

THE CORE VARIETY AND REPRESENTING MEASURES IN THE TRUNCATED MOMENT PROBLEM

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ABSTRACT. The truncated moment problem asks for conditions so that a linear functional L on the vector space of real n -variable polynomials of degree at most d can be written as integration with respect to a positive Borel measure μ on \mathbb{R}^n . More generally, let L act on a finite dimensional space of Borel-measurable functions defined on a T_1 topological space S . Using an iterative geometric construction, we associate to L a subset of S called the *core variety* $\mathcal{CV}(L)$. Our main result is that L has a representing measure μ if and only if $\mathcal{CV}(L)$ is nonempty. In this case, L has a finitely atomic representing measure, and the union of the supports of such measures is precisely $\mathcal{CV}(L)$.

KEYWORDS: *Truncated moment problems, representing measure, positive Riesz functional, moment matrix.*

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1. INTRODUCTION

Let S be a (nonempty) topological space in which points are closed, and let V be a finite dimensional vector space of Borel-measurable real-valued functions on S . Let V^* be the dual space of linear functionals on V . The main question of this paper is the following version of the generalized truncated moment problem: given $L \in V^*$, does L have a *representing measure*, i.e., a positive Borel measure μ on S such that $L(f) = \int_S f d\mu$ for all $f \in V$?

To motivate our results, we first recall the classical full and truncated moment problems. Let $\mathcal{P} \equiv \mathbb{R}[x_1, \dots, x_n]$ denote the ring of real polynomials in n real variables. For an n -dimensional multisequence $\beta \equiv \beta^{(\infty)} := \{\beta_i\}_{i \in \mathbb{Z}_+^n}$, $\beta_0 > 0$, let $L \equiv L_\beta : \mathcal{P} \rightarrow \mathbb{R}$ be the *Riesz linear functional* defined by $L(x^i) := \beta_i$ ($i \in \mathbb{Z}_+^n$), where $x \equiv (x_1, \dots, x_n) \in \mathbb{R}^n$, $i = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$, and $x^i := x_1^{i_1} \cdots x_n^{i_n}$. Given β and a closed set $K \subseteq \mathbb{R}^n$, the classical *full K -moment problem* seeks necessary and sufficient conditions for the existence of a positive Borel measure μ on \mathbb{R}^n

satisfying $\text{supp } \mu \subseteq K$ and $\beta_i = \int_{\mathbb{R}^n} x^i d\mu$, $i \in \mathbb{Z}_+^n$ [21], [24], [29], [31]. The theorems of M. Riesz [26] ($n = 1$) and E.K. Haviland [20] ($n > 1$) show that β has a representing measure supported in K if and only if L is *positive with respect to* K , i.e., $L(p) \geq 0$ whenever $p \in \mathcal{P}$ satisfies $p|_K \geq 0$. For general K , the Riesz–Haviland theorem is difficult to apply, but various concrete applications are known in special cases (cf. [1], [2], [21], [22]). We note particularly Schmüdgen’s solution to the full K -moment problem in the case when K is a compact basic semi-algebraic set [28]; Schmüdgen’s work established a close connection between moment theory and real algebraic geometry.

Now let \mathcal{P}_d denote the polynomials of degree at most d in n real variables, and let $\beta \equiv \beta^{(d)}$ denote an n -dimensional multisequence of degree d , i.e., $\beta = \{\beta_i\}_{i \in \mathbb{Z}_+^n, |i| \leq d}$. The classical *truncated K -moment problem* (TKMP) seeks conditions for β to have a representing measure supported in K . Although the direct analogue of the Riesz–Haviland theorem holds for TKMP when K is compact (this is essentially the content of the proof of Tchakaloff’s theorem [32]), such is not the case for general K . Instead, the truncated Riesz–Haviland theorem [11] shows that $\beta \equiv \beta^{(2d-2)}$ or $\beta \equiv \beta^{(2d-1)}$ admits a representing measure supported in a closed set K if and only if the corresponding Riesz functional L_β can be extended to a linear functional $\hat{L} : \mathcal{P}_{2d} \rightarrow \mathbb{R}$ positive with respect to K . Alternately, in [10], Curto and the second-named author formulated a solution in terms of *flat extensions* of moment matrices. Further, in [34], F.-H. Vasilescu introduced an approach to the truncated moment problem based on the study of elements p that are *idempotent with respect to a square-positive Riesz functional* L_β , i.e., $L_\beta(p^2) = L_\beta(p)$.

As in the full K -moment problem, the preceding solutions to the truncated problem are difficult to apply in general, although various concrete solutions are known in special cases [15]. Recently, the first-named author [6] used an approach based in convex geometry to establish the following concrete condition: $\beta^{(2d-1)}$ has a representing measure whenever the moment matrix M corresponding to $\beta^{(2d)}$ is positive semidefinite and satisfies $\text{rank } M \leq 3d - 3$.

In view of the difficulty of applying the extension results of [10] and [11] to general multisequences $\beta \equiv \beta^{(d)}$, in [16] we introduced an alternate approach based on a geometric invariant called the *core variety* of L_β . In [16], the core variety was used to establish the existence or nonexistence of representing measures in a number of special cases, and was used to illustrate some previously unknown phenomena in the moment problem (e.g. Theorem 3.8 of [16]). The core variety contains the support of each representing measure for β , but what emerged from [16] was the following question.

QUESTION. If the core variety of β is nonempty, does β have a representing measure?

In the present note, we resolve this question affirmatively, and in a very general setting. We also prove an extended version of the truncated Riesz–Haviland

theorem, and apply it to the full moment problem to obtain a generalized version of the Riesz–Haviland theorem. This also entails extending Stochel’s theorem [30], concerning the connection between the full and truncated moment problems. As a consequence of these results, we also obtain an analogue of our core variety results for the full moment problem. Our approach emphasizes convex-geometric aspects of the truncated moment problem, and we use a number of elementary results from finite-dimensional convex geometry.

REMARK 1.1. Finitely atomic measures correspond to positive linear combinations of point evaluation functionals in V^* . As we explain in more detail below (cf. Remark 2.22), our results concerning finitely atomic representing measures, when expressed equivalently in terms of conical combinations of point evaluations, actually hold in complete generality, with virtually no assumptions on S or on the functions in V . However, by working in complete generality, the connection to the classical moment problem is not as transparent as in our setting of a topological space S with points closed, where all point evaluation functionals have representing measures.

In the case where the linear functional L is known to come from a measure, the concept of core variety was studied by di Dio and Schmüdgen in [12], [29]. In particular, they independently obtained Theorem 2.10 in case L is known to come from a measure, and also Proposition 2.19 ([12], Theorem 29). Moreover, Gabardo did some work in this direction in the context of trigonometric moment problems in [19]. Note that the setting for our results includes a variation of TKMP in which the sequence $\beta^{(d)}$ is only partially defined, i.e., moments corresponding to certain monomials are not prescribed. A special case of this problem has been studied by Laurent and Mourrain [23], and a somewhat analogous partially defined trigonometric moment problem appears in [19].

The referee has pointed out a connection, at least thematically, between the present paper and the work of F.-H. Vasilescu in [33]. To apply our generalized truncated Riesz–Haviland theorem in Corollary 3.3, we assume the existence of $\rho \in V$ such that f/ρ vanishes at infinity for all f in the prescribed subspace U of V . In the polynomial case, where $U = \mathcal{P}_{2d-1}$ or $U = \mathcal{P}_{2d-2}$ and $V = \mathcal{P}_{2d}$, $\rho(x) := 1 + \|x\|^{2d}$ satisfies this requirement. A similar idea is used in the proof of the generalized Riesz–Haviland theorem for the full moment problem (Theorem 3.8); this permits us to obtain representing measures via weak* convergence in $C_0(S)$. In [33], Vasilescu studied the full multidimensional moment problem on \mathbb{R}^n by first studying the behavior of positive linear functionals on certain subspaces of the algebra of fractions $C(X)/\mathcal{Q}$, where X is a compact Hausdorff space and \mathcal{Q} is a *multiplicative family* of positive elements of $C(X)$ ([33], Theorems 3.2, 3.4, 3.7). By working with the sequence of denominators $q_k(x) = (1 + \|x\|^2)^k$ ($k \geq 0$), Vasilescu was able to use Theorems 3.2, 3.4, 3.7 of [33] to study the full moment problem via corresponding problems on compactifications of \mathbb{R}^n ([33], Theorem 4.8).

1.1. MAIN RESULTS. With S and V as above, let $P \equiv P_V$ be the set of all functions in V which are nonnegative on S :

$$P = \{f \in V : f(s) \geq 0 \text{ for all } s \in S\}.$$

Since V is finite dimensional, we may equip V with the Euclidean topology; note that P is then a closed convex cone in V . It also follows that P is a *pointed* cone, i.e., P does not contain lines. We will assume that P contains a strictly positive function. This assumption is important for our main results; see Remark 2.4 for additional discussion, and Remark 2.12 for a possible generalization. For a subset T of V we will denote by $\mathcal{Z}(T) \subset S$ the set of common zeros of functions in T :

$$\mathcal{Z}(T) = \{s \in S : f(s) = 0 \text{ for all } f \in T\}.$$

Let M be the cone of functionals in V^* which have representing measures. We observe that M is a convex cone in V^* , but M may fail to be closed [14]. Our main goal is to characterize linear functionals belonging to M .

Given a nonzero linear functional $L \in V^*$, let $S_0 \equiv S_0[L] = S$, and let $S_1 \equiv S_1[L]$ be the zero set of all nonnegative functions in the kernel of L :

$$S_1 = \mathcal{Z}(p \in P : L(p) = 0).$$

Then we iteratively define:

$$S_{i+1} \equiv S_{i+1}[L] = \mathcal{Z}(p \in V : L(p) = 0 \text{ and } p \text{ is nonnegative on } S_i).$$

The above construction eventually terminates, i.e., for some $k \geq 0$, we have $S_k = S_{k+1}$ (cf. Theorem 2.8 and Remark 2.12). This allows us to define the core variety of L , as follows.

DEFINITION 1.2. We call the terminal set $S_k (= S_{k+1} = S_{k+2} \dots)$ the *core variety* of L and denote it by $\mathcal{CV}(L)$.

The following is our main result on core varieties.

THEOREM (Theorem 2.10). *For any nonzero $L \in V^*$ exactly one of the following holds:*

- (i) *The core variety of L is empty and L does not have a representing measure.*
- (ii) *The core variety of L is non-empty, in which case, replacing L by $-L$ if necessary, we may assume that $L(p) > 0$ for a strictly positive $p \in P$. Then L has a representing measure, and the core variety of L is equal to the union of all supports of finitely atomic measures representing L . If S is a Hausdorff space, and the functions in V are continuous on S , then $\mathcal{CV}(L)$ is also equal to the union of supports of all Radon measures representing L .*

REMARK 1.3. We observe that if $L \in V^*$ is the zero linear functional, then $\mathcal{CV}(L) = \emptyset$, but L does have a positive representing measure: the zero measure.

REMARK 1.4. In part (ii) of the above theorem, concerning the case when the functions in V are continuous, the condition needed in the proof is that measures

μ satisfy $\mu(S \setminus \text{supp } \mu) = 0$. This is automatically satisfied if S is Hausdorff and μ is a Radon measure on S ([29], Appendix A.1, [4], p. 155). Therefore we can formulate part (ii) as: if the functions in V are continuous on S , then $\mathcal{CV}(L)$ is also equal to the union of supports of all measures representing L and satisfying $\mu(S \setminus \text{supp } \mu) = 0$.

As a corollary of the above theorem we prove a version of the Bayer–Teichmann theorem ([5], Theorem 2) on multivariable cubature. The result in [5] generalizes the classical cubature existence theorem of Tchakaloff [32] for the polynomial case when S is compact.

COROLLARY (Bayer–Teichmann theorem, Corollary 2.14). *Suppose that $L \in V^*$ has a representing measure μ . Then L has a finitely atomic representing measure ν with $\text{card supp } \nu \leq \dim V$. If, additionally, S is Hausdorff and μ is a Radon measure, then $\text{supp } \nu \subseteq \text{supp } \mu$.*

The core variety can be used to describe the *facial decomposition* of the cone M of functionals in V^* having representing measures. We use $\text{relint}(K)$ to denote the *relative interior* of a convex set K (cf. Part II, Section 6 of [27]).

THEOREM (Theorem 2.16). *Let $L \in M$. Let F_L be the set of all linear functionals in M whose core varieties are contained in the core variety of L :*

$$F_L = \{m \in M : \mathcal{CV}(m) \subseteq \mathcal{CV}(L)\}.$$

Then F_L is a face of M , and $m \in \text{relint } F_L$ if and only if $\mathcal{CV}(m) = \mathcal{CV}(L)$. Furthermore, any face of M has the form F_L for some $L \in M$.

Results of [16] illustrate some cases where the iterative construction of $\mathcal{CV}(L)$ can be carried out explicitly in examples, but in general it is difficult to compute the core variety. The following two results do, however, shed light on the behavior of the iterative construction. The first shows a geometric termination criterion for the iterative construction of the core variety for functionals in M .

PROPOSITION (Proposition 2.19). *Let $L \in V^*$ be a linear functional with a representing measure. Then $\mathcal{CV}(L) = S_1$ ($= \mathcal{Z}(p \in P : L(p) = 0)$) if and only if F_L is an exposed face of M .*

We next apply the preceding result to the case where some iterate S_k is finite.

PROPOSITION (Proposition 2.20). *Suppose that S_k is a finite set of points. Then $\mathcal{CV}(L) = S_k$ or $\mathcal{CV}(L) = S_{k+1}$, i.e., the iterative construction of the core variety is guaranteed to terminate in at most one more step.*

We prove a general version of the truncated Riesz–Haviland theorem in Theorem 3.1 and derive some applications for the full moment problem in a general setting. For the subsequent results, S is a σ -compact, locally compact metric space; see Chapter V, Appendix C of [9] for basic definition and properties. Let V be a linear subspace of the space $C(S)$ of continuous real-valued functions on

S which admits a decomposition as a countable union of increasing finite dimensional subspaces, i.e., $V = \bigcup_{j=1}^{\infty} V_j$, $V_{j-1} \subseteq V_j$, $\dim V_j < \infty$ ($j \geq 1$). A function $f : S \rightarrow \mathbb{R}$ *vanishes at infinity* if, for each $\varepsilon > 0$, there is a compact set $C_\varepsilon \subseteq S$ such that $S \setminus C_\varepsilon \subseteq \{x \in S : |f(x)| < \varepsilon\}$. Let $C_0(S)$ denote the Banach space of all continuous functions on S which vanish at infinity, equipped with the norm $\|f\|_\infty := \sup_{x \in S} |f(x)|$ (cf. p. 65 of [9]). We next present an extension of the Riesz–Haviland theorem to our general setting.

THEOREM (Theorem 3.8). *Suppose $1 \in V_1$ and suppose that for each $j \geq 2$, there exists a strictly positive function $\rho_j \in V_j$ such that $q/\rho_j \in C_0(S)$ for all $q \in V_{j-1}$. A linear functional $L : V \rightarrow \mathbb{R}$ has a representing measure if and only if $L(p) \geq 0$ for any function $p \in V$ nonnegative on S .*

To recover the Riesz–Haviland theorem from this result, let $S = \mathbb{R}^n$, $V = \mathbb{R}[x_1, \dots, x_n]$, and for $j \geq 1$, let $V_j = \mathcal{P}_{2j}$ and $\rho_j = 1 + \|x\|^{2j}$.

Let L be a linear functional on V . We now define the *core variety* of L by $\mathcal{CV}(L) := \bigcap_{j=1}^{\infty} \mathcal{CV}(L_j)$. We conclude with an application of the core variety to the full moment problem.

THEOREM (Theorem 3.10). *A linear functional $L : V \rightarrow \mathbb{R}$ has a representing measure if and only if $\mathcal{CV}(L) \neq \emptyset$.*

2. CORE VARIETY

We begin by establishing a few preliminaries. Since the cone P_V is pointed, the dual cone $P_V^* \subset V^*$ is full-dimensional in the dual vector space V^* ([3], p. 49). Let M be the cone of functionals in V^* which have representing measures:

$$M = \{L \in V^* : L \text{ has a representing measure}\}.$$

We observe that M is a convex cone in V^* , but M may fail to be closed [14].

For a point $s \in S$, let $L_s \in V^*$ denote the corresponding point evaluation functional on V : $L_s(f) = f(s)$. Since the points of S are closed, each linear functional L_s has a representing measure, namely the Dirac δ -measure concentrated at s . Let C be the set of all functionals in V^* coming from finitely atomic measures on S :

$$C = \text{ConicalHull}\{L_s : s \in S\}.$$

It is clear that C is a convex cone in V^* and $C \subseteq M$. Like M , the cone C may fail to be closed, and we will later show in Corollary 2.14 that $C = M$. Note that by Carathéodory’s theorem, an element of C may be represented as a conical combination of at most $\dim V$ point evaluations.

For a positive Borel measure μ on S , for which $V \subseteq L_1(\mu)$, we define the linear functional $L_\mu \in V^*$ to be integration with respect to μ : $L_\mu(f) = \int_S f d\mu$.

Recall that the *support* of μ , denoted by $\text{supp } \mu$, is defined as the set of all points $s \in S$ such that each open set U containing s satisfies $\mu(U) > 0$; note that $\text{supp } \mu$ is closed, hence Borel. Since points of S are closed, $\text{supp } \mu$ is finite if and only if μ is finitely atomic, i.e., L_μ belongs to C .

We begin by describing the closure and interior of C .

PROPOSITION 2.1. *The closure \overline{C} of C is equal to the dual cone of P :*

$$\overline{C} = P^*.$$

Proof. By the identification of V with V^{**} we can express the dual cone C^* as

$$C^* = \{f \in V : L_s(f) = f(s) \geq 0 \text{ for all } s \in S\} = P.$$

By the bipolarity theorem ([8], Exercise 2.31(f)) we have $\overline{C} = (C^*)^*$, so it follows that $\overline{C} = P^*$. ■

Recall that a linear functional $L \in V^*$ is said to be *strictly positive* if $L(f) > 0$ whenever f is a non-zero function in P (cf. [17]). Let SP denote the convex cone of all strictly positive functionals in V^* . We next show that the interiors of C and M agree, and that these interiors coincide with SP .

COROLLARY 2.2. *Let P be the cone of nonnegative functions in V , let M be the cone of linear functionals in V^* which have representing measures, and let C be the conical hull of the point evaluation functionals. Then $\text{int}(C)$ is non-empty, and*

$$\text{int}(C) = \text{int}(M) = \text{int}(P^*) = SP \equiv \{L \in V^* : L(f) > 0 \text{ for all non-zero } f \in P\}.$$

In particular, any linear functional L strictly positive on P has a finitely atomic representing measure, and the union of all supports of finitely atomic measures representing L is S .

Proof. Note that $C \subseteq M \subseteq P^*$, so Proposition 2.1 implies that

$$P^* = \overline{C} \subseteq \overline{M} \subseteq P^*.$$

Thus, $\overline{C} = \overline{M} = P^*$, and since C and M are convex, it follows that $\text{int}(C) = \text{int}(\overline{C}) = \text{int}(P^*) = \text{int}(\overline{M}) = \text{int}(M)$ (see Theorem 6.3 of [27] for the first equality). Since P^* is full-dimensional, $\text{int}(P^*)$ is non-empty ([3], p. 49). It now follows that $\text{int}(C)$ is non-empty. The fact that each strictly positive functional belongs to $\text{int}(P^*)$ is elementary convex geometry ([8], Exercise 2.31(d)). Conversely, suppose $L \in P^*$ is not strictly positive, and let $f \in P$ be a nonzero nonnegative function such that $L(f) = 0$. If $s \in S$ satisfies $f(s) > 0$, then for $\varepsilon > 0$, we have $(L - \varepsilon L_s)(f) = -\varepsilon f(s) < 0$, so L is not in $\text{int}(P^*)$. Thus, $\text{int}(C)$ and $\text{int}(M)$ coincide with the strictly positive functionals.

We will now prove the last point. Suppose that $L \in \text{int}(P^*)$. Then, we see that for any $s \in S$ there exists $\varepsilon > 0$ such that $L' := L - \varepsilon L_s \in \text{int}(P^*) = \text{int}(C)$.

Thus L' has a finitely atomic representing measure, whence $L = L' + \varepsilon L_s$ has a finitely atomic representing measure whose support includes s . ■

Given a linear functional $L \in V^*$, we define the following iterative construction: let $S_0 \equiv S_0[L] = S$ and let $S_1 \equiv S_1[L]$ be the zero set of all nonnegative functions in the kernel of L :

$$S_1 = \mathcal{Z}(p \in P : L(p) = 0).$$

Then we iteratively define:

$$S_{i+1} \equiv S_{i+1}[L] = \mathcal{Z}(p \in V : L(p) = 0 \text{ and } p \text{ is nonnegative on } S_i).$$

An easy induction argument shows that $S_{i+1} \subseteq S_i$ ($i \geq 0$). We now define the core variety of L as the intersection of the nested sets S_i .

DEFINITION 2.3. We call the intersection of the nested sets S_i the *core variety* of L and denote it by $\mathcal{CV}(L)$:

$$\mathcal{CV}(L) = \bigcap_{i=0}^{\infty} S_i.$$

We will show in Theorem 2.8 that the iterative construction stabilizes: we have $S_k = S_{k+1} = \dots$ for a sufficiently large k , and, in particular, we do not need to consider infinite intersections in dealing with the core variety.

REMARK 2.4. The above results did not use the existence of a strictly positive function $\rho \in P$. It is easy to show that a strictly positive function ρ exists if and only if the functions in P have no common zeros, i.e. $\mathcal{Z}(P) = \emptyset$.

We start by proving in Lemma 2.5 (below) two basic properties of the preceding construction. For convenience, we will make the following standing assumption on the linear functional L for all of the results up to and including Theorem 2.8. The assumption will be removed in Theorem 2.10, and not used in subsequent results.

STANDING ASSUMPTION. *There exists a strictly positive function $\rho \equiv \rho_L \in P$ such that $L(\rho) > 0$.*

We remark that this assumption is standard in the truncated moment problem literature. In the classical case when $V = \mathcal{P}_d$, all polynomials of degree at most d , we may take ρ to be the constant function $\mathbf{1}$, and the assumption $L(\mathbf{1}) > 0$ means that any representing measure μ satisfies $0 < \mu(S) < +\infty$; in general, since $0 < L(\rho) = \int_S \rho d\mu$, then $\mu(S) > 0$. We note that if there exists a strictly positive function f such that $L(f) = 0$, then $S_1 = \emptyset$.

LEMMA 2.5. (i) *If $S_1 = S$, then L has a finitely atomic representing measure. The union of all supports of finitely atomic representing measures for L is equal to S .*

(ii) If the functions in V are continuous on S , then $\text{supp } \mu \subseteq S_1$ for any measure μ representing L .

Proof. (i) We claim that L is strictly positive. Suppose that there exists a non-zero $q \in P$ with $L(q) \leq 0$. If $L(q) = 0$ and $q(s) > 0$ for some $s \in S$, then $s \notin S_1 = S$, a contradiction. If $L(q) < 0$, then let $a := L(\rho) > 0$, and let $b := L(g)$. Then $f := \rho - (a/b)g$ is a strictly positive element of P satisfying $L(f) = 0$, which implies $S_1 = \emptyset$, a contradiction. Thus L is strictly positive, so the result follows from Corollary 2.2.

(ii) Suppose that there exists $s \in \text{supp } \mu$ such that $s \notin S_1$. Then $S_1 \neq S$, and therefore there exists $p \in P$ such that $L(p) = 0$ and $p(s) > 0$. Since p is continuous, there exists an open set $U \subset S$ containing s , such that $p(u) > 0$ for all $u \in U$. Since $s \in \text{supp } \mu$, we have $\mu(U) > 0$. It follows that $L(p) = \int_S p \, d\mu \geq \int_U p \, d\mu > 0$, which is a contradiction. ■

In the following example we show that if the functions in V are merely Borel measurable, then the conclusion of Lemma 2.5 part (ii) need not hold.

EXAMPLE 2.6. Let $S = [-1, 1]$ with the usual Euclidean topology, and let μ denote the restriction of Lebesgue measure to S , so $\text{supp } \mu = S$. Define a function f on S by $f(x) = 0$ ($x \in S$, $x \neq 0$), and $f(0) = 1$. Let $g = 1 - f$. Now f and g are Borel measurable, and we let V denote the vector space spanned by f and g ; note that V contains $\rho := f + g = 1$ (strictly positive). Define a functional L on V by $L(f) = 0$ and $L(g) = 2$; we have $L(\rho) = 2$ (> 0). Since the only nonnegative functions in $\ker L$ are scalar multiples of f , it follows that $S_1 = S \setminus \{0\}$ and $S_1 = S_2 = \dots$. It is easy to directly check that 0 is not a support point of any finitely atomic representing measure for L . It is also easy to verify that for each nonzero $x \in S$, $2\delta_x$ is a representing measure for L , and the union of supports of these measures is $S_1 (= \mathcal{CV}(L))$. However, μ is a representing measure for L , and $\text{supp } \mu$ is not contained in S_1 .

We now make the following crucial observation.

LEMMA 2.7. If there exists $g \in V$ such that $L(g) \neq 0$ and g vanishes identically on S_1 , then there exists a function $f \in V$, strictly positive on S_1 , such that $L(f) = 0$, and therefore $S_2 = \emptyset$.

Proof. We have $a := L(\rho) > 0$. We may assume $b := L(g) < 0$, for otherwise we can consider $-g$. Then $f := \rho - (a/b)g$ is strictly positive on S_1 , and $L(f) = 0$, so $S_2 = \emptyset$. ■

We are now in position to show that the iterative construction will stabilize: $S_k = S_{k+1}$ for some k .

THEOREM 2.8. For some k , $0 \leq k \leq \dim V - 1$, we have $S_k = S_{k+1}$, and L has a representing measure if and only if S_k is non-empty. Moreover, if S_k is non-empty,

then S_k is the union of all supports of finitely atomic measures representing L . If S is a Hausdorff space, and the functions in V are continuous on S , then $\mathcal{CV}(L)$ is also equal to the union of supports of all Radon measures representing L .

REMARK 2.9. Now consider an arbitrary L in V^* . If L vanishes identically on P , then $\mathcal{CV}(L) = \emptyset$ in one step, since P contains a strictly positive function. Consider again an arbitrary functional $L \in V^*$. We observe that according to the construction, we have $\mathcal{CV}(L) = \mathcal{CV}(-L)$. If L does not vanish identically on P , then either L or $-L$ satisfies the Standing Assumption. The Standing Assumption is simply used to pick the correct sign. This follows, since $L(p) \neq 0$ for some p in the relative interior of P , and all functions in the relative interior of P are strictly positive. Thus, by Theorem 2.8, the iterative construction of the core variety terminates in a finite number of steps for any linear functional in V^* . It follows from the proof that $\mathcal{CV}(L) = S_k$ for $k \leq \dim V - 1$.

In the proof of Theorem 2.8 and elsewhere in the sequel, we sometimes pass from L to an induced functional \tilde{L} , as we next describe. Let $R \subset S$ and consider the subspace $W = \{f \in V : f|_R \equiv 0\}$. Let $\tilde{V} = V/W \equiv \{f|_R : f \in V\}$. Assuming that $W \subseteq \ker L$, L induces a well-defined element $\tilde{L} \in \tilde{V}^*$ by $\tilde{L}(f + W) \equiv \tilde{L}(f|_R) := L(f)$ ($f \in V$). Note that if \tilde{L} admits a finitely atomic representing measure, say $\tilde{L} = \sum a_i L_{r_i}$ ($a_i > 0, r_i \in R$), then since $L(f) = \tilde{L}(f|_R) = \sum a_i f(r_i)$ ($f \in V$), it follows that L admits a finitely atomic representing measure supported in R .

Proof of Theorem 2.8. We first show that the iterative construction stabilizes. For $i \geq 0$, let $V_i = \{f|_{S_i} : f \in V\} \equiv V / \{f \in V : f|_{S_i} \equiv 0\}$. Suppose S_{i+1} is nonempty and let $\tau : V_i \rightarrow V_{i+1}$ be the mapping given by $\tau(f|_{S_i}) = f|_{S_{i+1}}$. Since $S_{i+1} \subseteq S_i$, τ is a well-defined linear surjection, so $\dim V_{i+1} = \dim V_i - \dim \ker \tau$. Note that if there is some $x \in S_i \setminus S_{i+1}$, then there exists $f \in \ker L$ such that $f|_{S_i} \geq 0$ and $f(x) > 0$. Thus $f|_{S_i}$ is a nonzero element of $\ker \tau$, so $\dim V_{i+1} < \dim V_i$. It follows that for $i \geq 0$, either

- (a) $S_i = \emptyset$, or
- (b) $S_i \neq \emptyset$ and $S_{i+1} = \emptyset$, or
- (c) $S_{i+1} = S_i \neq \emptyset$, or
- (d) S_{i+1} is a proper nonempty subset of S_i , in which case $\dim V_{i+1} < \dim V_i$.

Since $V_0 \equiv V$ is finite dimensional, case (d) can occur at most a finite number of times, so there is a smallest k for which S_k satisfies (a), (b), or (c). In cases (a) and (c), we have $S_k = S_{k+1} = \dots$, and in case (b) we have $S_{k+1} = S_{k+2} = \dots$, so we have established stabilization. Since $\dim V_{i+1} < \dim V_i$ can occur at most $\dim V - 1$ times, stabilization occurs in at most $\dim V - 1$ steps.

We now show that if L has a representing measure μ , then the core variety is non-empty. Indeed, we will show by induction on $i \geq 0$ that S_i is a Borel set and that $\mu(S_i) = \mu(S_0) = \mu(S) > 0$. For the base case $S = S_0$, the Standing Assumption shows that $0 < L(\rho) = \int_S \rho d\mu$, so we must have $\mu(S) > 0$. Now suppose that

S_i is a Borel set and that $\mu(S_i) = \mu(S) > 0$. Let T be the span of the functions in the kernel of L that are nonnegative on S_i . If $T = \{0\}$, then $S_{i+1} = S_i$, so the result follows by our inductive assumption. Suppose then that T is nontrivial; since T is finite dimensional, we can choose a basis f_1, \dots, f_m of T with each f_i nonnegative on S_i . Then S_{i+1} is the common zero set of f_1, \dots, f_m . Let $F = f_1 + \dots + f_m$, so that $\mathcal{Z}(F) = \mathcal{Z}(f_1, \dots, f_m) = S_{i+1}$. Since F is Borel measurable, its zero set is Borel, so it follows that S_{i+1} is a Borel set. Now consider the complement C of S_{i+1} in S_i (a Borel set). We must have $\mu(C) = 0$. For otherwise, since $\mu(S) = \mu(S_i)$, $F|_{S_{i+1}} \equiv 0$, and $F|_C > 0$, then $L(F) = \int_S F d\mu = \int_{S_i} F d\mu = \int_C F d\mu > 0$, which is a contradiction to $L(F) = 0$. It follows that $\mu(S_{i+1}) = \mu(S_i) = \mu(S)$, as desired.

Now suppose that the core variety $\mathcal{CV}(L) \equiv S_k (= S_{k+1} = \dots)$ is nonempty. Exactly as in the proof of Lemma 2.7 (replacing S_1 by S_k and S_2 by S_{k+1}), we see that if $f \in V$ satisfies $f|_{S_k} \equiv 0$, then $L(f) = 0$. Then the mapping $\tilde{L}: V_k \rightarrow \mathbb{R}$ defined by $\tilde{L}(f|_{S_k}) := L(f)$ is a well-defined linear functional on V_k . Further, it follows exactly as in the proof of Lemma 2.5(i) (replacing S by S_k and S_1 by S_{k+1}), that \tilde{L} is strictly positive as an element of V_k^* . Corollary 2.2 thus implies that \tilde{L} has a finitely atomic representing measure, and that the union of all supports of such measures is precisely S_k . As noted in the remarks just preceding this proof, each such measure for \tilde{L} corresponds to a finitely atomic representing measure for L . We claim, conversely, that the support of each finitely atomic representing measure μ for L is contained in S_k . Indeed, since μ is a representing measure, we know (from just above) that $\mu(S) = \mu(S_k)$, so μ cannot have any atoms outside of S_k . Thus S_k is the union of supports of all finitely atomic representing measures for L .

Finally, we consider the case when S is Hausdorff, the measures are Radon and the functions in V are continuous on S . For properties of Radon measures we refer to Appendix A.1 of [29] and p. 155 of [4]. To complete the proof, it suffices to show that if μ is a representing Radon measure for L , then $\text{supp } \mu \subseteq S_i$ for each $i \geq 0$, and this is clear for $i = 0$. Suppose $\text{supp } \mu \subseteq S_i$ and there exists $x \in \text{supp } \mu \setminus S_{i+1}$. Since $x \in S_i \setminus S_{i+1}$, there exists $f \in V$ such that $f|_{S_i} \geq 0$, $f(x) > 0$, and $L(f) = 0$. From the continuity of f , there is an open set U such that $x \in U$ and $f|_U > 0$. Since $x \in \text{supp } \mu$, then $\mu(U) > 0$, and so $\int_U f d\mu > 0$. Also, since $f|_{S_i} \geq 0$, then $f|_{\text{supp } \mu} \geq 0$, so $\int_{\text{supp } \mu} f d\mu \geq \int_{U \cap \text{supp } \mu} f d\mu$. Since μ is Radon then $\mu(S \setminus \text{supp } \mu) = 0$. Now, $\int_S f d\mu = \int_{\text{supp } \mu} f d\mu$, so $0 = L(f) = \int_S f d\mu = \int_{\text{supp } \mu} f d\mu \geq \int_{U \cap \text{supp } \mu} f d\mu = \int_U f d\mu > 0$, and this contradiction completes the proof. ■

The following is the main result of this section. It is a slight generalization of Theorem 2.8 using the terminology of the core variety.

THEOREM 2.10. *For any nonzero $L \in V^*$ exactly one of the following holds:*

- (i) The core variety of L is empty and L does not have a representing measure.
- (ii) The core variety of L is non-empty, in which case, replacing L by $-L$ if necessary, we may assume that $L(\rho) > 0$ for a strictly positive $\rho \in P$. Then L has a representing measure, and the core variety of L is equal to the union of all supports of finitely atomic measures representing L . If S is a Hausdorff space, and the functions in V are continuous on S , then $\mathcal{CV}(L)$ is also equal to the union of supports of all Radon measures representing L .

Proof. If L vanishes identically on P , then $S_1 = \emptyset$, since by our assumption P contains a strictly positive function. If L does not vanish on P , then there exists a strictly positive function $\rho \in P$, such that $L(\rho) \neq 0$ (cf. Remark 2.9). Therefore, we can apply Theorem 2.8 to either L or $-L$ and the result follows. ■

We now consider the number of steps before stabilization occurs in the iterative construction of the core variety.

EXAMPLE 2.11. Consider the smallest value of k for which $\mathcal{CV}(L) = S_k$. In the examples of [16] and in the previous examples of this paper we have $k \leq 2$, but we now illustrate that $k = \dim V - 1$ is possible (i.e. it takes $\dim V$ steps to discover stabilization). This is the maximum number of steps possible, by the proof of Theorem 2.8. Let $\alpha_i = 4i$ ($0 \leq i \leq k+1$) and let $\beta_i = \alpha_i + 2$ ($0 \leq i \leq k$). Let S be the set obtained from $(0, +\infty)$ by deleting $\alpha_0, \dots, \alpha_{k-1}$. We define functions f_0, \dots, f_k on S as follows:

$$f_0(x) = -(x-4) \quad \text{with } 0 < x < 4, \quad \text{and} \quad f_0(x) = 0 \quad \text{with } x > 4.$$

For $1 \leq i \leq k$, we set $f_i(x) := 0$ for $x \in S$ with $0 < x < \alpha_{i-1}$ or $x > \alpha_{i+1}$, and

$$f_i(x) = -\frac{(x - \beta_{i-1})(x - \alpha_{i+1})}{(x - \alpha_{i-1})} \quad \text{for } \alpha_{i-1} < x < \alpha_{i+1}.$$

Let V denote the vector space spanned by f_0, \dots, f_k and $\rho(x) \equiv 1$. Since these functions are linearly independent in $C(S)$, we have $\dim V = k+2$, and we may define a linear functional L on V by $L(\rho) = 1$ and $L(f_i) = 0$ ($0 \leq i \leq k$). We claim that the cone P_0 of functions in $\ker L$ that are nonnegative on S consists precisely of the nonnegative multiples of f_0 . To see this, suppose $f := a_0 f_0 + \dots + a_k f_k$ is nonnegative on S . Since $0 \leq f(\beta_k) = a_k f_k(\beta_k)$ and $f_k(\beta_k) > 0$, we have $a_k \geq 0$. Now

$$0 \leq \lim_{x \rightarrow \alpha_{k-1}^+} f(x) = a_{k-1} \lim_{x \rightarrow \alpha_{k-1}^+} f_{k-1} + a_k \lim_{x \rightarrow \alpha_{k-1}^+} f_k(x).$$

If $a_k > 0$, then $\lim_{x \rightarrow \alpha_{k-1}^+} f(x) = -\infty$, and this contradiction implies that $a_k = 0$.

Repeating this argument, considering the behavior of each f_j at β_j and as $x \rightarrow \alpha_{j-1}^+$, we may successively show that $a_{k-1} = \dots = a_1 = 0$, and then $a_0 \geq 0$. It follows that $S_1 = \mathcal{Z}(f_0) = \{x \in S : x > \alpha_1\}$. By repeating the preceding arguments, we may next show that the functions in $\ker L$ that are nonnegative on S_1 are precisely of the form $a_0 f_0 + a_1 f_1$ with $a_0 \in \mathbb{R}$ and $a_1 \geq 0$, and from

this it follows that $S_2 = \mathcal{Z}(f_0, f_1) = \{x \in S : x > \alpha_2\}$. In this manner, we may show successively that for $i = 1, \dots, k$, $P_i := \{f \in \ker L : f|_{S_i} \geq 0\} = \{a_0 f_0 + \dots + a_{i-1} f_{i-1} + a_i f_i : a_0, \dots, a_{i-1} \in \mathbb{R}, a_i \geq 0\}$, from which it follows that $S_{i+1} = \mathcal{Z}(P_i) = \{x \in S : x > \alpha_{i+1}\}$. Thus S_0, \dots, S_{k+1} is a chain of strictly decreasing sets, and $\mathcal{CV}(L) = S_{k+1} = S_{k+2} = \dots$. Thus we have built a vector space of dimension $k+2$, where the core variety takes $k+2 = \dim V$ steps to stabilize, which is the largest number of steps possible, by the proof of Theorem 2.8.

REMARK 2.12. For $L \in V^*$, we proved Theorems 2.8 and 2.10 under the assumption that V contains a strictly positive function ρ . The argument in the proof of Theorem 2.8 shows, however, that stabilization of the core variety construction is a general phenomenon, which holds with no assumptions concerning strictly positive functions. In fact, it follows from the argument that $\mathcal{CV}(L) = S_k$ for $k \leq \dim V - 1$. Similarly, without positivity assumptions, the proof of Theorem 2.10 shows that for L nonzero, if L has a representing measure, then $\mathcal{CV}(L)$ is nonempty, and that if μ is a Radon representing measure, then $\text{supp } \mu \subseteq \mathcal{CV}(L)$. For the case of Theorem 2.10 where $\mathcal{CV}(L)$ is nonempty, we can sometimes get by with a slightly weaker assumption than the Standing Assumption, as follows.

Suppose that $\mathcal{CV}(L) \equiv S_k$ is nonempty, and consider the following possible property for L : $(\mathcal{S}(L, k))$ there exist $\rho \equiv \rho_L \in V$ such that $\rho|_{S_k}$ is nonnegative, $\rho|_{S_k} \not\equiv 0$, and $L(\rho) \geq 0$. Clearly, if L satisfies the Standing Assumption, then $\mathcal{S}(L, k)$ holds.

THEOREM 2.13. *If $\mathcal{CV}(L) \equiv S_k$ is nonempty and L satisfies $\mathcal{S}(L, k)$, then L has a finitely atomic representing measure, and the union of the supports of such measures coincides with $\mathcal{CV}(L)$.*

To prove this, we adapt the proof of Theorem 2.8: condition $\mathcal{S}(L, k)$ implies that the functional $\tilde{L} \in V_k^*$ is well-defined and strictly positive, so the conclusion follows as in the proof of Theorem 2.8 (by slightly modifying the statements of Lemmas 2.5(i) and 2.7).

The following example illustrates this approach. Let $S = [-1, 2]$; let $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$; let $g(x) = 1$ for $x \in [-1, 1]$ and $g(x) = -1$ for $x \in (1, 2]$; let $h(-1) = 1$ and $h(x) = 0$ for $-1 < x \leq 2$. Let $V = \langle f, g, h \rangle$. Define $L \in V^*$ by $L(f) = 0$, $L(g) = 2$, $L(h) = 1$. Then $P = \{af + bh : a, b \geq 0\}$, so there is no strictly positive element of P and Theorem 2.8 does not apply. However, $\mathcal{CV}(L) = S_1 = S \setminus \{0\}$ and L satisfies $\mathcal{S}(L, 1)$ with $\rho_L = h$, so the preceding theorem does apply. It is easy to directly check that $\mathcal{CV}(L)$ is the union of supports of 3-atomic representing measures for L , which is consistent with the theorem.

As a corollary of Theorem 2.10, we next prove a version of the Bayer–Teichmann theorem [5] on multivariable cubature (in [12], the result in [5] is attributed to H. Richter [25]). That result generalized the classical cubature existence theorem of Tchakaloff [32] for the polynomial case when S is compact.

COROLLARY 2.14 (Bayer–Teichmann theorem). *Suppose that $L \in V^*$ has a representing measure μ . Then L has a finitely atomic representing measure ν with $\text{card supp } \nu \leq \dim V$. Additionally, if S is Hausdorff and μ is a Radon measure, then $\text{supp } \nu \subseteq \text{supp } \mu$.*

Proof. The first part of the theorem follows directly from Theorem 2.10, and $\text{card supp } \nu \leq \dim V$ follows from Carathéodory’s theorem. Now assume that S is Hausdorff and μ is a Radon measure. Then we know that $\mu(S \setminus \text{supp } \mu) = 0$ ([29], Appendix A.1, [4], p. 155). Let $\tilde{V} := \{f|_{\text{supp } \mu} : f \in V\}$. Define \tilde{L} in \tilde{V}^* by $\tilde{L}(f|_{\text{supp } \mu}) := L(f)$. \tilde{L} is well-defined, since if $f|_{\text{supp } \mu} \equiv 0$, then $\mu(S \setminus \text{supp } \mu) = 0$ implies that $L(f) = \int f d\mu = \int_{\text{supp } \mu} f d\mu = 0$. Now, \tilde{L} has a representing measure, namely $\mu|_{\text{supp } \mu}$, so Theorem 2.10 (applied with S replaced by $\text{supp } \mu$) implies that \tilde{L} has a finitely atomic representing measure ν with $\text{supp } \nu \subseteq \text{supp } \mu$, and $\text{card supp } \nu \leq \dim \tilde{V} \leq \dim V$. Since ν is also a representing measure for L , the proof is complete. ■

REMARK 2.15. For the classical polynomial case, with $V = \mathcal{P}_d$, Theorem 2.10 shows that since the union of supports of all cubature rules for $L \equiv L_\mu$ coincides with $\mathcal{CV}(L)$, then this union is an algebraic variety. This is the motivation behind the terminology “core variety”.

We next show how the core variety can be used to describe the *facial decomposition* of the cone M of functionals in V^* having representing measures. Recall that for a general convex set K , a subset F is a *face* if F is convex and whenever $x, y \in K$ satisfy $tx + (1 - t)y \in F$ for some $t \in (0, 1)$, then $x, y \in F$. It is known that K is the disjoint union of the relative interiors of its faces ([27], Theorem 18.2); moreover, for $x \in K$, the unique face to which x belongs that is minimal with respect to set inclusion (among all faces of K) is the face G satisfying $x \in \text{relint } G$ (as described above).

THEOREM 2.16. *Let $L \in M$. Let F_L be the set of all linear functionals in M whose core varieties are contained in the core variety of L :*

$$F_L = \{m \in M : \mathcal{CV}(m) \subseteq \mathcal{CV}(L)\}.$$

Then F_L is a face of M , and $m \in \text{relint } F_L$ if and only if $\mathcal{CV}(m) = \mathcal{CV}(L)$. Furthermore, any face of M has the form F_L for some $L \in M$.

Proof. Let H be the subspace of V consisting all functions vanishing on $\mathcal{CV}(L)$. Let H^\perp be the subspace of V^* consisting of functionals that are identically 0 on H . Let G_L be the intersection of H^\perp with M . We claim that $F_L = G_L$. Note that this implies that F_L is a convex cone. The inclusion $F_L \subseteq G_L$ is immediate: any functional in F_L has a finitely atomic representing measure supported in $\mathcal{CV}(L)$ and therefore will be identically 0 on H . The inclusion $G_L \subseteq F_L$ follows from the iterative construction of the core variety. Indeed, let m be a linear

functional in G_L . Then m has a finitely atomic representing measure μ by Corollary 2.14. We claim that if the support of μ is not contained in $\mathcal{CV}(L) = S_k$, then it cannot be contained in S_{k-1} either. Arguing as in the proof of Theorem 2.8, there exists a function $f \in V$ identically 0 on S_k and strictly positive on $S_{k-1} \setminus S_k$. Since $f \in H$, we must have $m(f) = 0$, but if the support of μ contains a point in $S_{k-1} \setminus S_k$, then $m(f)$ is positive, which is a contradiction. Repeating the same argument shows that the support of μ cannot be contained in S_{k-1}, S_{k-2}, \dots until we get to $S = S_0$ and obtain a contradiction.

Now we show that F_L is a face of M . Suppose that there exist $\ell_1, \ell_2 \in M$ such that $\ell_1 + \ell_2 \in F_L$ but at least one of the functionals ℓ_i does not lie in F_L . Without loss of generality we may assume that $\ell_1 \notin F_L$. Then, by Theorem 2.10, ℓ_1 has a finitely atomic representing measure whose support is not contained in $\mathcal{CV}(L)$. Since ℓ_2 also has a finitely atomic representing measure, we see that $\ell_1 + \ell_2$ has a finitely atomic representing measure whose support is not contained in $\mathcal{CV}(L)$. This is a contradiction to Theorem 2.10 part (ii), since $\ell_1 + \ell_2 \in F_L$. It follows that F_L is a face of M .

Next we show that $L \in \text{relint } F_L$. Assume that $\mathcal{CV}(L) = S_k$, let $V_k = \{f|_{S_k} : f \in V\}$. and let C_k denote the cone consisting of functionals in V_k^* having a finitely atomic representing measure supported in S_k . For $m \in F_L$, define $\tilde{m} \in V_k^*$ by $\tilde{m}(f|_{S_k}) := m(f)$. To see that \tilde{m} is well-defined, note that from Theorem 2.10 and Corollary 2.14, m has a finitely atomic representing measure supported in $\mathcal{CV}(m) (\subseteq \mathcal{CV}(L))$. Thus m has the form $m = \sum a_i L_{s_i}$ with $a_i > 0$ and $s_i \in \mathcal{CV}(m)$. Now if $f \in V$ satisfies $f|_{S_k} \equiv 0$, then $m(f) = \sum a_i f(s_i) = 0$, so \tilde{m} is well-defined. The preceding remarks show that \tilde{L} is well-defined.

As in the proof of Theorem 2.8, \tilde{L} is strictly positive, so Corollary 2.2 implies that $\tilde{L} \in \text{int } C_k$, and, in particular, $\tilde{L} \in \text{relint } C_k$. Now suppose $J \in F_L$, so that \tilde{J} is well-defined. For $t \in \mathbb{R}$ sufficiently close to 1, $\tilde{K} := t\tilde{L} + (1-t)\tilde{J}$ satisfies $\tilde{K} \in C_k$. It follows that $K := tL + (1-t)J$ has a finitely atomic representing measure supported in S_k , and that if $f \in V$ satisfies $f|_{S_k} \equiv 0$, then $K(f) = 0$. Thus, from the above characterization of F_L , $K \in F_L$, so $L \in \text{relint } F_L$.

Let F be a face of M and let $L \in \text{relint } F$. Since L belongs to the relative interior of a unique face of M and $L \in \text{relint } F_L$, it follows that $F = F_L$. If $m \in M$ satisfies $\mathcal{CV}(m) = \mathcal{CV}(L)$, then $F_m = F_L$, so $m \in \text{relint } F_m = \text{relint } F_L$. Conversely, suppose $m \in \text{relint } F_L$. Since $m \in \text{relint } F_m$, it follows by the uniqueness property in the facial decomposition that $F_L = F_m$. Thus $L \in F_m$ and $m \in F_L$ from which it follows that $\mathcal{CV}(L) = \mathcal{CV}(m)$. ■

To illustrate Theorem 2.16, we give two examples, starting with a continuation of Example 2.6.

EXAMPLE 2.17. As in Example 2.6, let $S = [-1, 1]$ (with the Euclidean topology), and let f be the Borel measurable function on S given by $f(0) = 1$ and $f(x) = 0$ ($x \neq 0$). Let V be the vector space spanned by f and $g := 1 - f$. Note

that for nonzero points x and y in S , we have $L_x = L_y$ in V^* . A basis for V^* is therefore given by the functional L of Example 2.6, $L = 2L_1$ ($= 2L_x$ for $x \neq 0$), together with $J := L_0$. It is straightforward to verify that $P_V^* = M = \{H \equiv aL + bJ : a, b \geq 0\}$. The four faces of M are given as follows:

- (i) F_{J+L} ; thus $F_{J+L} = M$, and $H \in \text{relint } F_{J+L} \leftrightarrow a, b > 0$, in which case $\mathcal{CV}(H) = S$;
- (ii) F_L ; thus $H \in \text{relint } F_L \leftrightarrow a > 0, b = 0$, in which case $\mathcal{CV}(H) = S \setminus \{0\}$;
- (iii) F_J ; thus $H \in \text{relint } F_J \leftrightarrow a = 0, b > 0$, in which case $\mathcal{CV}(H) = \{0\}$;
- (iv) $F_{\{0\}} \equiv \{0\}$, where $\mathcal{CV}(0) = \emptyset$.

EXAMPLE 2.18. Let $S = \mathbb{R}^n$, and let V be the vector space of quadratic forms (homogeneous polynomials) in n variables. Then the cone P_V can be identified with the cone S_+^n of positive semidefinite matrices. Here we recover the usual facial decomposition of S_+^n ([3], p. 78): core varieties are subspaces of \mathbb{R}^n , and for a subspace of $M \subseteq \mathbb{R}^n$ the face F corresponding to M consists of positive semidefinite matrices with kernel contained in M . The relative interior of F consists of positive semidefinite matrices whose kernel is equal to M .

We next present a geometric termination criterion for the iterative construction of the core variety for functionals in M .

PROPOSITION 2.19. *Let $L \in V^*$ be a linear functional with a representing measure. Then $\mathcal{CV}(L) = S_1 = \mathcal{Z}(p \in P : L(p) = 0)$ if and only if F_L is an exposed face ([27], p. 162) of M .*

Proof. Let F_L^Δ be the conjugate (or dual) face of F_L in P :

$$F_L^\Delta = \{p \in P : m(p) = 0 \text{ for all } m \in F_L\}.$$

The minimal exposed face of M containing F_L is F_L' , which is the conjugate (dual) face of F_L^Δ :

$$F_L' = \{\ell \in M : \ell(p) = 0 \text{ for all } p \in F_L^\Delta\}.$$

If $F_L \subsetneq F_L'$, then by Theorem 2.16, F_L' corresponds to a strictly larger core variety, and therefore the iterative construction has not terminated.

Now suppose that $\mathcal{CV}(L) \subsetneq \mathcal{Z}(p \in P : L(p) = 0)$. Let $v \in \mathcal{Z}(p \in P : L(p) = 0) \setminus \mathcal{CV}(L)$ and let $L_v \in V^*$ be the point evaluation functional at v . Consider the functional $L' = L + L_v$. From Theorem 2.10 part (ii) it follows that $\{\mathcal{CV}(L) \cup v\} \subseteq \mathcal{CV}(L')$. Therefore L' has a strictly larger core variety, but we also have $L' \in F_L'$. Therefore by Theorem 2.16 we see that $F_L \subsetneq F_L'$. ■

PROPOSITION 2.20. *Suppose that S_k is a finite set of points. Then $\mathcal{CV}(L) = S_k$ or $\mathcal{CV}(L) = S_{k+1}$, i.e. the iterative construction of the core variety is guaranteed to terminate in at most one more step.*

Proof. Since S_k is a finite set of points, it follows that the cone M_k of linear functionals with a representing measure supported on S_k is simply the conical

hull of point evaluations L_s with $s \in S_k$. It follows that M_k is a polyhedral cone, and in particular, M_k is closed and all faces of M_k are exposed.

Suppose that L has a representing measure supported on S_k , which is equivalent to $L \in M_k$. If L lies in the interior of M_k , then $\mathcal{CV}(L) = S_k$ by Corollary 2.2. If L lies on the boundary of M_k , then $\mathcal{CV}(L) = S_{k+1}$, by Proposition 2.19. Finally if $L \notin M_k$ then, since M_k is closed, there exists p in the interior of P_k such that $L(p) = 0$. Such p is strictly positive on S_k , and therefore, $\mathcal{CV}(L) = S_{k+1} = \emptyset$. ■

EXAMPLE 2.21. Both cases described in Proposition 2.20 can arise. In Example 3.8 of [16] we have $S = S_0 = \mathbb{R}^2$ and $\mathcal{CV}(L) = S_1 = S_2 = \dots$, with $\text{card } \mathcal{CV}(L) = 10$. By contrast, Example 2.19 of [16] gives $S = \mathbb{R}^2$, S_1 is a set of 9 points, and $\mathcal{CV}(L) = S_2 = S_3 = \dots$, with $\text{card } \mathcal{CV}(L) = 8$.

REMARK 2.22. Note that the iterative construction of the core variety in terms of S_0, S_1, \dots , can be carried out in the setting of a general nonempty set S and a linear functional L acting on a finite dimensional vector space of real valued functions on S . All of our results concerning the connection between the core variety and the cone of finitely atomic representing measures can then be re-formulated in terms of C , the conical hull of the point evaluations.

2.1. COORDINATIZED PERSPECTIVE. A different way to think about V is to pick an explicit basis f_1, \dots, f_k of V where $k = \dim V$. Now we can identify V^* with \mathbb{R}^k by identifying a linear functional $L \in V^*$ with its vector of values $(L(f_1), \dots, L(f_k))$ on the basis. We will use S^* to denote the set of all linear functionals L_v which come from point evaluations on S . Define the map $\varphi_V : S \rightarrow \mathbb{R}^k$ by

$$\varphi_V(v) = (f_1(v), \dots, f_k(v)).$$

Under the above identification S^* is the image of S under the map φ_V :

$$S^* = \varphi_V(S).$$

For instance, when V is the vector space of polynomial functions of degree at most d and S is a subset of \mathbb{R}^n , the set S^* is the image of S under the d -th Veronese map ν_d , which sends $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ to all monomials x^α of degree at most d :

$$\nu_d : \mathbb{R}^n \rightarrow \mathbb{R}^{\binom{n+d}{d}}, \quad \nu_d(x_1, \dots, x_n) = (1, x_1, \dots, x^\alpha, \dots) \quad \text{where } |\alpha| \leq d.$$

3. TRUNCATED RIESZ–HAVILAND THEOREM

We now prove a generalized version of the truncated Riesz–Haviland theorem. Our setting is as follows: let U be a subspace of a finite dimensional vector space V of functions on a nonempty set S ; we equip V with Euclidean topology. Let P_U and P_V denote the respective cones of nonnegative functions. We will assume that $P \equiv P_V$ is full-dimensional, and thus has non-empty interior in V , and also that it contains a strictly positive function ρ . Since all functions in the

interior of P_V are then strictly positive, we may assume that ρ lies in the interior of P_V . Suppose that $L \in U^*$ can be extended to a functional $\hat{L} \in V^*$ such that \hat{L} is V -positive:

$$\hat{L}(p) \geq 0 \quad \text{for all } p \in P_V, \quad \text{and} \quad \hat{L}(f) = L(f) \quad \text{for all } f \in U.$$

We will show that with an additional hypothesis on V and U we may conclude that L is a nonnegative linear combination of point evaluations. In the context of moment problems, this will allow us to conclude that L has a representing measure.

Let $S^* \subset V^*$ be the set of linear functionals in V^* corresponding to point evaluations on S , and let C be the conical hull of S^* in V^* . From Proposition 2.1, we see that if C is closed, then $P^* = \overline{C} = C \subseteq P^*$, so $P^* = C$. If the set S^* is compact, then C is closed ([7], Exercise 4.17), and then V -positivity of \hat{L} already implies that \hat{L} is a nonnegative combination of point evaluations, and therefore the same holds for L . For the case where S is a compact subset of \mathbb{R}^n and $V = \{f|_S : f \in \mathcal{P}_d\}$, Tchakaloff [32] showed that C is closed. It can be shown more generally that if S is a compact metric space and the functions in V are continuous, then S^* is compact in V^* .

In case S^* is not compact, we need to define an appropriate compactification, using a strictly positive function ρ in the interior of P_V . Define a hyperplane $H \in V^*$ consisting of functionals which evaluate to 1 on ρ :

$$H = \{L \in V^* : L(\rho) = 1\}.$$

Let $\tilde{S}^* \subset H$ be the set of functionals $(1/\rho(y))\ell_y$, with $\ell_y \in S^*$. We note that \tilde{S}^* is bounded (\tilde{S}^* lies inside the set $P_V^* \cap H$, which is a base of a closed pointed cone P_V^* , and is therefore compact (cf. p. 66 of [3])), but it may fail to be closed. Finally, let $[\tilde{S}^*]$ be the closure of \tilde{S}^* in H . Now we can state the generalized truncated Riesz–Haviland theorem.

THEOREM 3.1. *Let V be a finite dimensional vector space of functions on S and let U be a subspace of V . Let $L \in U^*$ be a linear functional which admits a V -positive extension. Suppose that for all functionals $\ell \in [\tilde{S}^*] \setminus \tilde{S}^*$ we have $\ell(f) = 0$ for all $f \in U$. Then L is a nonnegative linear combination of point evaluations.*

Proof. From the preceding remarks, we may assume that S^* is not compact. Let M be the cone of representable functionals in V^* , and let C be the conical hull of S^* , so that $C = \text{ConicalHull}(\tilde{S}^*)$. Recall from Proposition 2.1 that $\overline{C} = \overline{M} = P_V^*$. Since \tilde{S}^* is bounded, it follows that $\text{ConicalHull}([\tilde{S}^*]) = \text{clos}(\text{ConicalHull}(\tilde{S}^*))$, whence P_V^* is simply the conical hull of $[\tilde{S}^*]$. Therefore, we can express the extension \hat{L} as a finite conical combination of functionals in $[\tilde{S}^*]$:

$$\hat{L} = \sum_{\ell_i \in \tilde{S}^*} \alpha_i \ell_i + \sum_{m_i \in [\tilde{S}^*] \setminus \tilde{S}^*} \beta_i m_i,$$

where $\alpha_i, \beta_i > 0$. When we restrict \widehat{L} to U we see that all functionals m_i evaluate to 0, and therefore the linear functional L can be written as:

$$L = \sum_{\ell_i \in \widetilde{S}^*} \alpha_i \ell_i.$$

Thus L is a nonnegative combination of point evaluations. ■

EXAMPLE 3.2. In the case when S is compact, the discussion preceding Theorem 3.1 implies that in Theorem 3.1 we may take $U = V$ and $[\widetilde{S}^*] = \widetilde{S}^*$. Thus, if S is compact, L has a representing measure if and only if L is U -positive. For example, the truncated trigonometric moment problem on the interval $[0, 2\pi]$ falls within the scope of this case of Theorem 3.1. Let $U \equiv V$ be the space of trigonometric polynomials p of degree at most n , i.e., $p(t) = a_0 + \sum_{j=1}^n [a_j \cos(jt) + b_j \sin(jt)]$, and let L be a linear functional on U . Classical results show that L has a representing measure if and only if an associated Toeplitz matrix M (the corresponding complex moment matrix) is positive semidefinite ([2], Theorem I.I.12). This problem may be reformulated in terms of trigonometric polynomials on the unite circle \mathbb{T} . Since, by the Fejér–Riesz theorem [13], each nonnegative trigonometric polynomial on \mathbb{T} of degree n is the complex square of an analytic polynomial of degree n , it follows that M is positive semi-definite if and only if L is a positive functional, so the case of Theorem 3.1 where S is compact and the functions in V are continuous is consistent with the classical solution of the truncated trigonometric moment problem.

We next use Theorem 3.1 to present a concrete realization of the truncated Riesz–Haviland theorem in a case where S has special properties. Let S be a locally compact Hausdorff space and let $C(S)$ denote the vector space of all continuous real-valued functions on S . A function $f : S \rightarrow \mathbb{R}$ *vanishes at infinity* if, for each $\varepsilon > 0$, there is a compact set $C_\varepsilon \subseteq S$ such that $S \setminus C_\varepsilon \subseteq \{x \in S : |f(x)| < \varepsilon\}$ (cf. p. 65 of [9]).

COROLLARY 3.3. *Suppose that there exists $\rho \in V$ in the interior of P_V such that f/ρ vanishes at infinity for all $f \in U$. Let $L \in U^*$ be a linear functional which admits a V -positive extension. Then L has a finitely atomic representing measure.*

Proof. Let ℓ be a linear functional in $[\widetilde{S}^*] \setminus \widetilde{S}^*$. Then ℓ is the limit of a sequence of rescaled point evaluations $(1/\rho(x_i))\ell_{x_i}$ with $x_i \in S$. Take $f \in U$ and consider the sequence of real numbers

$$\frac{1}{\rho(x_i)} \ell_{x_i}(f) = \frac{f(x_i)}{\rho(x_i)}.$$

If this sequence converges to 0 for all $\ell \in [\widetilde{S}^*] \setminus \widetilde{S}^*$ and all $f \in U$, then we can apply Theorem 3.1. Suppose that there exist $\ell \in [\widetilde{S}^*] \setminus \widetilde{S}^*$ and $f \in U$ such that the above sequence of values does not converge to 0. Then there exists a subsequence

y_i such that $|f(y_i)/\rho(y_i)| \geq \varepsilon$ for some $\varepsilon > 0$. Since f/ρ vanishes at infinity, the points y_i are contained in a compact subset of S , and some subsequence converges to a point $y \in S$. By continuity, it follows that $\ell = \ell_y/\rho(y)$ and therefore $\ell \in \tilde{S}^*$, which is a contradiction. ■

REMARK 3.4. Corollary 3.3 readily implies the polynomial case of the truncated Riesz–Haviland theorem [11]. Let P_k denote the vector space n -variate polynomials of degree at most k . To recover [11] from Corollary 3.3, let $S \subseteq \mathbb{R}^n$, $U = P_{2d-1}$ or $U = P_{2d-2}$, $V = P_{2d}$, and let $\rho(x) = 1 + \|x\|^{2d}$; clearly, for $f \in U$, $f/\rho \in C_0(S)$.

EXAMPLE 3.5. The exponents of moments for the polynomial case of the truncated Riesz–Haviland theorem in the preceding remark come from the dilations of the standard simplex. Using Theorem 3.1 and Corollary 3.3, this can be readily extended to moments coming from other polytopes as well. For instance, let $K \subset \mathbb{R}^n$ be the box $[0, a_1] \times \cdots \times [0, a_n]$, with $a_i \in \mathbb{N}$ and let $K' \subset \mathbb{R}^n$ be the box $K' = [0, b_1] \times \cdots \times [0, b_n]$, with each b_i the smallest even integer strictly greater than a_i . Let U be the vector space of n -variate polynomials with exponents coming from integer points of K , and let V be the vector space of n -variate polynomials with exponents coming from integer points of K' . Let $S \subseteq \mathbb{R}^n$, and let $\rho = 1 + \sum_{i=1}^n x_i^{b_i}$. Then ρ is in the interior of P_V and for $f \in U$, $f/\rho \in C_0(S)$, so Corollary 3.3 applies.

Continuing with the setting of locally compact Hausdorff spaces, we next derive analogues of the Riesz–Haviland theorem and of Theorem 2.10 for the generalized full moment problem. Let $C_0(S)$ denote the Banach space of all continuous functions on S which vanish at infinity, equipped with the norm $\|f\|_\infty := \sup_{x \in S} |f(x)|$. The space $C_c(S)$ of all continuous functions on S with compact support is dense in $C_0(S)$. The Riesz representation theorem ([9], Appendix C, Theorem 18) states that $C_0(S)^*$, the dual space of $C_0(S)$, is isometrically isomorphic to $M(S)$, the Banach space of finite regular Borel measures on S (equipped with the norm $\|\mu\| := |\mu|(S)$); under this duality, corresponding to $\mu \in M(S)$ is the functional $\hat{\mu}$ on $C_0(S)$ defined by $\hat{\mu}(f) = \int_S f d\mu$.

Recall that $B_1(C_0(S)^*)$, the closed unit ball of $C_0(S)^*$, is weak* compact (Alaoglu’s theorem, [9], Chapter V, Theorem 3.1). In the sequel we further assume that S is σ -compact and metrizable (e.g., $S = \mathbb{R}^n$). In this case, $C_0(S)$ is separable ([9], Chapter V, Section 5, Exercise 2, p. 136) and therefore $B_1(C_0(S)^*)$ is a weak* compact metric space ([9], Theorem V.3.1); thus each bounded sequence in $C_0(S)^*$ has a weak* convergent subsequence, a property that we will utilize below.

For the polynomial case and $S \subseteq \mathbb{R}^n$, in [30] Stochel proved that a multi-sequence $\beta^{(\infty)}$ has an S -representing measure if and only if each truncation $\beta^{(d)}$

($d \geq 1$) has an S -representing measure. We next present an analogue of this result for the generalized full moment problem.

In the sequel, S is a σ -compact, locally compact metric space. Let V be a linear subspace of $C(S)$ which admits a decomposition as a countable union of increasing finite dimensional subspaces, i.e., $V = \bigcup_{j=1}^{\infty} V_j$, $V_{j-1} \subseteq V_j$, $\dim V_j < \infty$ ($j \geq 1$). By a *representing measure* for a linear functional L on V we mean a finite positive regular Borel measure μ on S such that $L(f) = \int_S f d\mu$, ($f \in V$); in particular, μ is Radon.

THEOREM 3.6. *Suppose $1 \in V_1$ and suppose that for each $j \geq 2$, there exists a strictly positive function $\rho_j \in V_j$ such that $q/\rho_j \in C_0(S)$ for all $q \in V_{j-1}$. A linear functional $L : V \rightarrow \mathbb{R}$ has a representing measure if and only if each truncation $L_j := L|_{V_j}$ ($j \geq 1$) has a representing measure.*

Proof. If μ is a representing measure for L , then clearly μ is a representing measure for L_j ($j \geq 1$). For the converse, suppose μ_j is a representing measure for L_j ($j \geq 1$). For each j , $\int_S 1 d\mu_j = L_j(1) = L_1(1)$, so $\{\widehat{\mu_j}\}_{j=1}^{\infty}$ is bounded in $C_0(S)^*$. It follows that a subsequence, which we also denote by $\{\widehat{\mu_j}\}$, is weak* convergent in $C_0(S)^*$, say to μ , a positive finite regular Borel measure on S . Let $\{q_i\}_{i=1}^{\infty}$ denote a basis for V . To complete the proof it suffices to prove that $L(q_i) = \int_S q_i d\mu$ ($i \geq 1$). Fix $i \geq 1$, and let k be such that $q_i \in V_{k-1}$. Since the V_j 's are increasing, for $j \geq k-1$, we have $\int_S q_i d\mu_j = L_j(q_i) = L(q_i)$. Since $q_i/\rho_k \in C_0(S)$, we have

$$\lim_{j \rightarrow \infty} \int_S \frac{q_i}{\rho_k} d\mu_j = \int_S \frac{q_i}{\rho_k} d\mu.$$

For each $f \in C_c(S)$, $f \cdot \rho_k \in C_c(S)$, so

$$\lim_{j \rightarrow \infty} \int_S f \cdot \rho_k d\mu_j = \int_S f \cdot \rho_k d\mu.$$

Now

$$\lim_{j \rightarrow \infty} \|\rho_k d\mu_j\| = \lim_{j \rightarrow \infty} \int_S \rho_k d\mu_j = \int_S \rho_k d\mu_k = L(\rho_k) < +\infty,$$

so $\{\widehat{\rho_k d\mu_j}\}_{j=1}^{\infty}$ is bounded in $C_0(S)^*$ and convergent to $\widehat{\rho_k d\mu}$ on the dense subspace $C_c(S)$. Thus, $\{\widehat{\rho_k d\mu_j}\}_{j=1}^{\infty}$ is weak* convergent to $\widehat{\rho_k d\mu}$ in $C_0(S)^*$. It now follows that

$$L(q_i) = \int_S q_i d\mu_{k-1} = \lim_{j \rightarrow \infty} \int_S q_i d\mu_j = \lim_{j \rightarrow \infty} \int_S \frac{q_i}{\rho_k} (\rho_k d\mu_j) = \int_S q_i d\mu. \quad \blacksquare$$

To recover Stochel's result from Theorem 3.6, let $S = \mathbb{R}^n$, $V = \mathbb{R}[x] \equiv \mathbb{R}[x_1, \dots, x_n]$, and, for $j \geq 1$, let $V_j = \mathcal{P}_{2j}$ and $\rho_j = 1 + \|x\|^{2j}$.

EXAMPLE 3.7. Continuing Example 3.2, we see that in the case when S is a compact metric space and the functions on V are continuous, Corollary 3.3 may be reformulated without any requirement for the existence of the functions ρ_j , and similarly in Theorem 3.6; equivalently, we may take each $\rho_j \equiv 1$. The full trigonometric moment problem, with moment functional L , falls within the scope of these results. It is known that a representing measure exists in this problem if and only if the associated infinite Toeplitz matrix is positive semidefinite ([1], Theorem 5.1.2) and it follows as in Example 3.2 that this condition is equivalent to the positivity of L .

We next have an analogue of the Riesz–Haviland theorem for the generalized full moment problem.

THEOREM 3.8. *Let V be as in Theorem 3.6. A linear functional $L : V \rightarrow \mathbb{R}$ has a representing measure if and only if L is V -positive.*

Proof. It is clear that if L has a representing measure, then L is V -positive. For the converse, suppose L is V -positive. For each $j \geq 2$, $U \equiv V_{j-1}$, $V \equiv V_j$, $\rho \equiv \rho_j$, and $L \equiv L_j$ satisfy the conditions of Corollary 3.3, which implies that L_{j-1} has a finitely atomic representing measure. It now follows from Theorem 3.6 that L has a representing measure. ■

REMARK 3.9. When $V = C_0(S)$, Theorem 3.8 is essentially equivalent to the Riesz representation theorem for positive linear functionals ([9], Appendix C, Theorem 18). The latter result actually holds for *any* locally compact Hausdorff space without assuming σ -compactness.

Let V be as in Theorem 3.6 and let L be a linear functional on V . Since $V_{j-1} \subseteq V_j$, it follows that $\mathcal{CV}(L_j) \subseteq \mathcal{CV}(L_{j-1})$. We now define the *core variety* of L by $\mathcal{CV}(L) := \bigcap_{j=1}^{\infty} \mathcal{CV}(L_j)$. To show that $\mathcal{CV}(L)$ is well-defined, we show that it is independent of the decomposition of V as an increasing union of finite dimensional subspaces. Note first that if $\{V_{j_k}\}_{k=1}^{\infty}$ is a subsequence of $\{V_j\}$, then $\bigcap_{k=1}^{\infty} \mathcal{CV}(L|_{V_{j_k}}) = \bigcap_{j=1}^{\infty} \mathcal{CV}(L_j)$. Now suppose $V = \bigcup_{i=1}^{\infty} W_i$, where each W_i is finite dimensional and satisfies $W_i \subseteq W_{i+1}$. It follows from the finite dimensionality of the subspaces W_i and V_j that for each k , there exist i_k and j_k , with $i_k < i_{k+1}$, $j_k < j_{k+1}$, satisfying $W_{i_k} \subseteq V_{j_k} \subseteq W_{i_{k+1}}$, and from this we see that $\bigcap_{i=1}^{\infty} \mathcal{CV}(L|_{W_i}) = \bigcap_{k=1}^{\infty} \mathcal{CV}(L|_{W_{i_k}}) = \bigcap_{k=1}^{\infty} \mathcal{CV}(L|_{V_{j_k}}) = \bigcap_{j=1}^{\infty} \mathcal{CV}(L_j)$.

We conclude with an analogue of Theorem 2.10.

THEOREM 3.10. *A nonzero linear functional $L : V \rightarrow \mathbb{R}$ has a representing measure if and only if $\mathcal{CV}(L) \neq \emptyset$.*

Proof. Let μ be a representing measure for L . For each j , μ is a representing measure for L_j , and μ is Radon, so Theorem 2.10(ii) implies that $\text{supp } \mu \subseteq \mathcal{CV}(L_j)$. Thus, $\text{supp } \mu \subseteq \mathcal{CV}(L)$, so $\mathcal{CV}(L)$ is nonempty. For the converse, if $\mathcal{CV}(L) \neq \emptyset$, then for each j , $\mathcal{CV}(L_j)$ is nonempty, so Theorem 2.10 implies that L_j has a finitely atomic representing measure. It now follows from Theorem 3.6 that L has a representing measure. ■

EXAMPLE 3.11. In the classical full moment problem for $n = 1$, consider a real sequence $\beta \equiv \beta^{(\infty)} = \{\beta_i\}_{i=0}^\infty$, with Riesz functional L_β , for which the corresponding Hankel matrix $M := (\beta_{i+j})_{i,j=0}^\infty$ is positive definite. For each truncation $\beta^{(2d)}$, since the corresponding Hankel matrix (moment matrix) M_d is positive definite, ([16], Proposition 3.1) implies that $\mathcal{CV}(L_{\beta^{(2d)}}) = \mathbb{R}$. Thus $\mathcal{CV}(L_\beta) = \mathbb{R}$, so Theorem 3.10 implies that L_β has a representing measure, in keeping with Hamburger's theorem (cf. [1], [2]).

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