THE EXTREMAL TRUNCATED MOMENT PROBLEM

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Abstract. For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \le 2n}$ to have a *representing measure* μ, it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite and for the algebraic variety associated to β, $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy *rank* $\mathcal{M}(n) \le$ *card* $\mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \le 2n} a_i x^i$ vanishes on $\mathcal{V}$, then $\sum_{|i| \le 2n} a_i \beta_i = 0$. We prove that for the *extremal* case (*rank* $\mathcal{M}(n) =$ *card* $\mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, *rank* $\mathcal{M}(n)$-atomic) representing measure. We also show that in the preceding result, consistency cannot always be replaced by recursiveness of $\mathcal{M}(n)$.

1. Introduction

Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}_+^d, |i| \le 2n}$ denote a real d-dimensional multisequence of degree $2n$. The *truncated moment problem* for β concerns the existence of a positive Borel measure μ, supported in $\mathbb{R}^d$, such that

$$\beta_i = \int_{\mathbb{R}^d} x^i \, d\mu, \ |i| \le 2n; \tag{1.1}$$

(here, for $x \equiv (x_1, ..., x_d) \in \mathbb{R}^d$ and $i \equiv (i_1, ..., i_d) \in \mathbb{Z}_+^d$, we let $x^i := x_1^{i_1} \cdots x_d^{i_d}$). A measure μ as in (1.1) is a *representing measure* for β. The truncated moment problem is more general than the classical *full moment problem* (cf. [Akh] [AhKr] [KrNu] [ST] [PuVa] [StSz]). Indeed, a result of J. Stochel [Sto] shows that a full moment sequence $\beta^{(\infty)}$ has a representing measure supported in a prescribed closed set $K \subseteq \mathbb{R}^d$ if and only if for each n, $\beta^{(2n)}$ has a representing measure supported in K.

Let $\mathcal{P} \equiv \mathbb{R}^d[x] = \mathbb{R}[x_1, ..., x_d]$ denote the space of real valued d-variable polynomials, and for $k \ge 1$, let $\mathcal{P}_k \equiv \mathbb{R}_k^d[x]$ denote the subspace of $\mathcal{P}$ consisting of polynomials p with $\deg p \le k$. Corresponding to β we have the *Riesz functional* $\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}$, which associates to an element p of $\mathcal{P}_{2n}$, $p(x) \equiv \sum_{|i| \le 2n} a_i x^i$, the value $\Lambda(p) := \sum_{|i| \le 2n} a_i \beta_i$; of

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course, in the presence of a representing measure μ, we have $\Lambda(p) = \int p\, d\mu$. In the sequel, $\hat{p}$ denotes the coefficient vector (a_i) of p.

Following [CuFi2], we associate to β the *moment matrix* $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$, with rows and columns X^i indexed by the monomials of $\mathcal{P}_n$ in degree-lexicographic order; for example, with $d = n = 2$, the columns of $\mathcal{M}(2)$ are denoted as $1, X_1, X_2, X_1^2, X_2X_1, X_2^2$. The entry in row X^i, column X^j of $\mathcal{M}(n)$ is β_{i+j}, so $\mathcal{M}(n)$ is a real symmetric matrix characterized by

$$\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = \Lambda(pq) \;\; (p, q \in \mathcal{P}_n). \tag{1.2}$$

If μ is a representing measure for β, then $\langle \mathcal{M}(n)\hat{p}, \hat{p}\rangle = \Lambda(p^2) = \int p^2 d\mu \geq 0$; since $\mathcal{M}(n)$ is real symmetric, it follows that $\mathcal{M}(n)$ is positive semidefinite (in symbols, $\mathcal{M}(n) \geq 0$). The *algebraic variety* of β (or of $\mathcal{M}(n)(\beta)$) is defined by

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) := \{x \in \mathbb{R}^d : p(x) = 0\}$. (We sometimes denote $\mathcal{V}_\beta$ as $\mathcal{V}(\mathcal{M}(n)(\beta))$.) Each element of *ker* $\mathcal{M}(n)$ is of the form $\hat{p}$ for some $p \in \mathcal{P}_n$, and corresponds to a column dependence relation that we denote by $p(X) = 0$. If β admits a representing measure μ, then $p \in \mathcal{P}_n$ satisfies $p(X) = 0$ if and only if *supp* $\mu \subseteq \mathcal{Z}(p)$ [CuFi2, Proposition 3.1]. Thus *supp* $\mu \subseteq \mathcal{V}$, and it follows from [CuFi4,(1.7)] that $r := rank\; \mathcal{M}(n)$ and $v := card\; \mathcal{V}$ satisfy $r \leq card\; supp\; \mu \leq v$. Further, in this case, if $p, q, pq \in \mathcal{P}_n$ and $p(X) = 0$ in the column space of $\mathcal{M}(n)$, then $(pq)(X) = 0$. To summarize the preceding discussion, we have the following basic necessary conditions for the existence of a representing measure for $\beta^{(2n)}$:

$$\text{(Positivity) } \mathcal{M}(n) \geq 0 \tag{1.3}$$

$$\text{(Recursiveness) } p, q, pq \in \mathcal{P}_n,\; p(X) = 0 \Longrightarrow (pq)(X) = 0. \tag{1.4}$$

$$\text{(Variety Condition) } r \leq v, \text{ i.e., } rank\; \mathcal{M}(n) \leq card\; \mathcal{V}_\beta. \tag{1.5}$$

For $d = 2$ (the plane), there exists $\mathcal{M}(3) > 0$ (positive definite) for which β has no representing measure [CuFi3, Section 4]. Since an invertible moment matrix satisfies (1.4) and (1.5) vacuously, it follows that in general (1.3)-(1.5) are not sufficient for the existence of representing measures. Nevertheless, it is of interest to identify cases where (1.3)-(1.5) are sufficient, because these conditions are "concrete". Indeed, only elementary linear algebra is needed to compute $r \equiv rank\; \mathcal{M}(n)$ and to check positivity and recursivenesss, and for moderate values of n, mathematical software can be used to estimate $v \equiv card\; \mathcal{V}_\beta$.

In [] we showed that for $d = 1$ (the *truncated Hamburger Moment Problem*), positivity and recursiveness of the associated Hankel matrix are sufficient for the existence of a representing measure supported in $\mathbb{R}$. For $d = 2$ and $deg\; p(x, y) \leq 2$, the results of [] together show that $\beta^{(2n)}$ has a representing measure supported in the curve $p(x, y) = 0$ if and only if $\mathcal{M}(n)$ has a column dependence relation $p(X, Y) = 0$ and (1.3)-(1.5) hold. Further, in the *truncated complex moment problem*, a planar complex multisequence $\gamma^{(2n)}$ admits a representing measure supported in the (finite) variety $z^k = q(z, \bar{z}) = 0$ (where $deg\; q < k \leq [n/2] + 1$) if and only if the associated complex moment matrix $M(n)(\gamma)$ is positive and recursively generated, and has a column relation $Z^k = q(Z, \bar{Z})$. The preceding results motivate the following question (cf. [F3, Conjecture 3]).

Question 1.1. *Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, recursively generated, and $r \leq v$, does β admit a representing measure?*

In the sequel, we study Question 1.1 primarily in the *extremal* case, when $r \equiv rank\ \mathcal{M}(n)(\beta)$ and $v \equiv card\ \mathcal{V}_\beta$ satisfy $r = v$. We focus on the planar $\mathcal{M}(3)$ moment problem with $\mathcal{M}(2) > 0$ and a column relation $Y = X^3$ (so that representing measures are necessarily supported in the curve $y = x^3$). In this setting, the extremal case can occur only with $r = v = 7$ or $r = v = 8$. For the former case, we show in Section 4 (Theorem 4.1) that positivity and recursiveness are sufficient for representing measures. Our main result, for the case $r = v = 8$, provides a perhaps surprising negative answer to Question 1.1.

Theorem 1.2. *There exists a planar $\mathcal{M}(3)$ with a column relation $Y = X^3$ such that $\mathcal{M}(3)$ is positive and recursively generated, with $r = v = 8$, but $\beta^{(6)}$ does not admit a representing measure.*

In view of Theorem 1.2 we consider a necessary condition for representing measures that is stronger than recursiveness:

$$\text{(Consistency) } p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}_\beta} = 0 \Longrightarrow \Lambda_\beta(p) = 0. \tag{1.6}$$

We then have the following general solution to the extremal truncated moment problem.

Theorem 1.3. *Let $d \geq 1$. For $\beta \equiv \beta^{(2n)}$ extremal, i.e., $r = v$, the following are equivalent:*
(i) β has a representing measure;
(ii) β has a unique representing measure, which is rank $\mathcal{M}(n)$-atomic;
(iii) $\mathcal{M}(n)$ is positive semidefinite and β is consistent.

Theorem 1.3 suggests the following question, which remains open.

Question 1.4. *Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, $r \leq v$, and β is consistent, does β admit a representing measure?*

The results of Sections 4 and 5 provide some positive evidence for Question 1.4.

Theorem 1.5. *Let $d = 2$. Suppose $\mathcal{M}(3) \geq 0$ satisfies $Y = X^3$.*
i) If $r \leq v \leq 7$, $\beta^{(6)}$ has a representing measure if and only if $\mathcal{M}(3)$ is recursively generated.
ii) If $r \leq v = 8$, $\beta^{(6)}$ has a representing measure if and only if β is consistent.

Much remains to be learned as to why recursiveness is sufficient in some cases, but consistency is required in others. The pathology exhibited by Theorem 1.2 results from the presence of zeros of higher multiplicity in $\mathcal{V}(\mathcal{M}(3))$. With this kind of variety, other new phenomena can be discerned. For example, in the full moment problem, it is known that if $\beta^{(\infty)}$ has a representing measure, then *ker* $\mathcal{M}(\infty)$ is a *real ideal* (cf. Section 3). By contrast, in Proposition 5.1 we illustrate a sequence $\beta^{(6)}$ having a representing measure, but for which the ideal generated by *ker* $\mathcal{M}(3)$ is not a real ideal.

If the points of $\mathcal{V}(\mathcal{M}(n))$ are known exactly, then only elementary linear algebra is needed to check whether or not β is consistent (cf. Section 2). If the points of $\mathcal{V}(\mathcal{M}(n))$ are not known exactly, then it may be difficult to verify consistency directly. For this reason, we seek to identify instances when the test for consistency can be simplified. This is the case in the $\mathcal{M}(3)$ moment problem with $\mathcal{M}(2) > 0$, $Y = X^3$ and $r = v = 8$. We show in Theorem 5.3 that for this problem, consistency reduces to checking that $\Lambda_\beta(h) = 0$ for a particular polynomial $h \in \mathbb{R}[x, y]$ of degree 4 that we associate to β.

We observe that the extremal case is inherent in the truncated moment problem. A recent result of C. Bayer and J. Teichmann [BaTe] (extending a classical theorem of V. Tchakaloff [Tch] and its successive generalizations in [Mys], [Put] and [CuFi9] implies that if $\beta^{(2n)}$ has a representing measure, then it has a finitely atomic representing measure. In [CuFi4] it was shown that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $\mathcal{M}(n)$ admits an extension to a positive moment matrix $\mathcal{M}(n+k)$ (for some $k \geq 0$), which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $\mathcal{M}(n + k + 1)$. Further, [CuFi11, Theorem 1.2] shows that any flat extension $\mathcal{M}(n+k+1)$ is an extremal moment matrix for which there is a computable *rank* $\mathcal{M}(n + k)$-atomic representing measure μ. Clearly, μ is also a finitely atomic representing measure for $\beta^{(2n)}$, and every finitely atomic representing measure for $\beta^{(2n)}$ arises in this way. In this sense, the existence of a representing measure for $\beta^{(2n)}$ is intimately related to the solution of an extremal truncated moment problem.

2. Moment matrices, consistency and the extremal moment problem

In this section we study the implications of consistency, leading to Theorem 2.8, which includes a proof of Theorem 1.3. Recall that the columns of $\mathcal{M}(n)$ are denoted as X^i, $|i| \leq n$, following the degree-lexicographic ordering of the monomials x^i in $\mathcal{P}_n$. Let $p \in \mathcal{P}_n$, $p(x) \equiv \sum_{|i|\leq n} a_i x^i$; the general element of $\mathcal{C}_{\mathcal{M}(n)}$, the column space of $\mathcal{M}(n)$, may thus be denoted as $p(X) := \sum_{|i|\leq n} a_i X^i$. Let $\hat{p} \equiv (a_i)$ denote the coefficient vector of p relative to the basis of monomials of $\mathcal{P}_n$ in degree-lexicographic order, and note that $p(X) = \mathcal{M}(n)\hat{p}$. Now recall the variety of β, $\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p\in\mathcal{P}_n, p(X)=0} \mathcal{Z}(p)$, where $\mathcal{Z}(p) := \{x \in \mathbb{R}^d : p(x) = 0\}$. Let $\mathcal{P}_n|_\mathcal{V}$ denote the restriction to $\mathcal{V}$ of the polynomials in $\mathcal{P}_n$, and consider the mapping $\phi_\beta : \mathcal{C}_{\mathcal{M}(n)} \to \mathcal{P}_n|_\mathcal{V}$ given by $p(X) \mapsto p|_\mathcal{V}$. The map ϕ_β is well-defined, for if $p, q \in \mathcal{P}_n$ with $p(X) = q(X)$, then $\mathcal{V} \subseteq \mathcal{Z}(p - q)$, whence $p|_\mathcal{V} = q|_\mathcal{V}$. In the sequel, we say that β is *weakly consistent* if ϕ_β is 1-1, i.e., $p \in \mathcal{P}_n$, $p|\mathcal{V} \equiv 0 \implies p(X) = 0$. Note that if β has a representing measure μ, then β is weakly consistent; for, if $p \in \mathcal{P}_n$ and $p|_\mathcal{V} \equiv 0$, then since *supp* $\mu \subseteq \mathcal{V}$ (cf. Section 1), we have $p|_{supp\ \mu} \equiv 0$, whence [CuFi2, Proposition 3.1] implies $p(X) = 0$. Consider also the following property of β:

$$p \in \mathcal{P}_n, q \in \mathcal{P}, pq \in \mathcal{P}_{2n}, p(X) = 0 \Rightarrow \Lambda_\beta(pq) = 0 \tag{2.1}$$

The following result will be used in the proof of Theorem 2.8 and in Sections 4 and 5.

Proposition 2.1. *Let β, ϕ_β and $\mathcal{M}(n)(\beta)$ be as above. Then*
(i) β consistent $\Longrightarrow$ β weakly consistent $\Longrightarrow$ $\mathcal{M}(n)(\beta)$ recursively generated.
(ii) β consistent $\Longrightarrow$ β satisfies (2.1) $\Longrightarrow$ $\mathcal{M}(n)(\beta)$ recursively generated.

Proof. (i) Suppose β is consistent. Let $p \in \mathcal{P}_n$ with $p|_{\mathcal{V}_\beta} \equiv 0$. For every $q \in \mathcal{P}_n$, $(pq)|_{\mathcal{V}_\beta} \equiv 0$, so consistency implies $\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = \Lambda(pq) = 0$. Thus $p(X) = \mathcal{M}(n)\hat{p} = 0$, whence β is weakly consistent. We next assume that β is weakly consistent and we show that $\mathcal{M}(n)$

is recursively generated. Let $p,\ q,\ pq \in \mathcal{P}_n$ and suppose $p(X) = 0$. Since $\mathcal{V} \subseteq \mathcal{Z}(p)$, then $p|_{\mathcal{V}} \equiv 0$, whence $pq|_{\mathcal{V}} \equiv 0$. Since $pq \in \mathcal{P}_n$ and β is weakly consistent, it follows that $(pq)(X) = 0$.

(ii) Suppose β is consistent. Let $p \in \mathcal{P}_n$ and let $q \in \mathcal{P}$, with $pq \in \mathcal{P}_{2n}$. If $p(X) = 0$, then clearly $\mathcal{V}_\beta \subseteq \mathcal{Z}(p)$, whence $(pq)|_{\mathcal{V}_\beta} \equiv 0$. Now, consistency implies that $\Lambda_\beta(pq) = 0$, so (2.1) holds. Assume now that (2.1) holds and suppose $p, q, pq \in \mathcal{P}_n$ with $p(X) = 0$. Now, for each $s \in \mathcal{P}_n$, $p(qs) \in \mathcal{P}_{2n}$, so

$$\langle \mathcal{M}(n)\widehat{pq}, \hat{s}\rangle = \Lambda_\beta((pq)s) = \Lambda_\beta(p(qs)) = 0 \text{ (by (2.1)).}$$

Thus $(pq)(X) = \mathcal{M}(n)\widehat{pq} = 0$, so $\mathcal{M}(n)$ is recursively generated. $\square$

Remark 2.2. For the case when $\mathcal{V} \equiv \mathcal{V}_\beta$ is finite and the elements of $\mathcal{V}$ can be computed exactly, we next describe an elementary procedure for determining whether or not β is consistent. Denote the distinct points of $\mathcal{V}$ as $\{w_j\}_{j=1}^m$. Let $W \equiv W_{2n}[\mathcal{V}_\beta]$ denote the matrix with m rows, with columns indexed by the monomials in $\mathcal{P}_{2n}$ (in degree-lexicographic order), and whose entry in row k, column x^i is w_k^i. Clearly, a polynomial $p(x) \equiv \sum_{|i|\le 2n} a_i x^i \in \mathcal{P}_{2n}$ satisfies $p|_{\mathcal{V}} \equiv 0$ if and only if $W\hat{p} = 0$. Using Gaussian elimination, we may row-reduce W so as to find a basis for $\ker W$, say $\{\hat{p}_1, ..., \hat{p}_s\}$. It follows that $\{p_1, ..., p_s\}$ is a basis for $\{p \in \mathcal{P}_{2n} : p|_{\mathcal{V}} \equiv 0\}$. Let $\hat{p}_j := (a_{ji})_{|i|\le 2n}$ $(1 \le j \le s)$. We now see that β is consistent if, and only if, for each j, $\Lambda_\beta(p_j) \equiv \sum_{|i|\le 2n} a_{ji}\beta_i = 0$.

As the following lemma will show, consistency is a very strong condition, already yielding an atomic measure (though one which may have some negative densities).

Lemma 2.3. *Let $\Lambda : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional and let $\mathcal{V} \subseteq \mathbb{R}^d$. The following statements are equivalent.*
(a) There exist $\alpha_1, ..., \alpha_m \in \mathbb{R}$ and there exist $w_1, ..., w_m \in \mathcal{V}$ such that $\Lambda(p) = \sum_{i=1}^m \alpha_i p(w_i)$ $(p \in \mathcal{P}_{2n})$.
(b) If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = 0$.

Proof. The implication (a) $\Rightarrow$ (b) is obvious. Therefore assume that (b) holds, and fix the basis of monomials x^i of $\mathcal{P}_{2n}$. For notational convenience, denote this basis by $t_1, \ldots, t_K$. Then b) is equivalent to

$$\text{(c)} \quad \text{For all } c_1, ..., c_K \in \mathbb{R}^K : \sum_{j=1}^K c_j t_j(w) = 0 \ \text{ (all } w \in \mathcal{V}) \Rightarrow \sum_{j=1}^K c_j \Lambda(t_j) = 0.$$

Using $\hat{c} := (c_1, \ldots, c_K)$, $t(w) := (t_1(w), \ldots, t_K(w))$, and $\hat{\Lambda} := (\Lambda(t_1), \ldots, \Lambda(t_K))$, (b) is thus equivalent to

$$\text{(d)} \quad \text{For all } \hat{c} \in \mathbb{R}^K : \hat{c} \perp t(w) \ \text{ (all } w \in \mathcal{V}) \Rightarrow \hat{c} \perp \hat{\Lambda}.$$

Recall that for subspaces $\mathcal{R}$ and $\mathcal{S}$ of $\mathbb{R}^K$, $\mathcal{R}^\perp \subseteq \mathcal{S}^\perp \Leftrightarrow \mathcal{S} \subseteq \mathcal{R}$. Hence $\hat{\Lambda}$ is in the $\mathbb{R}$-linear subspace of $\mathbb{R}^K$ spanned by $\{t(w) : w \in \mathcal{V}\}$. As such, this subspace has a basis of m $(\le K)$ vectors $t(w_1), ..., t(w_m)$, where $w_1, ..., w_m \in \mathcal{V}$. Hence there exist $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that

$\hat{\Lambda} = \sum_{i=1}^{m} \alpha_i t(w_i)$, or equivalently,

$$\Lambda(t_j) = \sum_{i=1}^{m} \alpha_i t_j(w_i) \ \ (1 \le j \le K).$$

This is a linear relation holding for a basis of $\mathcal{P}_{2n}$, hence it holds for all $p \in \mathcal{P}_{2n}$, that is,

$$p \in \mathcal{P}_{2n} \Rightarrow \Lambda(p) = \sum_{i=1}^{m} \alpha_i p(w_i).$$

□

Remark 2.4. (i)If Λ is the Riesz functional Λ_β corresponding to $\beta \equiv \beta^{(2n)}$, then Lemma 2.3(b) is the consistency condition (1.6).
(ii) In the sequel we say that a linear functional $\Lambda : \mathcal{P}_{2n} \to \mathbb{R}$ is *square positive* if $\Lambda(p^2) \ge 0$ $(p \in \mathcal{P}_n)$. If $\mathcal{M}(n)$ denotes the moment matrix corresponding to $\beta_i := \Lambda(x^i)$ $(|i| \le 2n)$, then Λ is square positive if and only if $\mathcal{M}(n)$ is positive semidefinite. We note that in the proof of Lemma 2.3(b) we did not assume the square positivity of Λ.

When $\Lambda = \Lambda_\beta$, $m = rank\ \mathcal{M}(n)$, and $\{w_1, ..., w_m\} \subseteq \mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n))$, we next show that in the representation of Lemma 2.3(a), square positivity of Λ is equivalent to positivity of the α_i's; we have noted above that Λ is square positive if and only if $\mathcal{M}(n)$ is positive semidefinite.

Lemma 2.5. *Let $\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}$ be given by*

$$\Lambda(p) := \sum_{i=1}^{m} \alpha_i p(w_i) \ \ (p \in \mathcal{P}_{2n}),$$

with $m = rank\ \mathcal{M}(n)$ and $\{w_1, ..., w_m\} \subseteq \mathcal{V}_\beta$. The following statements are equivalent:
(i) $\alpha_i > 0$ (all $i = 1, ..., m$);
(ii) Λ is square positive.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. Conversely, assume that Λ is square positive, i.e., $\mathcal{M}(n)$ is positive semi-definite. Let $t_1, ..., t_N$ be the basis of monomials in $\mathcal{P}_n$ in degree-lexicographic order, so that the (j,k)-entry of $\mathcal{M}(n)$ is $\Lambda(t_j t_k)$. It follows that $\mathcal{M}(n)$ can be decomposed as

$$\mathcal{M}(n) = W_n^T \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_m \end{pmatrix} W_n, \tag{2.2}$$

where W_n is the $m \times N$ matrix with rows $t(w_i) \equiv (t_1(w_i), ..., t_N(w_i))$ $(1 \le i \le m)$. Since $rank\ \mathcal{M}(n) = m$, (2.2) implies that $rank\ W_n = m$, so the columns of W_n span $\mathbb{R}^m$. This implies that there exist polynomials $\ell_i \in \mathcal{P}_n$ satisfying $\ell_i(w_j) = \delta_{ij}$ $(1 \le i, j \le m)$, where δ_{ij} denotes the Kronecker symbol. Now, $\alpha_i = \Lambda(\ell_i^2) = \left\langle \mathcal{M}(n)\hat{\ell}_i, \hat{\ell}_i \right\rangle \ge 0$ (since

$\mathcal{M}(n) \geq 0$). Finally, no α_i can be zero, because otherwise (2.2) implies $rank\ \mathcal{M}(n) < m$, a contradiction. $\square$

Remark 2.6. A decomposition similar to (2.2) was used by Laurent [Lau2] in a study of the full moment problem for $\beta^{(\infty)}$ in the case when $card\ \mathcal{V}(\mathcal{M}(\infty)) < +\infty$.

Assume that $\beta \equiv \beta^{(2n)}$ is extremal, i.e., $r := rank\ \mathcal{M}(n)$ and $v := card\ \mathcal{V}_\beta$ satisfy $r = v$. Let $\mathcal{V} \equiv \{w_1, ..., w_r\}$ denote the distinct points of $\mathcal{V}_\beta$. If μ is a representing measure for β, then $supp\ \mu \subseteq \mathcal{V}$ and $r \leq card\ supp\ \mu \leq v$ (cf. Section 1), so the extremal hypothesis $r = v$ implies that $supp\ \mu = \mathcal{V}$. Thus μ is necessarily is of the form

$$\mu = \sum_{i=1}^{r} \rho_i \delta_{w_i}. \tag{2.3}$$

We next establish a criterion which allows us to compute the densities ρ_i. Let $p_1, ..., p_r$ be polynomials in $\mathcal{P}_n$ such that $\mathcal{B} \equiv \{p_1(X), ..., p_r(X)\}$ is a basis for the column space of $\mathcal{M}(n)$, and set

$$V \equiv V_{\mathcal{B}}[\mathcal{V}] := \begin{pmatrix} p_1(w_1) & . & . & . & p_1(w_r) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ p_r(w_1) & . & . & . & p_r(w_r) \end{pmatrix}.$$

Note that for a representing measure μ as in (2.3),

$$V(\rho_1, ..., \rho_r)^T = (\int p_1 d\mu_{\mathcal{B}}, ..., \int p_r d\mu_{\mathcal{B}})^T = (\Lambda_\beta(p_1), ..., \Lambda_\beta(p_r))^T. \tag{2.4}$$

Lemma 2.7. *The following are equivalent for β extremal:*
i) β is weakly consistent, i.e., $p \in \mathcal{P}_n$, $p|_{\mathcal{V}} \equiv 0 \Longrightarrow p(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$;
ii) For any basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$, V is invertible;
iii) There exists a basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$ such that V is invertible.

Proof. Let $\mathcal{B}$ (as above) be a basis for $\mathcal{C}_{\mathcal{M}(n)}$. V is singular if and only if there exist scalars $\alpha_1, ..., \alpha_r$, not all 0, such that $\alpha_1 p_1(w_i) + \cdots \alpha_r p_r(w_i) = 0$ $(1 \leq i \leq r)$. Equivalently, the polynomial $p \in \mathcal{P}_n$ defined by $p := \alpha_1 p_1 + \cdots + \alpha_r p_r$ satisfies $p|_{\mathcal{V}} \equiv 0$. Since $\mathcal{B}$ is a basis, it follows that $p(X) \equiv \alpha_1 p_1(X) + \cdots + \alpha_r p_r(X) \neq 0$, so β is not weakly consistent. Conversely, suppose β is not weakly consistent, i.e., there exists $q \in \mathcal{P}_n$ with $q|_{\mathcal{V}} \equiv 0$ and $q(X) \neq 0$ in $\mathcal{C}_{\mathcal{M}(n)}$. Since $\mathcal{B}$ is a basis, there exist scalars $a_1, ..., a_r$, not all 0, such that $q(X) = \sum_{i=1}^{r} a_i p_i(X)$, and since ϕ_β is well-defined, we may assume that $q = \sum_{i=1}^{r} a_i p_i$. Now $q|_{\mathcal{V}} \equiv 0$ implies that $\sum_{i=1}^{r} a_i p_i(w_j) = 0$ $(1 \leq j \leq r)$, whence V is singular. $\square$

Suppose now that β is extremal and let $\mathcal{B}$ be any basis for $\mathcal{C}_{\mathcal{M}(n)}$; thus there exist polynomials $p_1, ..., p_r \in \mathcal{P}_n$ such that $\mathcal{B} = \{p_1(X), ..., p_r(X)\}$. If β is weakly consistent, then V is invertible, and we let $\mu_{\mathcal{B}}$ denote the signed measure defined by (2.3) and

$$(\rho_1, ..., \rho_r)^T = V^{-1}(\Lambda_\beta(p_1), ..., \Lambda_\beta(p_r))^T. \tag{2.5}$$

In the sequel, we say that a signed Borel measure ν is *interpolating* for β if $\int p d\nu = \Lambda_\beta(p)$ $(p \in \mathcal{P}_{2n})$.

The main result of this section, which follows, includes a proof of Theorem 1.3.

Theorem 2.8. *For $\beta \equiv \beta^{(2n)}$ extremal, the following are equivalent:*
(i) β has a representing measure;
(ii) β has a unique representing measure, which is rank $\mathcal{M}(n)$-atomic;
(iii) For some (respectively, for every) basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$, $V \equiv V_{\mathcal{B}}[\mathcal{V}]$ is invertible and $\mu_{\mathcal{B}}$ is a representing measure for β;
(iv) $\mathcal{M}(n) \geq 0$ and for some (respectively, for every) basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$, $V \equiv V_{\mathcal{B}}[\mathcal{V}]$ is invertible and $\mu_{\mathcal{B}}$ is an interpolating measure for β;
(v) β is consistent and $\mathcal{M}(n) \geq 0$;
(vi) $\mathcal{M}(n) \geq 0$ has a flat extension $\mathcal{M}(n+1)$;
(vii) $\mathcal{M}(n) \geq 0$ has a unique flat extension $\mathcal{M}(n+1)$.

Proof. Let $r := rank\ \mathcal{M}(n)$ and $v := card\ \mathcal{V}_\beta$, so that $r = v$. The implications (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (v) are clear, so it suffices to prove (v) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii), (iii) $\Longleftrightarrow$ (iv), and (ii) $\Longrightarrow$ (vii) $\Longrightarrow$ (vi) $\Longrightarrow$ (i) ($\Longleftrightarrow$ (ii)) . We begin with (v) $\Longrightarrow$ (iii). Assume that β is consistent and $\mathcal{M}(n) \geq 0$. Since β is consistent, Lemma 2.3 implies that β has a signed, m-atomic representing measure μ with *supp* $\mu \subset \mathcal{V}_\beta$, so that $m \leq v$ (cf. Remark 2.4). In the corresponding decomposition (2.2), considerations of rank show that $m \geq r$, whence $m = r$. Lemma 2.5 now implies the μ is positive, so that μ is an r-atomic representing measure for β. Since β is consistent, Proposition 2.1 implies that β is weakly consistent, whence $V \equiv V_{\mathcal{B}}[\mathcal{V}]$ is invertible for any column basis $\mathcal{B}$ by Lemma 2.7. Now, (2.4) and (2.5) show that $\mu = \mu_{\mathcal{B}}$, so $\mu_{\mathcal{B}}$ is a representing measure; thus, (iii) holds. Now assume (iii) holds, so that for any column basis $\mathcal{B}$, $V_{\mathcal{B}}$ is invertible and $\mu_{\mathcal{B}}$ is a representing measure. If μ is any representing measure, then it follows as above (from (2.4) and (2.5)) that $\mu = \mu_{\mathcal{B}}$, whence (ii) holds. This completes the equivalence of (i), (ii), (iii) and (v).

Now recall that β has a *rank* $\mathcal{M}(n)$-atomic representing measure if and only if $\mathcal{M}(n)$ is positive and admits a flat extension $\mathcal{M}(n+1)$ [CF2, Theorem 5.13], and clearly distinct flat extensions correspond to distinct *rank* $\mathcal{M}(n)$-atomic representing measures. Thus we have (ii) $\Rightarrow$ (vii) $\Rightarrow$ (vi) $\Rightarrow$ (i), and since (i) $\Longleftrightarrow$ (ii), the proof is complete. □

Remark 2.9. For a positive, extremal $\mathcal{M}(n)$ for which the points of the variety are known exactly, Theorem 2.8 provides two ways to determine whether or not β has a representing measure. Following Theorem 2.8(iv) one can use the method of Remark 2.2 to determine whether or not β is consistent. Alternatively, one can select any basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(n)}$ and check whether $V \equiv V_{\mathcal{B}}$. is invertible. If V is not invertible, there is no representing measure. If V is invertible, then $\mu_{\mathcal{B}}$ automatically interpolates all moments up to degree n, so β has a representing measure if and only if the densities in (2.5) are positive and $\mu_{\mathcal{B}}$ interpolates all moments of degrees $n+1, n+2, ..., 2n$. In a given numerical problem, one approach or the other may be easier to implement, depending on the size of n and the value of *rank* $\mathcal{M}(n)$.

3. Real ideals and moment matrices

In this section we discuss some connections between polynomial ideals and consistency of Riesz functionals that we will utilize in Sections 4 and 5. If $\beta \equiv \beta^{(2n)}$ has a representing measure μ, then the Riesz functional $\Lambda \equiv \Lambda_\beta$ is *square positive* (equivalently, $\mathcal{M}(n)(\beta)$ is positive semidefinite, cf. Remark 2.4(ii)). If we assume, in addition, that all moments $\int_{\mathbb{R}^d} x^i d\mu$ $(i \in \mathbb{Z}_+^d)$ are convergent, then we can extend Λ to $\mathcal{P}$ by letting $\Lambda(x^i) := \int_{\mathbb{R}^d} x^i d\mu, \quad i \in \mathbb{Z}_+^d$, thus obtaining a square positive functional over $\mathcal{P}$ (e.g., if μ is an m-atomic measure with support $\{w_1, ..., w_m\} \subseteq \mathbb{R}^d$, then $\Lambda(p) = \sum_{i=1}^m p(w_i)\mu(\{w_i\})$ for all polynomials p). If Λ_β does extend to a square positive linear functional Λ on $\mathcal{P}$, then, as shown in [Moe1], the set

$$\mathcal{I} := \{p \in \mathcal{P} : \Lambda(p^2) = 0\}$$

is a *real ideal*, i.e., it is an ideal ($p_1, p_2 \in \mathcal{I} \Rightarrow p_1 + p_2 \in \mathcal{I}$ and $p \in \mathcal{I},\ q \in \mathcal{P} \Rightarrow pq \in \mathcal{I}$) and satisfies one of the following two equivalent conditions:

(i) For $s \in \mathbb{Z}_+, p_1, \dots, p_s \in \mathcal{P} : \sum_{i=1}^s p_i^2 \in \mathcal{I} \Rightarrow \{p_1, \dots, p_s\} \subseteq \mathcal{I}$;
(ii) There exists $G \subseteq \mathbb{R}^d$ such that for all $p \in \mathcal{P} : p|_G \equiv 0 \Rightarrow p \in \mathcal{I}$.

By contrast, we will show in Section 5 that when β has a representing measure, the ideal in $\mathcal{P}$ generated by $\{p \in \mathcal{P}_n : \Lambda_\beta(p^2) = 0\}$ is not necessarily a real ideal.

If $\mathcal{I}$ is a real ideal, then one may take for G the *real variety*

$$V_{\mathbb{R}}(\mathcal{I}) := \{w \in \mathbb{R}^d : f(w) = 0 \quad (\text{all } f \in \mathcal{I})\}.$$

But one may also take for G any subset of $V_{\mathbb{R}}(\mathcal{I})$ containing sufficiently many points, such that

$$p \in \mathcal{P},\ p|_G \equiv 0 \Rightarrow p|_{V_{\mathbb{R}}(\mathcal{I})} \equiv 0.$$

For instance, if the real variety is a (real) line, one may take for G a subset of infinitely many points on that line. On the other hand, if $V_{\mathbb{R}}(\mathcal{I})$ is a finite set of points, then necessarily $G = V_{\mathbb{R}}(\mathcal{I})$. (We note that in the full moment problem for $\beta \equiv \beta^{(\infty)}$, M. Laurent [Lau2] independently showed that $\mathcal{J} := \{p \in \mathcal{P} : M(\infty)\hat{p} = 0\}$ is a *radical* ideal; equivalently, $p \in \mathcal{J} \Leftrightarrow p^2 \in \mathcal{J}$.)

If $\mathcal{I}$ is an ideal in $\mathcal{P}$, its subset $\mathcal{I}_k := \mathcal{I} \cap \mathcal{P}_k$ is an $\mathbb{R}$-vector subspace of $\mathcal{P}_k$. One can then introduce the *Hilbert function* of $\mathcal{I}$ by

$$H_{\mathcal{I}}(k) := \dim \mathcal{P}_k - \dim \mathcal{I}_k, \quad k \in \mathbb{Z}_+;$$

in [CLO] this is called the *affine Hilbert function*. As shown for instance in [CLO], both $k \mapsto \dim \mathcal{I}_k$ and $k \mapsto H_{\mathcal{I}}(k)$ are nondecreasing functions, and for sufficiently large k, say $k \geq k_0$, $H_{\mathcal{I}}(k)$ becomes a polynomial in k, the so-called *Hilbert polynomial of* $\mathcal{I}$, whose degree equals the *dimension* of $\mathcal{I}$.

Example 3.1. Let $G \equiv \{w_1, \dots, w_m\} \subseteq \mathbb{R}^d$. Then $\mathcal{I} := \{f \in \mathcal{P} : f|_G \equiv 0\}$ is a real ideal with $V_{\mathbb{R}}(\mathcal{I}) = G$. Let $t_1, t_2, t_3, \dots$ denote the monomials x^i in degree-lexicographic order, so that for each $k \in \mathbb{Z}_+$, $t_1, \dots, t_K$ (with $K := \dim \mathcal{P}_k$) form a basis of the $\mathbb{R}$-vector space

$\mathcal{P}_k$. For $p \in \mathcal{P}_k$, $p \equiv \sum_{i=1}^{K} a_i t_i$, let $\hat{p} := (a_1, \ldots, a_K)$ (the coefficient vector of p). Then $p(x)$ can be written as

$$p(x) = \langle \hat{p}, t(x) \rangle ,$$

where $t(x) := (t_1(x), ..., t_K(x))$, so

$$p \in \mathcal{I} \cap \mathcal{P}_k \Leftrightarrow \hat{p} \perp t(w_i), \ i = 1, \ldots, m.$$

Arranging the rows $t(w_i)$ $(= (t_1(w_i), \ldots, t_K(w_i)))$ in a matrix

$$W_k \equiv W_k[G] := (t_j(w_i))_{i=1,\ldots,m, \ j=1,\ldots,K},$$

one gets $p \in \mathcal{I} \cap \mathcal{P}_k \Leftrightarrow \hat{p} \in \ker \ W_k$, whence $\dim \mathcal{I}_k + rank \ W_k = \dim \mathcal{P}_k$, or using the Hilbert function,

$$H_{\mathcal{I}}(k) = rank \ W_k, \quad k \in \mathbb{Z}_+ \ .$$

By construction, W_k is a submatrix of W_{k+1}. Hence $rank \ W_k \leq rank \ W_{k+1}$, reflecting the fact that the Hilbert function increases. If, for a given k, the rank of W_k is less than m, then one row of W_k, say the last one, depends on the others. This means that every polynomial which vanishes in $w_1, \ldots, w_{m-1}$ also vanishes in w_m. Using Lagrange interpolation polynomials, we see that for all sufficiently large k this cannot happen. Hence $rank \ W_k = m$ for all sufficiently large k. This m is the constant (degree-0) polynomial in k which coincides with $H_{\mathcal{I}}(k)$ for all $k \geq k_0$; hence, $\mathcal{I}$ is a zero dimensional ideal. □

We next present some ideal-theoretic necessary conditions for weak consistency. Given $\beta \equiv \beta^{(2n)}$, let $\mathcal{V} \equiv \mathcal{V}_\beta$ (or $\mathcal{V}(\mathcal{M}(n))$). One can then define the ideal

$$\mathcal{I}(\mathcal{V}) := \{p \in \mathcal{P} : p|_{\mathcal{V}} \equiv 0\}. \tag{3.1}$$

Since $\mathcal{V}$ is a set of real points, $\mathcal{I}(\mathcal{V})$ is a real ideal, which we will call the *real ideal* of β. Let

$$\mathcal{N}_n := \{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\} \tag{3.2}$$

If $p \in \mathcal{P}_n$ and $\mathcal{M}(n)\hat{p} = 0$, then $p|_{\mathcal{V}} \equiv 0$ by the definition of $\mathcal{V}$. Hence $p \in \mathcal{I}(\mathcal{V})$, so we always have $\mathcal{N}_n \subseteq \mathcal{I}(\mathcal{V})$.

Proposition 3.2. *Suppose $\beta \equiv \beta^{(2n)}$ is weakly consistent.*
(i) $\mathcal{N}_n = \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_n$.
(ii) If $t_1, ..., t_N$ denote the monomials $x^i \in \mathcal{P}_n$ in degree-lexicographic order, then the row vectors of $\mathcal{M}(n)$ and the row vectors $\{t(w) := (t_1(w), ..., t_N(w)) : w \in \mathcal{V}\}$, span the same subspace of $\mathbb{R}^N$; in particular, rank $\mathcal{M}(n) = H_{\mathcal{I}(\mathcal{V})}(n)$.

Proof. (i) We have $\mathcal{N}_n \subseteq \mathcal{I}(\mathcal{V})$, and the reverse inclusion follows directly from the definition of weak consistency.
(ii) Using (i) and proceeding as in Example 3.1, we see that

$$\begin{aligned} \hat{p} &\in \ker \ \mathcal{M}(n) \Leftrightarrow p \in \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_n \\ &\Leftrightarrow \hat{p} \perp t(w) \ \text{ (all } w \in \mathcal{V}). \end{aligned}$$

This means that the rows of $\mathcal{M}(n)$ span the same space (namely, $\mathbb{R}^N \ominus \ker\ \mathcal{M}(n)$) as the rows $(t_1(w), \ldots, t_N(w))$, $w \in \mathcal{V}$. It also follows that

$$rank\ \mathcal{M}(n) = \dim\ \mathcal{P}_n - \dim\ \ker\ \mathcal{M}(n) = \dim\ \mathcal{P}_n - \dim\ \mathcal{I}(\mathcal{V}) \cap \mathcal{P}_n = H_{\mathcal{I}(\mathcal{V})}(n).$$

$\square$

Given $\beta \equiv \beta^{(2n)}$, let $\{p_1, ..., p_s\}$ denote a basis for $\mathcal{N}_n := \{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\}$. Denote by $\mathcal{J} \equiv \mathcal{J}_\beta$ the smallest ideal containing the polynomials $p_1, ..., p_s$. Since $\mathcal{V} \equiv \mathcal{V}_\beta$ is the set of all real common zeros of $p_1, ..., p_s$, we have $\mathcal{J} \subseteq \mathcal{I}(\mathcal{V})$, whence

$$\dim(\mathcal{J} \bigcap \mathcal{P}_k) \le \dim(\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_k) \ \ (k \ge 0). \tag{3.3}$$

If β is weakly consistent, then Lemma 3.2(i) implies equality in (3.2) when $k = 0, ..., n$. For general β, the consistency condition (1.6) can be rephrased in terms of $\mathcal{I}(\mathcal{V})$ as

$$p \in \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n} \Rightarrow \Lambda_\beta(p) = 0.$$

Now, since $\mathcal{J} \bigcap \mathcal{P}_{2n}$ is a subset of $\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}$, we can find

$$M := \dim(\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}) - \dim(\mathcal{J} \bigcap \mathcal{P}_{2n}) \ (H_{\mathcal{J}}$$

polynomials $h_1, ..., h_M \in \mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}$ enlarging a basis for $\mathcal{J} \bigcap \mathcal{P}_{2n}$ to a basis for $\mathcal{I}(\mathcal{V}) \bigcap \mathcal{P}_{2n}$. Then consistency can be rephrased again as

$$p \in \mathcal{J} \bigcap \mathcal{P}_{2n} \Longrightarrow \Lambda(p) = 0, \text{ and } \Lambda(h_i) = 0 \ \ (1 \le i \le M). \tag{3.4}$$

Note that if $f \in \mathcal{N}_n$ and $g \in \mathcal{P}_n$, then $p := fg \in \mathcal{J} \bigcap \mathcal{P}_{2n}$ and $\Lambda(p) = \left\langle \mathcal{M}(n)\hat{f}, \hat{g} \right\rangle = 0$. In Section 5 we will identify situations in which $p \in \mathcal{J} \bigcap \mathcal{P}_{2n}$ always implies $\Lambda(p) = 0$, so that consistency reduces to the test $\Lambda(h_i) = 0$ $(1 \le i \le M)$.

Suppose $\mathcal{M}(n)$ admits a positive extension $\mathcal{M}(n+1)$. If $f \in \mathcal{P}_n$ and $f(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, then $f(X) = 0$ in $\mathcal{C}_{\mathcal{M}(n+1)}$, i.e., $\mathcal{N}_n \subseteq \mathcal{N}_{n+1}$ [Fia1]. If, further, $\mathcal{M}(n+1)$ is recursively generated, then it follows that $\mathcal{J}_\beta \bigcap \mathcal{P}_{n+1} \subseteq \mathcal{N}_{n+1}$. Motivated by [Moe2], we say that $\mathcal{M}(n+1)$ is a *tight* extension of $\mathcal{M}(n)$ if $\mathcal{N}_{n+1} = \mathcal{J}_\beta \bigcap \mathcal{P}_{n+1}$. ([Moe2] discusses "tight extensions" of linear functionals on $\mathcal{P}_n$.) Examination of the proofs of [CuFi7], [CuFi8], [CuFi10], and [Fia2] reveals that in each extremal case studied therein, $\mathcal{M}(n)$ admits a tight flat extension $\mathcal{M}(n+1)$. (Conversely, in [Fia4] it is shown that if $\mathcal{M}(n) \ge 0$ admits a tight flat extension, then $\mathcal{M}(n)$ is extremal.) Despite these results, we will show in Section 5 (Proposition 5.1) that there exists a positive, recursively generated, extremal $\mathcal{M}(3)$, which admits a flat extension, but has no tight flat extension.

4. Solution of the $\mathcal{M}(3)$ Moment Problem with $Y = X^3$ and $r \le v \le 7$

In this section (and the next) we return to the question as to whether a positive, extremal, recursively generated moment matrix has a representing measure (cf., Question 1.1). We also consider the extent to which recursiveness implies consistency. Our motivation is the observation that it is generally much easier to verify recursiveness than consistency. We examine these issues in detail for a planar moment matrix $\mathcal{M}(3)$ with $\mathcal{M}(3) \ge 0$ and $Y = X^3$

in $\mathcal{C}_{\mathcal{M}(3)}$. For the case $r \le v \le 7$ we show that recursiveness does imply consistency; the following result proves Theorem 1.5(i).

Theorem 4.1. *Let $d = 2$. Suppose $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$. If $\mathcal{M}(3)$ is positive, recursively generated, and $r \le v \le 7$, then $\beta^{(6)}$ has a representing measure; in particular, $\beta^{(6)}$ is consistent.*

Example 4.2. We illustrate Theorem 5.1 with the following moment matrix:

$$\mathcal{M}(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 200 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 200 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 200 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 200 & 5868 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 200 & 5868 & 386568 \\ 0 & 42 & 200 & 0 & 0 & 0 & 200 & 5868 & 386568 & 26992856 \end{pmatrix}.$$

$\mathcal{M}(3)$ is positive and recursively generated, with column basis $\mathcal{B}$ $:= \{1,\ X,\ Y,\ X^2, YX,\ Y^2,\ YX^2\}$, and column relations $Y = X^3$, $Y^2X = 208X - 282Y + 74YX^2$, and $Y^3 = 15392X - 20660Y + 5194YX^2$. A calculation shows that $\mathcal{V}_\beta$ consists of exactly 7 points in $\mathbb{R}^2$, $\{(x_i, x_i^3)\}_{i=1}^7$, with $x_1 = 0$, $x_2 \cong 8.36748$, $x_3 \cong 0.996357$, $x_4 \cong 1.7299$, and $x_{4+j} = -x_{j+1}$ $(1 \le j \le 3)$. Thus β is extremal, so Theorem 4.1 implies that β has a representing measure. Since β is extremal, Theorem 2.8 and (2.5) show that $V_{\mathcal{B}}$ is invertible, whence $\mu_{\mathcal{B}} \equiv \sum_{i=1}^7 \rho_i \delta_{(x_i, x_i^3)}$ has densities $\rho_1 \cong 0.331731$, $\rho_2 \cong 3.3378229 \times 10^{-10}$, $\rho_3 \cong 0.249980$, $\rho_4 \cong 0.08415439$, and $\rho_{4+j} = \rho_{j+1}$ $(1 \le j \le 3)$. $\square$

We begin the proof of Theorem 4.1 with some preliminary results. Recall from Section 2 the map $\phi_\beta : \mathcal{C}_{\mathcal{M}(n)} \to \mathcal{P}_n|_{\mathcal{V}_\beta}$, given by $p(X) \mapsto p|_{\mathcal{V}_\beta}$ $(p \in \mathcal{P}_n)$. As noted in Section 3, ϕ_β is 1-1 (and β is weakly consistent) if and only if $\mathcal{N}_n = \mathcal{K}_n$ (where $\mathcal{N}_n := \{p \in \mathcal{P}_n : p(X) = 0\}$ and $\mathcal{K}_n := \mathcal{I}(\mathcal{V}_\beta) \bigcap \mathcal{P}_n = \{p \in \mathcal{P}_n : p|_{\mathcal{V}_\beta} \equiv 0\}$); we always have $\mathcal{N}_n \subseteq \mathcal{K}_n$.

Lemma 4.3. *If $\mathcal{M}(n)(\beta)$ satisfies $r \le v$ and $\dim \mathcal{K}_n \le \dim \mathcal{P}_n - v$, then $\mathcal{M}(n)(\beta)$ is extremal and β is weakly consistent.*

Proof. We have $v \le \dim \mathcal{P}_n - \dim \mathcal{K}_n \le \dim \mathcal{P}_n - \dim \mathcal{N}_n = r \le v$. It follows that $r = v$ and $\mathcal{N}_n = \mathcal{K}_n$, so $\mathcal{M}(n)(\beta)$ is extremal and β is weakly consistent. $\square$

Lemma 4.4. *If $\mathcal{M}(3)(\beta)$ satisfies $Y = X^3$ and $r \le v = 7$, then β is weakly consistent.*

Proof. Suppose $p(x, y) \equiv c_1 + c_2x + c_3y + c_4x^2 + c_5yx + c_6y^2 + c_7x^3 + c_8yx^2 + c_9y^2x + c_{10}y^3$ is an element of $\mathcal{K}_3$, i.e., $p|_{\mathcal{V}_\beta} \equiv 0$. Denote the distinct points of $\mathcal{V}_\beta$ by $\{(x_i, y_i)\}_{i=1}^7$; since $y_i = x_i^3$ $(1 \le i \le 7)$, the x_i's are distinct. Consider the linear map $\Psi : \mathcal{K}_3 \to \mathbb{R}^3$ defined by $\Psi(p) = (c_7, c_9, c_{10})$. We claim that Ψ is 1-1; for, suppose $c_7 = c_9 = c_{10} = 0$ and let $f(x) := p(x, x^3) \equiv c_1 + c_2x + c_3x^3 + c_4x^2 + c_5x^4 + c_6x^6 + c_8x^5$. Since f has the seven distinct roots $\{x_i\}_{i=1}^7$, it follows that $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_8 = 0$, whence $p \equiv 0$ and Ψ

is 1-1. Thus $\dim \mathcal{K}_3 \leq \dim \mathbb{R}^3 = 3 = 10 - 7 = \dim \mathcal{P}_3 - v$, so Lemma 4.3 implies that β is weakly consistent. $\square$

Proposition 4.5. *Let $\mathcal{M}(3)(\beta) \geq 0$, with $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$. If $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2\}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$ and $v = r$, then $\beta^{(6)}$ has a representing measure.*

Proof. Let $\mathcal{V} \equiv \mathcal{V}_\beta$; Lemmas 4.4 and 2.7 imply that $V_\mathcal{B}[\mathcal{V}]$ is invertible, so, from Theorem 2.8 (iii) $\iff$ (iv), to prove that $\mu_\mathcal{B}$ is a representing measure, it suffices to prove that $\mu_\mathcal{B}$ is interpolating for $\beta^{(6)}$, i.e., $\beta_{ij} = \int y^j x^i \, d\mu_\mathcal{B}$ $(i,j \geq 0,\ i+j \leq 6)$. Relation (2.4) shows that $\mu_\mathcal{B}$ interpolates the moments corresponding to elements of $\mathcal{B}$, namely β_{00}, β_{10}, β_{01}, β_{20}, β_{11}, β_{02}, and β_{21}. From the hypothesis, we have

$$Y = X^3. \tag{4.1}$$

Also, there exist α, $\gamma \in \mathbb{R}$ and $p, q \in \mathcal{P}_2$, such that we have column relations

$$Y^2X = \alpha YX^2 + p(X,Y), \tag{4.2}$$

and

$$Y^3 = \gamma YX^2 + q(X,Y). \tag{4.3}$$

In the sequel, for $f, g \in \mathcal{P}_3$ we denote $\langle \mathcal{M}(3)\hat{f}, \hat{g} \rangle$ by $\langle f(X,Y), g(X,Y) \rangle$. In *supp* $\mu_\mathcal{B}$ we have $y = x^3$, so $\int x^3 d\mu_\mathcal{B} = \int y d\mu_\mathcal{B} = \beta_{01} = \langle Y, 1 \rangle = \left\langle X^3, 1 \right\rangle = \beta_{30}$ (by (4.1)); thus $\int x^3 d\mu_\mathcal{B} = \beta_{30}$. Similarly,

$$\begin{aligned} \int y^2 x \, d\mu_\mathcal{B} &= \int (\alpha y x^2 + p(x,y)) d\mu_\mathcal{B} = \alpha \beta_{21} + \Lambda_\beta(p) \\ &= \left\langle \alpha YX^2 + p(X,Y), 1 \right\rangle = \left\langle Y^2X, 1 \right\rangle = \beta_{12} \end{aligned}$$

(by (2.5) and (4.2)), and

$$\begin{aligned} \int y^3 d\mu_\mathcal{B} &= \int (\gamma y x^2 + q(x,y)) d\mu_\mathcal{B} = \gamma \beta_{21} + \Lambda_\beta(q) \\ &= \left\langle \gamma YX^2 + q(X,Y), 1 \right\rangle = \left\langle Y^3, 1 \right\rangle = \beta_{03} \end{aligned}$$

(by (2.5) and (4.3)). Thus, $\mu_\mathcal{B}$ interpolates all moments up to degree 3.

The proof now continues inductively, using the results for all degrees $< k$ to obtain the result for degree k, and using (4.1)-(4.3) in successive rows of $\mathcal{M}(3)$. For example, to obtain results for degree 4, we start with the relations $y = x^3$, $y^2x = \alpha y x^2 + p(x,y)$, and $y^3 = \gamma y x^2 + q(x,y)$, valid in $\mathcal{V}_\beta$, to get new relations of degree 4 in $\mathcal{V}_\beta$: $x^4 = yx$, $yx^3 = y^2$, $y^2x^2 = \alpha y x^3 + xp(x,y)$, $y^3x = \gamma y x^3 + xq(x,y)$, $y^4 = \gamma y^2 x^2 + yq(x,y)$. Using (4.1)-(4.3) and the results for degrees 1, 2 and 3, we may now successively integrate these new relations to obtain $\beta_{i+j} = \int y^j x^i \, d\mu_\mathcal{B}$ $(i,j \geq 0,\ i+j = 4)$. For example, $\int x^4 d\mu_\mathcal{B} = \int yx d\mu_\mathcal{B} = \beta_{11} = \langle Y, X \rangle = \left\langle X^3, X \right\rangle = \beta_{40}$; $\int x^3 y d\mu_\mathcal{B} = \int y^2 d\mu_\mathcal{B} = \beta_{02} = \langle Y, Y \rangle = \left\langle X^3, Y \right\rangle = \beta_{31}$; $\int x^2 y^2 d\mu_\mathcal{B} = \int (\alpha y x^3 + xp(x,y)) d\mu_\mathcal{B} = \alpha \Lambda_\beta(yx^3) + \Lambda_\beta(xp(x,y)) = \alpha \left\langle X^2Y, X \right\rangle + \langle p(X,Y), X \rangle$ $= \left\langle \alpha X^2 Y + p(X,Y), X \right\rangle = \langle XY^2, X \rangle = \beta_{22}$, etc. Degrees 5 and 6 are treated similarly. $\square$

Proof of Theorem 4.1. The results in [?], [CF5] and [CF7] show that if $\mathcal{M}(n)$ is positive, recursively generated, satisfies $r \le v$ and has a column relation of degree one or two, then $\beta^{(2n)}$ admits a representing measure. We may thus assume that $\mathcal{M}(2)$ is positive and invertible; indeed, positivity in $\mathcal{M}(3)$ implies that any dependence relation in the columns of $\mathcal{M}(2)$ extends to the columns of $\mathcal{M}(3)$ [?]. In particular, we may assume in the sequel that a basis $\mathcal{B}$ of $\mathcal{C}_{\mathcal{M}(3)}$ includes $\{1, X, Y, X^2, YX, Y^2\}$, whence $r \ge 6$. If $r = 6$, then $\mathcal{M}(3)$ is flat, i.e., *rank* $\mathcal{M}(3) =$ *rank* $\mathcal{M}(2)$, so [] implies that β has a unique, 6-atomic representing measure. We may thus assume that $r = v = 7$.

Lemma 4.4 implies that β is weakly consistent. From Lemma 2.7, we may thus form $\mu_{\mathcal{B}}$, and from Theorem 2.8(vii), it suffices to show that $\mu_{\mathcal{B}}$ is interpolating for β. The proof of Proposition 4.5 shows that this is the case if $\mathcal{B} = \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ YX^2\}$. This proof shows, more generally, that $\mu_{\mathcal{B}}$ is interpolating if $\mathcal{B}$ contains $\{1,\ X,\ Y,\ X^2,\ YX,\ Y^2\}$ and there exist column relations of the form (4.2) and (4.3).

We consider next the case when $\mathcal{B} = \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ Y^2X\}$, with column relations $YX^2 = u(X,Y) + \gamma Y^2X$ $(\gamma \in \mathbb{R},\ \deg\ u \le 2)$ and $Y^3 = \delta Y^2X + t(X,Y)$ $(\delta \in \mathbb{R}, \deg\ t \le 2)$. Let $h(x,y) := x^2y - u(x,y) - \gamma xy^2$, so that $h(X,Y) = 0$ and $\mathcal{V}_\beta \subseteq \{(x,y) \in \mathbb{R}^2 : y = x^3 \text{ and } h(x,y) = 0\}$. If $\gamma = 0$, then $h(x,y) = 0$ has at most 6 real roots of the form (x,x^3), contradicting $r = v = 7$. Thus $\gamma \ne 0$, and we may derive a system as in (4.2)-(4.3); indeed, $Y^2X = \frac{1}{\gamma}YX^2 - \frac{1}{\gamma}u(X,Y)$ and $Y^3 = \frac{\delta}{\gamma}YX^2 + (t - \frac{\delta}{\gamma}u)(X,Y)$. Using this system, we may now proceed as in the proof of Proposition 4.5 to conclude that $\mu_{\mathcal{B}}$ is interpolating. Finally, we consider the case $\mathcal{B} = \{1,\ X,\ Y,\ X^2,\ YX,\ Y^2,\ Y^3\}$, with relations

$$YX^2 = s(X,Y) + \delta Y^3 \quad (\delta \in \mathbb{R},\ \deg s \le 2) \tag{4.4}$$

and

$$Y^2X = t(X,Y) + \epsilon Y^3 \quad (\epsilon \in \mathbb{R},\ \deg t \le 2). \tag{4.5}$$

Since $h(x,y) := yx^2 - s(x,y)$ has at most 6 roots of the form (x,x^3), then $v = 7$ implies $\delta \ne 0$. We may now successively transform (4.4) and (4.5) into (4.2) and (4.3) and then apply the method of the proof of Proposition 4.5. $\square$

5. Solution of the $\mathcal{M}(3)$ Moment Problem with $Y = X^3$ and $r \le v = 8$

In this section we study the extremal moment problem for a moment matrix $\mathcal{M}(3)$ satisfying

$$\mathcal{M}(3) \ge 0, \mathcal{M}(2) > 0, Y = X^3 \text{ in } \mathcal{C}_{\mathcal{M}(3)}, \text{ and } r = v = 8. \tag{5.1}$$

In Proposition 5.1 we illustrate (5.1) with the first example of an extremal moment matrix $\mathcal{M}(n)$, which admits a representing measure, but for which (i) the ideal $\mathcal{J}_\beta$ corresponding to ker $\mathcal{M}(n)$ is *not* a real ideal, and (ii) the unique flat extension $\mathcal{M}(n+1)$ is *not* a tight flat extension. In Theorem 5.2 we resolve Question 1.1 in the negative, by constructing a moment matrix $\mathcal{M}(3)$ which satisfies (5.1), but is not consistent, and thus admits no representing measure. In Theorem 5.3 we provide a simplified consistency test for moment matrices satisfying (5.1), and in Theorem 5.6 we complete the analysis of the moment problem for $\mathcal{M}(3)$ with $Y = X^3$ and $r \le v = 8$.

We begin by introducing the objects that we will use in our examples. Let $f(x,y) := y-x^3$. Recall from Bezout's Theorem ([CLO, Theorem 8.7.10] that if deg $g=3$, then f and g have exactly 9 common zeros (counting multiplicity), including complex zeros and zeros at infinity. To construct a variety that will serve as $\mathcal{V}(\mathcal{M}(3))$ in Proposition 5.1 and Theorem 5.2, we first seek a polynomial $g \in \mathbb{R}[x,y]$ of degree 3 such that f and g have exactly 8 distinct common real affine zeros, one of which is a zero of multiplicity 2. For this, let $\ell_i(x,y)=0$ $(i=1,2,3)$ be lines in the plane such that ℓ_1 intersects $y=x^3$ in 3 distinct points $((x_i,y_i), 1 \le i \le 3)$, ℓ_2 intersects $y=x^3$ in 3 additional distinct points $((x_i,y_i), 4 \le i \le 6)$, and ℓ_3 intersects $y=x^3$ in 2 additional distinct points $((x_i,y_i), 7 \le i \le 8)$, such that ℓ_3 is the tangent line to $y=x^3$ at (x_8,y_8). Setting $g(x,y) := \ell_1(x,y)\ell_2(x,y)\ell_3(x,y)$, we have $\mathcal{V}((f,g)) = \{(x_i,y_i)\}_{i=1}^8$, and (x_8,y_8) is a common zero of f and g with multiplicity 2. Indeed, (x_8,y_8) is a multiple zero since $\ell_3(x,y)=0$ is a common tangent line for $f(x,y)=0$ and $g(x,y)=0$ at (x_8,y_8); equivalently, there exist $a,b \in \mathbb{R}$ such that the differential $D:\mathcal{P} \to \mathbb{R}$ defined by

$$D(p) := a\frac{\partial p}{\partial x}(x_8,y_8) + b\frac{\partial p}{\partial y}(x_8,y_8) \tag{5.2}$$

satisfies $D(f)=D(g)=0$ (cf. [CLO, Proposition 3.4.2], [MMM]). We next introduce some ideals which will be referenced in the sequel. Let $\mathcal{V} \equiv \mathcal{V}((f,g))$ $(=\{(x_i,y_i)\}_{i=1}^8)$ and set $\mathcal{A} := \mathcal{I}(\mathcal{V}) \equiv \{p \in \mathcal{P} : p|_{\mathcal{V}} \equiv 0\}$ and $\mathcal{D} := \{p \in \mathcal{A} : D(p)=0\}$; $\mathcal{A}$ is a real ideal (cf. Section 2), and $\mathcal{D}$ is an ideal (which contains f and g). For the last assertion, note that if $p \in \mathcal{D}$ and $q \in \mathcal{P}$, then $(pq)|_{\mathcal{V}} \equiv 0$ and $D(pq) = q(x_8,y_8)D(p) + p(x_8,y_8)D(q) = 0$ (since $D(p)=0$ and $p|_{\mathcal{V}} \equiv 0$).

As we show below, the conditions of (5.1) imply that $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$, so we will further require that the points of $\mathcal{V}$ are in "general position" relative to the monomials $1, x, y, x^2, yx, y^2, yx^2$ and y^2x, i.e., we will require that $V \equiv V_{\mathcal{B}}[\mathcal{V}]$ is invertible (cf. Lemma 2.7). Let $W \equiv W_{\mathcal{B}}[\mathcal{V}] := V^T$. Now, if $H(x,y)$ is any real-valued function defined on $\mathcal{V}$, then there exist scalars $\alpha_1, ..., \alpha_8 \in \mathbb{R}$ such that

$$H(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 yx + \alpha_6 y^2 + \alpha_7 yx^2 + \alpha_8 y^2 x \quad ((x,y) \in \mathcal{V});$$

indeed, $\alpha \equiv (\alpha_1, ..., \alpha_8)$ is uniquely determined from

$$\alpha^T = W^{-1}(H(x_1,y_1), .., H(x_8,y_8))^T. \tag{5.3}$$

In particular, there exist unique real numbers $a_1, ..., a_8$ such that

$$h(x,y) := y^2x^2 - (a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 yx + a_6 y^2 + a_7 yx^2 + a_8 y^2 x) \tag{5.4}$$

vanishes on $\mathcal{V}$.

For the sake of definiteness, let

$$
\begin{aligned}
&\ell_1(x,y) := y - 4x\\
((x_1,y_1) = (-2,-8), &(x_2,y_2) = (0,0), (x_3,y_3) = (2,8)),\\
\\
&\ell_2(x,y) := y - 4x + 3\\
((x_4,y_4) = (1,1), &(x_5,y_5) = (-\frac{1}{2} + \frac{\sqrt{13}}{2}, -5 + 2\sqrt{13}),\\
(x_6,y_6) = (-\frac{1}{2} &- \frac{\sqrt{13}}{2}, -5 - 2\sqrt{13})),\\
\\
&\ell_3(x,y) := y - \frac{3}{4}x + \frac{1}{4}\\
((x_7,y_7) = (-1,-1), &(x_8,y_8) = (\frac{1}{2}, \frac{1}{8})).
\end{aligned}
\tag{5.5}
$$

Then

$$
\begin{aligned}
4g(x,y) &= 4(y-4x)(y-4x+3)(y - \frac{3}{4}x + \frac{1}{4})\\
&= -48x^3 + 88yx^2 - 35y^2x + 4y^3 + 52x^2 - 65yx + 13y^2 - 12x + 3y.
\end{aligned}
\tag{5.6}
$$

A calculation shows that ℓ_3 is tangent to both f and g at (x_8, y_8); indeed, $D(f) = D(g) = 0$, where D is the functional given by (5.2) with $a = 1, b = \frac{3}{4}$. Further, $\det V = \frac{98415}{4}\sqrt{13}$ ($\neq 0$), so *rank* $V = 8$. Applying (5.3) with $H(x,y) = y^2x^2$, we see that in (5.4) we have

$$
h(x,y) = y^2x^2 + 6x - 14x^2 - \frac{11}{2}y + \frac{43}{2}yx - yx^2 - \frac{17}{2}y^2 + \frac{1}{2}y^2x,
\tag{5.7}
$$

and $h|_{\mathcal{V}} \equiv 0$. A calculation shows that $D(h) = -\frac{405}{128}$ ($\neq 0$), so $h \in (\mathcal{A} \bigcap \mathcal{P}_4) \setminus \mathcal{D}$.

Proposition 5.1. *Let* $\mu := \sum_{i=1}^{8} \delta_{(x_i,y_i)}$ *(with* (x_i,y_i) *from (5.5)) and let* $\mathcal{M}(3) := \mathcal{M}(3)[\mu]$. *Then* $\mathcal{M}(3)$ *satisfies (5.1) and has the following additional properties:*
(i) The ideal $\mathcal{J}_{\beta^{(6)}}$ *generated by* $\mathcal{N}_3 \equiv \{p \in \mathcal{P}_3 : \mathcal{M}(3)\hat{p} = 0\}$ *is not a real ideal;*
(ii) $\mathcal{M}(3)$ *has a flat extension, but* $\mathcal{M}(3)$ *does not admit a tight flat extension.*

Proof. A direct calculation using the points in (5.5) shows that $V_{\mathcal{B}}[\mathcal{V}]$ is invertible, so it follows as in the proof of Lemma 2.7 that $\mathcal{B}$ is independent in $\mathcal{C}_{\mathcal{M}(3)}$. Also, since μ is a representing measure for $\mathcal{M}(3)[\mu]$, *supp* $\mu = \mathcal{V}$, $f|_{\mathcal{V}} \equiv 0$ and $g|_{\mathcal{V}} \equiv 0$, we have $Y = X^3$ and $g(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$, whence $\mathcal{B}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$, *rank* $\mathcal{M}(3) = 8$, and $\mathcal{V}(\mathcal{M}(3)) = \mathcal{Z}(f) \bigcap \mathcal{Z}(g) = \mathcal{V}$. Thus, $\mathcal{M}(3)$ satisfies (5.1).

(i) Let $\mathcal{J} \equiv \mathcal{J}_{\beta^{(6)}}$ denote the ideal generated by $\{p \in \mathcal{P}_3 : \mathcal{M}(3)\hat{p} = 0\}$, so that $\mathcal{J} = (f, g)$. We claim that $\mathcal{J}$ is not a real ideal. For, otherwise, there would exist $G \subseteq \mathbb{R}^2$ such that for $p \in \mathcal{P}$, $p|_G \equiv 0 \iff p \in \mathcal{J}$ (cf. Section 2). In this case, since $f^2 + g^2 \in \mathcal{J}$, then $(f^2 + g^2)|_G \equiv 0$, whence $G \subseteq \mathcal{V}$. Recall that the function h given by (5.7) satisfies $h|_{\mathcal{V}} \equiv 0$

and $D(h) \neq 0$. Since $p \in \mathcal{J} \Rightarrow D(p) = 0$, we see that $h \notin \mathcal{J}$; but since $h|_{\mathcal{V}} \equiv 0$, then $h|_G \equiv 0$, contradicting the defining property of G. Thus, $\mathcal{J}$ is not a real ideal.

(ii) Since $\mathcal{M}(3)$ is extremal and has a representing measure (that is, μ), it has a unique flat extension $\mathcal{M}(4)$, namely $\mathcal{M}(4)[\mu]$. Since $h|_{\mathcal{V}} \equiv 0$, we have $h(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(4)}$ [CF2, Proposition 3.1], so $h \in \mathcal{N}_4$. Since we have shown in the proof of (i) that $h \notin \mathcal{J}$, we must have $\mathcal{N}_4 \nsubseteq \mathcal{J} \bigcap \mathcal{P}_4$, so $\mathcal{M}(4)$ is not a tight flat extension. $\square$

We next present an example of $\mathcal{M}(3)$ satisfying (5.1), but not consistent, so that $\beta^{(6)}$ has no representing measure; this provides a negative answer to Question 1.1. We define a linear functional $L : \mathcal{P}_6 \to \mathbb{R}$ by

$$L(p) := a_0 D(p) + \sum_{i=1}^{8} a_i p(x_i, y_i) \quad (p \in \mathcal{P}_6) \tag{5.8}$$

(with D and $\{(x_i, y_i)\}_{i=1}^8$ as defined just previous to Proposition 5.1, and $a_i \in \mathbb{R}$ $(0 \leq i \leq 8)$). Let $\beta^{(6)}$ be the sequence corresponding to L, i.e., $\beta_{ij} := L(x^i y^j)$ $(i, j \geq 0, i + j \leq 6)$. Let $M \equiv \mathcal{M}(3)$ be the corresponding moment matrix, which is real symmetric since

$$\left\langle M\widehat{x^i y^j}, \widehat{x^k y^\ell} \right\rangle = L(x^{i+k} y^{j+\ell}) = \left\langle M\widehat{x^k y^\ell}, \widehat{x^i y^j} \right\rangle.$$

Recall $f(x,y) := y - x^3$ and note that $f(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$. Indeed, for $p \in \mathcal{P}_3$,

$$\langle f(X,Y), \hat{p} \rangle = \left\langle \mathcal{M}(3)\hat{f}, \hat{p} \right\rangle = L(fp) = 0$$

(since $D(f) = 0 \Longrightarrow D(pf) = 0$ and since $f|_{\mathcal{V}} \equiv 0 \Longrightarrow fp|_{\mathcal{V}} \equiv 0$). Similarly, for g as defined earlier, since $D(g) = 0$ and $g|_{\mathcal{V}} \equiv 0$, we have $g(X,Y) = 0$. For the sake of definiteness, let $a_i := 1$ $(0 \leq i \leq 7)$.

Theorem 5.2. *There exists α $(\cong 6.97093)$ such that if $a_8 > \alpha$, then $\mathcal{M}(3)$ satisfies (5.1) (and is thus positive, recursively generated, and extremal), but $\beta^{(6)}$ has no representing measure. In particular, β is weakly consistent, but β is not consistent.*

Proof. Consider $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$. Since $Y = X^3$ and $g(X,Y) = 0$, $\mathcal{B}$ spans $\mathcal{C}_{\mathcal{M}(3)}$. It follows from Smul'jan's Theorem that $\mathcal{M}(3)$ is positive semi-definite if and only if $M_{\mathcal{B}}$, the compression of $\mathcal{M}(3)$ to rows and columns indexed by $\mathcal{B}$, is positive semi-definite. Calculating nested determinants, we see that $M_{\mathcal{B}}$ is positive definite if and only if $a_8 > \alpha$, where $\alpha := \frac{6012817451}{862617600}$. In this case, since $M_8 > 0$ and $f(X,Y) = 0 = g(X,Y)$, it follows that *rank* $\mathcal{M}(3) = 8$ and $\mathcal{V}(\mathcal{M}(3)) = \mathcal{Z}(f) \bigcap \mathcal{Z}(g) = \mathcal{V}$. In particular, $\mathcal{M}(3)$ satisfies (5.1) (and is thus also recursively generated). Further, β is weakly consistent (see the proof of Proposition 5.1, or use Lemma 5.4 below). We claim that $\beta^{(6)}$ is not consistent. Indeed, the Riesz functional for $\beta^{(6)}$ is L. The function h from (5.6) satisfies $h|_{\mathcal{V}} \equiv 0$ and $D(h) \neq 0$, whence $L(h) = D(h) \neq 0$. Now β is not consistent and thus has no representing measure. $\square$

In view of Theorem 2.8, the existence of a representing measure in the extremal moment problem (5.1) is equivalent to establishing that the Riesz functional Λ_β vanishes on a basis for $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$, and we will show below that dim $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) = 20$. The substance of the next result is that, following (3.4) and the remarks following (3.4), the test for consistency in (5.1) can be reduced to checking that $\Lambda_\beta(h) = 0$ for h given by (5.4).

Theorem 5.3. *Suppose $\mathcal{M}(3)$ satisfies (5.1), with $\mathcal{V}(\mathcal{M}(3)) = \{(x_i, y_i)\}_{i=1}^8$ and column basis $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$. Let h be as in (5.4). Then $\beta^{(6)}$ has a representing measure if and only if $\Lambda_\beta(h) = 0$.*

We require the following preliminary result.

Lemma 5.4. *If $Y = X^3$ in $\mathcal{C}_{\mathcal{M}(3)}$ and $r \le v = 8$, then β is weakly consistent.*

Proof. Let $\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(3))$ and for $f \in \mathcal{K}_3 := \mathcal{P}_3 \bigcap \mathcal{I}(\mathcal{V})$, write

$$(5.9) \qquad \begin{aligned} f(x, y) \;&\equiv\; a_1 + a_2x + a_3y + a_4x^2 + a_5yx + a_6y^2 \\ &\quad +a_7x^3 + a_8yx^2 + a_9y^2x + a_{10}y^3. \end{aligned}$$

Define a linear map $\Psi : \mathcal{K}_3 \to \mathbb{R}^2$ by $\Psi(f) := (a_7, a_{10})$. We claim that Ψ is 1-1. Suppose $a_7 = a_{10} = 0$ and define

$$p(x) := f(x, x^3) \equiv a_1 + a_2x + a_4x^2 + a_3x^3 + a_5x^4 + a_8x^5 + a_6x^6 + a_9x^7.$$

Since $\mathcal{V} \subseteq \mathcal{Z}(y - x^3)$, the eight points of $\mathcal{V}$ have distinct x-coordinates, and $f|_{\mathcal{V}} \equiv 0$, it follows that p has at least 8 distinct real roots. Since deg $p \le 7$, we must have $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_8 = a_9 = 0$, whence $f \equiv 0$, so Ψ is 1-1. Now $\dim \mathcal{K}_3 \le \dim \mathbb{R}^2 = 10 - 8 = \dim \mathcal{P}_3 - v$, so Lemma 4.3 implies that β is weakly consistent. $\square$

Proof of Theorem 5.3. If $\beta \equiv \beta^{(6)}$ has a representing measure, then β is consistent, and since $h \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$, it follows that $\Lambda_\beta(h) = 0$. For the converse, we suppose that $\Lambda_\beta(h) = 0$ and we will show that β is consistent, i.e., $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \subseteq \ker\ \Lambda_\beta$ (cf. Theorem 2.8). To this end, we first compute $\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}))$. Consider $W \equiv W_6[\mathcal{V}]$ (cf. Section 2); clearly, $p \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \iff \hat{p} \in \ker\ W$, so $\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})) = \dim\ \ker\ W = \dim\ \mathcal{P}_6 - rank\ W$. Lemma 5.4 shows that β is weakly consistent, so Lemma 2.7 implies that $W_{\mathcal{B}}[\mathcal{V}]$ $(\equiv V_{\mathcal{B}}[\mathcal{V}]^T)$ is invertible. Now $W_{\mathcal{B}}[\mathcal{V}]$ is the compression of W to columns indexed by the monomials corresponding to elements of $\mathcal{B}$, so $8 \ge$ row rank $W = rank\ W \ge rank\ W_{\mathcal{B}}[\mathcal{V}] = 8$, whence $rank\ W = 8$. Thus,

$$\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})) = \dim\ \mathcal{P}_6 - rank\ W = 28 - 8 = 20.$$

Let $f(x, y) := y - x^3$, so that $\mathcal{M}(3)\hat{f} = f(X, Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$. Also, there exist $a, b \in \mathbb{R}$ and $p \in \mathcal{P}_2$ such that $g(x, y) := y^3 + ayx^2 + by^2x + p(x, y)$ satisfies $\mathcal{M}(3)\hat{g} = g(X, Y) = 0$ in $\mathcal{C}_{\mathcal{M}(3)}$. Since $r = 8$, it follows that $\mathcal{V} = \mathcal{Z}(f) \bigcap \mathcal{Z}(g)$, and clearly $f, g \in \mathcal{I}(\mathcal{V})$. Now, if $s, t \in \mathcal{P}_3$, then $sf + tg \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$ and $\Lambda_\beta(sf + tg) = \left\langle \mathcal{M}(3)\hat{f}, \hat{s} \right\rangle + \langle \mathcal{M}(3)\hat{g}, \hat{t} \rangle = 0$, whence $sf + tg \in \ker\ \Lambda_\beta$ (see the remarks following (3.4)).

We next identify 19 linearly independent elements of $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})$ of the form $sf + tg$ $(s, t \in \mathcal{P}_3)$. Consider the following 20 polynomials:

$$
\begin{array}{cc}
f_1 := g \equiv y^3 + ayx^2 + by^2x + p(x,y) & f_2 := f \equiv y - x^3 \\
f_3 := xg \equiv y^3x + ayx^3 + by^2x^2 + xp(x,y) & f_4 := xf \equiv yx - x^4 \\
f_5 := yg \equiv y^4 + ay^2x^2 + by^3x + yp(x,y) & f_6 := yf \equiv y^2 - yx^3 \\
f_7 := x^2g \equiv y^3x^2 + ayx^4 + by^2x^3 + x^2p(x,y) & f_8 := x^2f \equiv yx^2 - x^5 \\
f_9 := yxg \equiv y^4x + ay^2x^3 + by^3x^2 + yxp(x,y) & f_{10} := yxf \equiv y^2x - yx^4 \\
f_{11} := y^2g \equiv y^5 + ay^3x^2 + by^4x + y^2p(x,y) & f_{12} := y^2f \equiv y^3 - y^2x^3 \\
f_{13} := x^3g \equiv y^3x^3 + ayx^5 + by^2x^4 + x^3p(x,y) & f_{14} := x^3f \equiv yx^3 - x^6 \\
f_{15} := yx^2g \equiv y^4x^2 + ay^2x^4 + by^3x^3 + yx^2p(x,y) & f_{16} := yx^2f \equiv y^2x^2 - yx^5 \\
f_{17} := y^2xg \equiv y^5x + ay^3x^3 + by^4x^2 + y^2xp(x,y) & f_{18} := y^2xf \equiv y^3x - y^2x^4 \\
f_{19} := y^3g \equiv y^6 + ay^4x^2 + by^5x + y^3p(x,y) & f_{20} := y^3f \equiv y^4 - y^3x^3.
\end{array}
$$

We assert that $\mathcal{F} := \{f_i\}_{i=1}^{19}$ is linearly independent in $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta$. For $1 \le k \le 19$, set $\mathcal{F}_k := \{f_i\}_{i=1}^{k}$. Proceeding inductively, let $2 \le k \le 19$ and assume that $\mathcal{F}_{k-1}$ is linearly independent. Observe that, except when $k = 6, 10, 12, 16, 18$, f_k contains a monomial of highest degree that does not appear in any polynomial in $\mathcal{F}_{k-1}$, whence $\mathcal{F}_k$ is linearly independent. In the remaining cases, note that
(i) f_k contains a monomial of highest degree that also appears, among the elements of $\mathcal{F}_k$, only in f_{k-2};
(ii) f_{k-2} has a different monomial that also appears, among the elements of $\mathcal{F}_k$, only in f_{k-1};
(iii) f_{k-1} has a monomial of highest degree that appears in no other element of $\mathcal{F}_k$.
We thus see that $\mathcal{F}_k$ is independent in these cases. (Observe also that $\mathcal{F}_{20}$ is dependent, since

$$f_{20} = -f_{13} - af_{16} - bf_{18} + f_5 + p(y - x^3),$$

and $p(y - x^3) = pf_2 \in \langle f_2, f_4, f_6, f_8, f_{10}, f_{12}\rangle$.)

Now, $\dim[\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta] \ge 19$ and $\dim(\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V})) = 20$. Since $h \in \mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta$, to complete the proof that $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \subseteq \ker\ \Lambda_\beta$, it suffices to verify that $h \notin \left\langle \{f_i\}_{i=1}^{19} \right\rangle$. Let $2 \le k \le 19$ and assume by induction that $h \notin \left\langle \{f_i\}_{i=1}^{k-1} \right\rangle$. Consider a linear combination $q := \alpha_1 f_1 + \cdots + \alpha_k f_k$, with $\alpha_k \ne 0$. Except when $k = 6, 10, 12, 16, 18$, f_k contains a monomial term of highest degree that does not appear in h or in any element of $\mathcal{F}_{k-1}$, so $q \ne h$. In the remaining cases, if $q = h$, then proceeding as in the proof that $\mathcal{F}$ is independent, we see that $\alpha_{k-2} \ne 0$, and then that $\alpha_{k-1} \ne 0$. Now f_{k-1} contains a monomial of highest degree that does not appear in h or in any other element of $\mathcal{F}_k$, so we arrive at a contradiction. Thus $q \ne h$ in these cases also. Now, following (3.4), $\{f_i\}_{i=1}^{19} \bigcup \{h\}$ forms a basis for $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \bigcap \ker\ \Lambda_\beta$, whence $\mathcal{P}_6 \bigcap \mathcal{I}(\mathcal{V}) \subseteq \ker\ \Lambda_\beta$, so β is consistent. The proof is now complete. $\square$

Remark 5.5. (i) If the points of the variety $\{(x_i, y_i)\}_{i=1}^{8}$ are known explicitly, then h can be computed as in (5.4). In this case, Theorem 5.3 provides an effective test for the existence of a representing measure in the extremal problem (5.1). If the points of the variety are not known explicitly, there is still available a concrete test for the non-existence of a representing measure, as follows:

If $\mathcal{M}(3)$ (as in (5.1)) has a representing measure, then there is a unique flat extension $\mathcal{M}(4)$, and $\mathcal{V}(\mathcal{M}(4)) = \mathcal{V}(\mathcal{M}(3)) = \mathcal{V}$. In this case, there is a column relation in $\mathcal{C}_{\mathcal{M}(4)}$ of the form

$$Y^2X^2 = \alpha_1 1 + \alpha_2 X + \alpha_3 Y + \alpha_4 X^2 + \alpha_5 YX + \alpha_6 Y^2 + \alpha_7 YX^2 + \alpha_8 Y^2 X.$$

To compute $\alpha_1, \cdots, \alpha_8$, let $\mathbf{v}$ denote the compression of column Y^2X^2 in $\mathcal{M}(4)$ to rows indexed by the basis $\mathcal{B}$, i.e., $\mathbf{v} := (\beta_{22}, \beta_{32}, \beta_{23}, \beta_{42}, \beta_{33}, \beta_{24}, \beta_{43}, \beta_{34})^T$. Since $\mathcal{M}(4)$ is recursively generated, we have $X^4 = YX$ and $YX^3 = Y^2$ in $\mathcal{C}_{\mathcal{M}(4)}$, whence $\beta_{43} = \beta_{14}$ and $\beta_{34} = \beta_{05}$. Thus $\mathbf{v}$ is expressed in terms of the original data from $\beta^{(6)}$. Let J denote the compression of $\mathcal{M}(3)$ to rows and columns indexed by elements of $\mathcal{B}$; then J is invertible and $\alpha := (\alpha_1, \cdots, \alpha_8)$ is uniquely determined by

$$\alpha^T = J^{-1}\mathbf{v}. \tag{5.10}$$

Now let

$$k(x,y) := y^2x^2 - (\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 yx + \alpha_6 y^2 + \alpha_7 yx^2 + \alpha_8 y^2 x.$$

Since $k(X,Y) = 0$ in $\mathcal{C}_{\mathcal{M}(4)}$, then $k|_{\mathcal{V}} \equiv 0$, so it follows from (5.3) and (5.4) that $k \equiv h$, whence $\Lambda_\beta(k) = 0$. Thus, if k is computed as above (using (5.10)) and $\Lambda_\beta(k) \neq 0$, then β has no representing measure.
(ii) Let $k(x,y)$ be computed as above. Even without knowing the points of $\mathcal{V}$ explicitly, if we know that $k|_{\mathcal{V}} \equiv 0$, then from Lemma 5.4, Lemma 2.7, and (5.4) it follows that $k = h$, so β has representing measure if and only if $\Lambda_\beta(k) = 0$.

We conclude with a proof of Theorem 1.5(ii), which we re-state for convenience.

Theorem 5.6. *Let $d = 2$. Suppose $\mathcal{M}(3) \geq 0$ satisfies $Y = X^3$. If $r \leq v = 8$, then $\beta \equiv \beta^{(6)}$ has a representing measure if and only if β is consistent.*

Proof. The necessity of consistency is clear, so we focus on sufficiency. Exactly as in the proof of Theorem 4.9, we may assume $\mathcal{M}(2) > 0$ and $r \geq 7$. We first show that the case $r = 7$, $v = 8$ cannot occur. Since $\mathcal{M}(2) > 0$, we may assume that the column space of $\mathcal{M}(3)$ has a basis of the form $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, f(X,Y)\}$, where $f(x,y)$ is either x^2y, xy^2, or y^3. Suppose first that $f(x,y) = xy^2$. Then there is a column relation of the form $X^2Y = \alpha XY^2 + p(X,Y)$ with *deg* $p \leq 2$. Thus both $x^2y - (\alpha xy^2 + p(x,y))$ and $y - x^3$ vanish on $\mathcal{V}(\mathcal{M}(3))$, which implies that $x^5 - (\alpha x^7 + p(x,x^3) = 0$ admits 7 distinct roots, a contradiction. Similar arguments show that the cases $f(x,y) = x^2y$ and $f(x,y) = y^3$ also cannot arise.

We may now assume that $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$, $Y = X^3$, and $r = v = 8$. The case when $\mathcal{B} := \{1, X, Y, X^2, YX, Y^2, YX^2, Y^2X\}$ is a basis for $\mathcal{C}_{\mathcal{M}(3)}$ is covered by Theorem 5.3. To complete the proof, it suffices to show that problem (5.1) can always be reduced to this case. Indeed, suppose that a maximal linearly independent set of columns is $\{1, X, Y, X^2, YX, Y^2, YX^2, Y^3\}$. Then there is a column relation of the form $Y^2X = \alpha_1 YX^2 + \alpha_2 Y^3 + p(X,Y)$ (deg $p \leq 2$). If $\alpha_2 = 0$, then (since $Y = X^3$), $\mathcal{V}(\mathcal{M}(3))$ is a subset of the zeros of $x^7 = \alpha_1 x^5 + p(x,x^3)$, whence $v \leq 7$, a contradiction. Thus, $\alpha_2 \neq 0$, and since $r = 8$, it follows that $\mathcal{B}$ is a basis. A similar argument can be used in the case when $\{1, X, Y, X^2, YX, Y^2, Y^2X, Y^3\}$ is a basis. This completes the proof. $\square$

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