49+25s$, $1045-36s$, $-18536+58s$, 727, $-31+33s$, $669-34s$, $-49+45s$, $889-60s$, $-41+87s$, $17842-140s$, 639, $-67+65s$, $413-66s$, $-133+85s$, $391-108s$, $-287+145s$, $359-210s$, $-12080+338s$. Degree 8 moments: 2218898, 65541, $-189280+2s$, 49171, $-1+3s$, $556018-4s$, 16407, $-7+5s$, $16391-6s$, $-152200+10s$, 9271, $-7+9s$, $12321-10s$, $-7+15s$, $185074-24s$, 4191, $-19+17s$, $4133-18s$, $-49+25s$, $4117-36s$, $-84240+58s$, 2311, $-31+33s$, $2397-34s$, $49+45s$, $3203-60s$, $-41+87s$, $77554-140s$, 1407, $-67+65s$, $1181-66s$, $-133+85s$, $1159-108s$, $-287+145s$, $1127-210s$, $-47544+338s$, 1291, $-127+129s$, $861-130s$, $-217+165s$, $935-204s$, $-287+261s$, $1473-350s$, $-239+507s$, $42002-816s$.

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