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The core variety of a multisequence in the truncated moment problem

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ABSTRACT

Let $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\le m}$, $\beta_0 > 0$, denote a real n-dimensional multisequence of finite degree m. The Truncated Moment Problem concerns the existence of a positive Borel measure μ, supported in $\mathbb{R}^n$, such that

$$\beta_i = \int_{\mathbb{R}^n} x^i d\mu \qquad (i \in \mathbb{Z}_+^n, \quad |i| \le m). \tag{0.1}$$

We associate to $\beta \equiv \beta^{(2d)}$ an algebraic variety in $\mathbb{R}^n$ called the *core variety*, $\mathcal{V} \equiv \mathcal{V}(\beta)$. The core variety contains the support of each representing measure μ. We show that if $\mathcal{V}$ is nonempty, then $\beta^{(2d-1)}$ has a representing measure. Moreover, if $\mathcal{V}$ is a nonempty compact or determining set, then $\beta^{(2d)}$ has a representing measure. We also use the core variety to exhibit a sequence β, with positive definite moment matrix and positive Riesz functional, which fails to have a representing measure.

1. Introduction

Let $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\le m}$, $\beta_0 > 0$, denote a real n-dimensional multisequence of finite degree m, and let K denote a closed subset of $\mathbb{R}^n$. The *Truncated K-Moment Problem* for β (TKMP) concerns the existence of a positive Borel measure μ, supported in K, such that

$$\beta_i = \int_K x^i d\mu \qquad (i \in \mathbb{Z}_+^n, \quad |i| \le m). \tag{1.1}$$

(Here, for $x \equiv (x_1, \dots, x_n) \in \mathbb{R}^n$ and $i \equiv (i_1, \dots, i_n) \in \mathbb{Z}_+^n$, we set $|i| = i_1 + \cdots + i_n$ and $x^i = x_1^{i_1} \cdots x_n^{i_n}$.) A measure μ as in (1.1) is a *K-representing measure* for β; for $K = \mathbb{R}^n$, we refer to TKMP simply as the *Truncated Moment Problem* (TMP) and to μ as a *representing measure*. In the sequel, unless otherwise

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noted, we assume m is even, $m \equiv 2d$ (so the moment data completely define a moment matrix, as described below). By a *concrete solution* to the truncated K-moment problem we mean a set of necessary and sufficient conditions for K-representing measures that can be effectively applied in numerical examples. Concrete solutions, valid for all $d \geq 1$, are known only in a few cases. These include, for $n = 1$, $K = \mathbb{R}$, $[0, +\infty)$, and $[a, b]$ (cf. [10]), and for $n = 2$, when K is a curve $p(x, y) = 0$ with $deg\ p \leq 2$ (cf. [15,17]), and for certain curves of higher degree [25]. TKMP is motivated in part by the much-studied *Full K-Moment Problem* for $\beta^{(\infty)} \equiv \{\beta_i\}_{i \in \mathbb{Z}_+^n}$ (cf. [1,2,34,38,43–45,48]). A result of J. Stochel [47] shows that $\beta^{(\infty)}$ has a K-representing measure if and only if $\beta^{(m)}$ has a K-representing measure for every $m \geq 1$. Furthermore, there are close connections between truncated moment problems, polynomial optimization, and positive polynomials, as described in several recent surveys (cf. [32,35,36]).

In previous work with R.E. Curto [16] we developed necessary and sufficient conditions for representing measures based on flat extensions of positive moment matrices (cf. Theorems A.1 and A.2 below). The moment matrix approach yields concrete results in a number of significant cases of TMP (e.g.., [11,15,17,19, 21,25]), but at present it is not known how to apply this approach to a general multisequence $\beta^{(2d)}$. In the sequel we introduce an alternate, more geometric, approach to TMP. In Section 2 we associate to $\beta \equiv \beta^{(2d)}$ an algebraic variety in $\mathbb{R}^n$ called the *core variety*, denoted by $\mathcal{V} \equiv \mathcal{V}(\beta)$, and use this to develop certain new necessary or sufficient conditions for representing measures. The core variety contains the support of each representing measure for β. Our results imply that if $\mathcal{V}$ is nonempty, then $\beta^{(2d-1)}$ has a representing measure. Furthermore, β itself has a representing measure if $\mathcal{V}$ is a nonempty compact or determining set. We illustrate techniques for computing $\mathcal{V}$ in certain cases, using sums-of-squares techniques and linear programming, but in general it may be quite difficult to compute the core variety. It appears that advances in the computation of $\mathcal{V}$ may depend on additional techniques from Real Algebraic Geometry, perhaps involving semi-definite programming and semi-algebraic sets, linear matrix inequalities, etc. [3,35].

We begin by recalling several basic necessary conditions for representing measures. Let $M_d \equiv M_d(\beta)$ denote the moment matrix associated with β (cf. Appendix A.1 for additional notation). The rows and columns of M_d are denoted by X^i and are indexed (in degree-lexicographic order) by the monomials x^i in $\mathcal{P}_d \equiv \{p \in \mathbb{R}[x_1, \ldots, x_n] : \ deg\ p \leq d\}$. Corresponding to $p \equiv \sum_{i \in \mathbb{Z}_+^n, |i| \leq d} a_i x^i \in \mathcal{P}_d$ is the element $p(X) \equiv \sum a_i X^i$ of $Col\ M_d$, the column space of M_d; M_d is *recursively generated* if whenever $p,\ q,\ pq \in \mathcal{P}_d$ and $p(X) = 0$, then $(pq)(X) = 0$. Positivity and recursiveness of M_d are necessary conditions for representing measures [10,13].

Let $V \equiv V(M_d)$ denote the *algebraic variety* corresponding to M_d, i.e., $V = \bigcap_{p \in \mathcal{P}_d, p(X) = 0} \mathcal{Z}_p$ (where $\mathcal{Z}_p = \{x \in \mathbb{R}^n : p(x) = 0\}$). The core variety is contained in V and encodes more information as to the existence of representing measures than does V (see the remarks following Theorem 1.3). Nevertheless, $V(M_d)$ is itself a useful invariant. It is known that if $\beta \equiv \beta^{(2d)}$ has a representing measure μ, then $supp\ \mu \subseteq V(M_d)$ and $rank\ M_d \leq card\ supp\ \mu$ [13]. It follows that another necessary condition for representing measures is the "variety condition", i.e., $rank\ M_d \leq card\ V(M_d)$ (cf. [13, (1.7), p. 6]). In some cases, certain combinations of the preceding conditions are sufficient for representing measures. Thus, for $n = 1$, β has a representing measure if and only if M_d is positive semidefinite and recursively generated (cf. [10] and Theorem 2.21). Further, for $n = 2$, with $K = \mathcal{Z}_p$ for a polynomial $p(x, y)$ of degree 1 or 2, and $d \geq deg\ p$, β has a K-representing measure if and only if M_d is positive, recursively generated, satisfies the variety condition, and has a column relation $p(X, Y) = 0$ (cf. [12,15,17,26]). In general, however, the preceding conditions are not sufficient for representing measures [20,25].

To motivate our results, we recall the role of positive Riesz functionals in the moment problem. For $\beta \equiv \beta^{(m)}$, the *Riesz functional* $L_\beta : \mathcal{P}_m \longrightarrow \mathbb{R}$ is defined by $L_\beta(\sum_{i \in \mathbb{Z}_+^n, |i| \leq m} a_i x^i) = \sum a_i \beta_i$. If μ is a K-representing measure for β and $p \in \mathcal{P}_m$ satisfies $p|_K \geq 0$, then $L_\beta(p) = \int_K p\ d\mu \geq 0$, so in this sense L_β is K*-positive*; for $K = \mathbb{R}^n$, we say simply that L_β is *positive*. In the Full K-Moment Problem for $\beta \equiv \beta^{(\infty)}$, a classical theorem of M. Riesz ($n = 1$) [41] and E.K. Haviland ($n > 1$) [30] shows that β has a K-representing measure if and only if the corresponding functional L_β is K*-positive*, i.e., for $p \in \mathbb{R}[x_1, \ldots, x_n]$, if $p|_K \geq 0$, then $L_\beta(p) \geq 0$. The *proof* of Tchakaloff's Theorem [49] shows that the direct analogue of Riesz–Haviland holds in TKMP if K is *compact* (see Theorem A.3 below), but in general the direct analogue of Riesz–Haviland for TKMP is not valid (cf. (A.3) [22]). Nevertheless, there is an appropriate analogue, as we next describe.

Theorem 1.1 *(Truncated Riesz–Haviland theorem [18]). Let $\beta \equiv \beta^{(2d)}$ or $\beta \equiv \beta^{(2d+1)}$. β has a K-representing measure if and only if β admits an extension to a sequence $\widetilde{\beta} \equiv \widetilde{\beta}^{(2d+2)}$ such that $L_{\widetilde{\beta}}$ is K-positive.*

Theorem 1.1 is not, by itself, a concrete solution to TKMP because in numerical examples it may be very difficult to verify K-positivity. For example, with $K = \mathbb{R}^n$, $n \geq 2$, and $d \geq 3$, there is no concrete description of the polynomials $p \in \mathcal{P}_{2d}$ satisfying $p|K \geq 0$ (cf. Appendix A), so there is no direct method for verifying that $L_{\widetilde{\beta}}$ is K-positive. Several authors have addressed this issue from a variety of viewpoints (cf. [28,31,37,50]). In the sequel we use the core variety to circumvent this difficulty, at least in some cases (cf. Section 3). The approach we take is based in part on a refinement of K-positivity. Following [27], for $\beta \equiv \beta^{(m)}$, we say that L_β is *strictly K-positive* if L_β is K-positive and the conditions $p \in \mathcal{P}_m$, $p|_K \geq 0$, and $p|_K \not\equiv 0$ imply $L_\beta(p) > 0$. If $K = \mathbb{R}^n$ and L_β is strictly K-positive, we say that L_β is *strictly positive*. Further, K is a *determining set* for $\mathcal{P}_m$ if the conditions $p \in \mathcal{P}_m$ and $p|_K \equiv 0$ imply $p \equiv 0$. (If K has nonempty interior, then K is a determining set, but certain finite sets are also determining sets (see Appendix A.2 below).) Strict positivity leads to the following existence criterion.

Theorem 1.2. *([27, Theorem 1.3]) For $\beta \equiv \beta^{(m)}$, if K is a determining set for $\mathcal{P}_m$ and L_β is strictly K-positive, then β has a K-representing measure.*

Our main result provides the following sufficient conditions for positivity of L_β and representing measures for β.

Theorem 1.3. *Let $\beta \equiv \beta^{(2d)}$. If the core variety $\mathcal{V} \equiv \mathcal{V}(\beta)$ is nonempty, then L_β is strictly $\mathcal{V}$-positive and $\beta^{(2d-1)}$ has a $\mathcal{V}$-representing measure. Moreover, if $\mathcal{V}$ is nonempty and is either compact or a determining set for $\mathcal{P}_{2d}$, then $\beta^{(2d)}$ has a $\mathcal{V}$-representing measure.*

A theorem of Bayer and Teichmann [4], generalizing Tchakaloff's Theorem on multivariable cubature [49] (cf. Appendix A.2), implies that if m is finite and $\beta \equiv \beta^{(m)}$ has a K-representing measure, then β has a *finitely atomic* K-representing measure (cf. [36,25]). It follows that the representing measures in Theorem 1.3 can always be taken to be finitely atomic. By way of contrast with Theorem 1.3, examples are known in which the variety $V(M_d)$ is a nonempty determining set, but L_β is not positive (cf. Example 3.5-iii)), or in which $V(M_d)$ is nonempty and compact, but β has no representing measure (cf. (A.3) below).

Several results in the literature relate the rank of a singular positive moment matrix M_d to the existence of a representing measure. The heuristic here is that the more dependence relations there are in the columns of M_d, the easier it is to determine whether or not β has a representing measure. Indeed, a basic result of [11,16] implies that if $M_d \succeq 0$ and $rank\ M_d = rank\ M_{d-1}$ (i.e., M_d is *flat* in the sense of [11]), then β has a unique representing measure. Moreover, a remarkable recent result of G. Blekherman [7] implies that if $M_d \succeq 0$ and $rank\ M_d \leq 3d - 3$ (with $d \geq 3$), then L_β is positive (whence $\beta^{(2d-1)}$ has a representing measure by Theorem 1.1). By contrast, relatively little is known about the case when M_d is positive definite ($M_d \succ 0$), which is the focus of Section 3. The cases of $M_d \succ 0$ within the scope of Hilbert's theorem on sums of squares (cf. Appendix A.2) have representing measures, but the first case beyond the scope of Hilbert's theorem, when $n = 2$ and $d = 3$, is largely unsolved. A result of K. Schmüdgen [42] implies an example for $n = 2$, $d = 3$ in which $M_3 \succ 0$, but $L_{\beta^{(6)}}$ is not even positive (cf. Example 3.5-iii)). For the case $n = 2$, $d = 3$ and $M_3 \succ 0$, we show in Theorem 3.4 that either $\mathcal{V} = \mathbb{R}^2$ (and there is a measure), *card* $\mathcal{V} = 10$ (and there is a measure), or $\mathcal{V} = \emptyset$ (and there is no measure). Theorem 1.2 implies that if $M_d \succ 0$ and L_β is strictly positive, then β has a representing measure, and in [27, Question 1.2] we asked whether the same conclusion holds if $M_d \succ 0$ and L_β is merely positive. In Section 3 (Theorem 3.8), we use the core variety to resolve this question as follows.

Theorem 1.4. *For $n = 2$, $d = 3$, there exists $\beta \equiv \beta^{(6)}$ such that $M_3 \succ 0$ and L_β is positive, but β has no representing measure.*

In Section 4 we discuss some open questions concerning the core variety. The existence results that we have discussed above are based on convex analysis and basic algebraic geometry; however, the explicit constructions of measures in the examples in the sequel are based as well on positivity and extension properties of moment matrices. To make the exposition largely self-contained, we have collected some background material concerning these topics in Appendix subsections A.1 and A.2.

2. The core variety of a multisequence

In this section we introduce the core variety $\mathcal{V} \equiv \mathcal{V}(\beta)$ of a multisequence $\beta \equiv \beta^{(2d)}$ and derive its basic properties, leading to a proof of Theorem 1.3. We assume that β satisfies the most basic requirement for a representing measure, $M_d \succeq 0$, which is equivalent to the condition that L_β be *square positive*, i.e., $L_\beta(p^2) \geq 0$ $(p \in \mathcal{P}_d)$. Recall the variety $V(M_d) := \bigcap_{p \in \mathcal{P}_d,\ M_d \widehat{p} = 0} \mathcal{Z}_p$, which we now designate by $\mathcal{V}^{(0)}$. In [27, Section 2], we introduced $\mathcal{V}^{(1)} := \bigcap_{p \in ker L_\beta,\ p|_{\mathcal{V}^{(0)}} \geq 0} \mathcal{Z}_p$ as an initial attempt to refine $V(M_d)$. Now, for $i \geq 0$, let

$$\mathcal{V}^{(i+1)} := \bigcap_{p \in ker L_\beta,\ p|_{\mathcal{V}^{(i)}} \geq 0} \mathcal{Z}_p.$$

We define the *core variety* of β (or of $M_d(\beta)$) by $\mathcal{V} \equiv \mathcal{V}(\beta) := \bigcap_{i=0}^{\infty} \mathcal{V}^{(i)}$; we also denote this by $\mathcal{V}(M_d)$.

In the definition of $\mathcal{V}$, we used $\mathcal{V}^{(0)} = V(M_d)$, primarily because $V(M_d)$ is easy to compute. An alternate definition for a core variety can be based on setting $\mathcal{W}^{(0)} := \bigcap_{p \in ker\ L_\beta, p|_{\mathbb{R}^n} \geq 0} \mathcal{Z}_p$, then setting

$$\mathcal{W}^{(i+1)} := \bigcap_{p \in ker L_\beta,\ p|_{\mathcal{W}^{(i)}} \geq 0} \mathcal{Z}_p \ (i \geq 0),$$

and, finally, defining $\mathcal{W} \equiv \mathcal{W}(\beta) := \bigcap_{i=0}^{\infty} \mathcal{W}^{(i)}$. We claim that $\mathcal{W} = \mathcal{V}$. Since $M_d \succeq 0$, for $p \in \mathcal{P}_d$, $M_d(\widehat{p}) = 0 \iff L_\beta(p^2) = 0$, and since $\mathcal{Z}_p = \mathcal{Z}_{p^2}$, it follows that $\mathcal{W}^{(0)} \subseteq \mathcal{V}^{(0)}$. Now, for $p \in ker\ L_\beta$, $p|\mathcal{V}^{(0)} \geq 0 \Longrightarrow p|\mathcal{W}^{(0)} \geq 0$, so $\mathcal{W}^{(1)} \subseteq \mathcal{V}^{(1)}$. Further, if $p \in ker\ L_\beta$ is *psd* ($p|\mathbb{R}^n \geq 0$, cf. Section A.2), then $p|\mathcal{V}^{(0)} \geq 0$, so $\mathcal{V}^{(1)} \subseteq \mathcal{W}^{(0)}$. Thus, $\mathcal{W}^{(1)} \subseteq \mathcal{V}^{(1)} \subseteq \mathcal{W}^{(0)}$, and it is not difficult to prove (by induction) that for $i \geq 1$,

$$\mathcal{W}^{(i)} \subseteq \mathcal{V}^{(i)} \subseteq \mathcal{W}^{(i-1)},$$

whence $\mathcal{W} = \mathcal{V}$. In the sequel we always compute the core variety using $\mathcal{V}$, because $\mathcal{V}^{(0)}$ is much easier to compute than $\mathcal{W}^{(0)}$.

The usefulness of the variety $V(M_d)$ lies in the fact that it contains the support of any representing measure (see the remarks following (A.1)). Our first result shows that the core variety has the same inclusion property as $V(M_d)$, and since it is contained in $V(M_d)$, it provides a better indication of the location of the support.

Proposition 2.1. *If μ is a representing measure for β, then supp $\mu \subseteq \mathcal{V}$.*

The proof of Proposition 2.1 follows immediately from the following result.

Lemma 2.2. *If β has a representing measure μ, then for each $i \geq 0$, supp $\mu \subseteq \mathcal{V}^{(i)}$.*

Proof. The proof is by induction on $i \geq 0$, and for $i = 0$ the result follows from (A.1). Suppose $i \geq 0$ and $supp\ \mu \subseteq \mathcal{V}^{(i)}$. We claim that $supp\ \mu \subseteq \mathcal{V}^{(i+1)}$. If not, there exists $x \in supp\ \mu$ with $x \notin \mathcal{V}^{(i+1)}$. Thus there exists $p \in ker L_\beta$, $p|_{\mathcal{V}^{(i)}} \geq 0$, with $p(x) \neq 0$. Since $x \in supp\ \mu \subseteq \mathcal{V}^{(i)}$, then $p(x) > 0$, whence $0 < \displaystyle\int_{supp\ \mu} p d\mu = L_\beta(p) = 0$, a contradiction. $\square$

Corollary 2.3. *i) If β has a representing measure, then rank $M_d \leq$ card $\mathcal{V}$.*
ii) If μ is a representing measure for β with int(supp μ) $\neq \emptyset$, then $\mathcal{V}(\beta) = \mathbb{R}^n$.

Proof. i) (A.2) shows that if μ is a representing measure for β, then $rank\ M_d \leq card\ supp\ \mu$, so the result follows from Proposition 2.1.

ii) Since a proper affine variety has empty interior in $\mathbb{R}^n$, the result follows from Proposition 2.1. $\square$

We next turn to several results related to computing the core variety.

Lemma 2.4. *For $i \geq 0$, $\mathcal{V}^{(i+1)} \subseteq \mathcal{V}^{(i)}$.*

Proof. The proof is by induction on $i \geq 0$. Let $\mathcal{Q}^{(0)} := \{p \in \mathcal{P}_d : M_d\widehat{p} = 0\}$, and let $\mathcal{Q}^{(i+1)} := \{p \in \mathcal{P}_{2d} : L_\beta(p) = 0,\ p|_{\mathcal{V}^{(i)}} \geq 0\}$ $(i \geq 0)$. Thus, $\mathcal{V}^{(i)} = \bigcap_{p \in \mathcal{Q}^{(i)}} \mathcal{Z}_p$ $(i \geq 0)$. Suppose $q \in \mathcal{Q}^{(0)}$; then $\mathcal{V}^{(0)} \subseteq \mathcal{Z}_q$, $q|_{\mathcal{V}^{(0)}} \equiv 0$, and $L_\beta(q) = L_\beta(q \cdot 1) = \langle M_d\hat{q}, \hat{1}\rangle = 0$. Thus, $q \in \mathcal{Q}^{(1)}$, so $\mathcal{V}^{(1)} = \bigcap_{p \in \mathcal{Q}^{(1)}} \mathcal{Z}_p \subseteq \bigcap_{q \in \mathcal{Q}^{(0)}} \mathcal{Z}_q = \mathcal{V}^{(0)}$. Now assume $i \geq 1$ and $\mathcal{V}^{(i)} \subseteq \mathcal{V}^{(i-1)}$. Let $q \in \mathcal{Q}^{(i)}$, so $q|_{\mathcal{V}^{(i-1)}} \geq 0$. Thus, $q|_{\mathcal{V}^{(i)}} \geq 0$, whence $q \in \mathcal{Q}^{(i+1)}$. Now, $\mathcal{V}^{(i+1)} = \bigcap_{p \in \mathcal{Q}^{(i+1)}} \mathcal{Z}_p \subseteq \bigcap_{q \in \mathcal{Q}^{(i)}} \mathcal{Z}_q = \mathcal{V}^{(i)}$, so the result follows. $\square$

We note for future reference the following implications that are implicit in the proof of Lemma 2.4:

$$p \in kerL_\beta,\ p|_{\mathcal{V}^{(i)}} \geq 0 \Longrightarrow \mathcal{V}^{(i+1)} \subseteq \mathcal{Z}_p \bigcap \mathcal{V}^{(i)} \tag{2.1}$$

$$p \in ker\ L_\beta,\ p|_{\mathcal{V}^{(i)}} > 0 \Longrightarrow \mathcal{V} = \emptyset \tag{2.2}$$

$$\mathcal{V}^{(i)} = \mathcal{V}^{(i+1)} \Longrightarrow \mathcal{V} = \mathcal{V}^{(i)} \tag{2.3}$$

The following result shows that there always exists i for which the hypothesis of (2.3) holds. In the sequel, for a subset $\mathcal{S} \subseteq \mathbb{R}[x] \equiv \mathbb{R}[x_1, \ldots, x_n]$, $Var(\mathcal{S})$ denotes the *variety* of $\mathcal{S}$, i.e., $Var(\mathcal{S}) := \{x \in \mathbb{R}^n : s(x) = 0\ \forall s \in \mathcal{S}\}$ $(= \bigcap_{s \in \mathcal{S}} \mathcal{Z}_s)$; any set of the form $Var(\mathcal{S})$ is an *algebraic* set. For $X \subseteq \mathbb{R}^n$, let $I(X)$ denote the *ideal* of X, i.e., $I(X) = \{p \in \mathbb{R}[x] : p|X \equiv 0\}$. It follows from [8, Ch. 4, Section 2, Theorem 7] [30, page 11, (9)] that for $\mathcal{S} \subseteq \mathbb{R}[x]$, $Var(I(Var(\mathcal{S}))) = Var(\mathcal{S})$.

Proposition 2.5. *There exists $i \geq 0$ such that $\mathcal{V} = \mathcal{V}^{(i)}$; equivalently, $\mathcal{V}^{(i)} = \mathcal{V}^{(i+1)} = \cdots = \mathcal{V}^{(i+k)} = \cdots$ $(k \geq 0)$; equivalently, if $p \in ker\ L_\beta(p)$, and $p|_{\mathcal{V}^{(i)}} \geq 0$, then $\mathcal{V}^{(i)} \subseteq \mathcal{Z}_p$.*

Proof. For $j \geq 0$, let $\mathcal{I}^{(j)} = I(\mathcal{V}^{(j)}) := \{p \in \mathbb{R}[x] : p|_{\mathcal{V}^{(j)}} \equiv 0\}$. Since $\mathcal{V}^{(j+1)} \subseteq \mathcal{V}^{(j)}$ (Lemma 2.4), then $\mathcal{I}^{(j)} \subseteq \mathcal{I}^{(j+1)}$; thus $\mathcal{I}^{(0)} \subseteq \mathcal{I}^{(1)} \subseteq \cdots \subseteq \mathcal{I}^{(j)} \subseteq \mathcal{I}^{(j+1)} \cdots$. Since $\mathbb{R}[x]$ is Noetherian [29, page 13] [9, Ch. 2.5, Thm. 4], it satisfies the ascending chain condition for ideals [9, Thm. 7]. Thus there exists $i \geq 0$ such that $\mathcal{I}^{(i)} = \mathcal{I}^{(i+1)} = \cdots = \mathcal{I}^{(i+k)} = \cdots$ $(k \geq 0)$, whence $Var(\mathcal{I}^{(i)}) = Var(\mathcal{I}^{(i+1)}) = \cdots = Var(\mathcal{I}^{(i+k)}) = \cdots$. Note that $\mathcal{V}^{(j)}$ is an algebraic set since, by definition, $\mathcal{V}^{(j)} = Var(\mathcal{S}_j)$ for $\mathcal{S}_j := \{p \in ker\ L_\beta : p|\mathcal{V}^{(j-1)} \geq 0\}$. It now follows from [29, page 11, (9)] that $Var(\mathcal{I}^{(j)}) = Var(I(\mathcal{V}^{(j)})) = \mathcal{V}^{(j)}$, and thus $\mathcal{V}^{(i)} = \mathcal{V}^{(i+1)} = \cdots = \mathcal{V}^{(i+k)} = \cdots$. Now, from (2.3), $\mathcal{V} = \mathcal{V}^{(i)}$. Finally, it is clear that $\mathcal{V} = \mathcal{V}^{(i)}$ if and only if $\mathcal{V}^{(i)} = \mathcal{V}^{(i+1)}$, or, equivalently, if $p \in ker\ L_\beta$ and $p|_{\mathcal{V}^{(i)}} \geq 0$ together imply $p|_{\mathcal{V}^{(i)}} \equiv 0$. □

We next illustrate the core variety and the preceding results with an example. In the sequel, we set $r := rank\ M_d$, $v := card\ V(M_d)$, and $\nu := card\ \mathcal{V}$. Recall from [17,26] that for $n = 2$, if M_d is positive, recursively generated, $r \leq v$, and M_2 is singular, then $\beta^{(2d)}$ has a representing measure supported in a planar curve of degree 1 or 2. In [25] we solved TKMP for the case when K is the planar curve $y = x^3$; in particular, we showed that the results of [17] do not extend to planar curves of degree 3. We next compute the core variety for a family of moment matrices with $Y = X^3$ column relations.

Example 2.6. With $n = 2$, consider M_3 given by

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 430 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 430 & S \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 430 & S & T \end{pmatrix}.$$

It is straightforward to check that for each $S \in \mathbb{R}$, M_3 is positive, with $rank\ M_3 = 9$ and a column relation $Y = X^3$, if and only if

$$T > \frac{2048888 + 2856S + S^2}{2}. \tag{2.4}$$

Assuming (2.4) holds, we have $\mathcal{V}^{(0)} = \Gamma \equiv \mathcal{Z}_{y-x^3}$. To compute $\mathcal{V}^{(1)}$ we next describe the polynomials in $ker\ L_\beta$ that are nonnegative on $\mathcal{V}^{(0)}$. By considering a basis for $ker\ L_\beta$, it is not difficult to see that if $Q(x,y) \in ker\ L_\beta$, then Q admits a representation of the form

$$Q(x,y) = F(x,y) + H(x,y)(y - x^3), \tag{2.5}$$

where $deg\ H \leq 3$ and

$$\begin{aligned} F(x,y) := {} & ax + by + c(x^2 - 1) + d(xy - 2) + e(y^2 - 5) + fx^2y + gxy^2 + hy^3 \\ & + j(x^2y^2 - 14) + k(xy^3 - 42) + m(y^2 - 132) + nx^2y^3 + pxy^4 + qy^5 \\ & + r(x^2y^4 - 430) + s(xy^5 - S) + t(y^6 - T), \end{aligned}$$

for scalars $a,\ b,\ c,\ldots$. Clearly, $Q|\Gamma \geq 0$ if and only if $F|\Gamma \geq 0$. Now $P(x) := F(x, x^3)$ is nonnegative for all real x if and only if P can be represented as

$$P(x) = R(x)^2 + U(x)^2, \tag{2.6}$$

for polynomials $R(x) = a_0 + \cdots + a_9x^9$ and $U(x) = b_0 + \cdots + b_9x^9$.

For each choice of R and U, let $W := P - (R^2 + U^2)$. By setting the coefficients of W equal to 0, we see that, starting with the degree 18 term and working downwards to the degree 1 term, each coefficient of F is uniquely determined as a quadratic expression in certain of the a_i and b_i; thus, $t = a_9^2 + b_9^2$, $s = a_8^2 + 2a_7a_9 + b_8^2 + 2b_7b_9, \ldots,\ a = 2a_0a_1 + 2b_0b_1$. By considering the x^{17} term of W we also see that it is necessary that

$$a_8a_9 + b_8b_9 = 0. \tag{2.7}$$

To satisfy (2.6) it thus suffices to chose $\widehat{a} := (a_0, \ldots, a_9)$ and $\widehat{b} := (b_0, \ldots, b_9)$ so that (2.7) holds and the constant term of W equals 0. Consider the real symmetric matrix

$$J := \begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\ 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 \\ 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 \\ 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 \\ 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 \\ 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 430 \\ 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 430 & 0 \\ 0 & 14 & 0 & 42 & 0 & 132 & 0 & 430 & 0 & S \\ 14 & 0 & 42 & 0 & 132 & 0 & 430 & 0 & S & 0 \\ 0 & 42 & 0 & 132 & 0 & 430 & 0 & S & 0 & T \end{pmatrix}.$$

A calculation shows that the constant term of W can be expressed as

$$\kappa := \langle J\widehat{a}, \widehat{a}\rangle + \langle J\widehat{b}, b\rangle. \tag{2.8}$$

Thus, (2.6) holds if and only if there is a choice of $\widehat{a}$ and $\widehat{b}$ such that (2.7) holds and $\kappa = 0$. Since (2.4) holds, a further calculation shows that J is positive semidefinite if and only if $S \geq 1444$.

We first consider the case $S > 1444$. In this case, J is positive definite, so the only solution to (2.8) is $\widehat{a} = \widehat{b} = 0$, which implies $F(x, y) \equiv 0$ (because the coefficients of F are, from (2.6), quadratic polynomials in the a_i and b_i). In this case, $\mathcal{Z}_{Q(x,y)|\Gamma} = \Gamma$, so we have $\mathcal{V}^{(1)} = \mathcal{V}^{(0)}$, whence (2.3) implies $\mathcal{V} = \Gamma$.

For the case $S = 1444$, *dim ker* $J = 1$, and a calculation shows that the non-trivial solutions to (2.6) are of the form $P(x) = \alpha Z(x)^2$ ($\alpha > 0$), where

$$Z(x) = x^8 - 8x^6 + 20x^4 - 16x^2 + 2. \tag{2.9}$$

$Z(x)$ has 8 distinct real roots, given by $x_1 = -\sqrt{2 - \sqrt{2 - \sqrt{2}}}$, $x_2 = \sqrt{2 - \sqrt{2 - \sqrt{2}}}$, $x_3 = -\sqrt{2 + \sqrt{2 - \sqrt{2}}}$, $x_4 = \sqrt{2 + \sqrt{2 - \sqrt{2}}}$, and with x_{4+i} obtained from x_i by replacing the innermost $2 - \sqrt{2}$ by $2 + \sqrt{2}$ ($1 \leq i \leq 4$). Thus, if $Q|\Gamma \geq 0$, then $\mathcal{Z}_Q \bigcap \mathcal{V}^{(0)} = \mathcal{S} := \{(x_i, x_i^3)\}_{i=1}^8$, whence $\mathcal{V}^{(1)} = \mathcal{S}$. Now, *card* $\mathcal{V} \leq$ *card* $\mathcal{V}^{(1)} \leq 8 < 9 =$ *rank* M_3, so Corollary 2.3-i) implies that β has no representing measure.

Although we already know that there is no representing measure, we next show explicitly that $\mathcal{V} = \mathcal{V}^{(2)} = \emptyset$. To do this, we use the form of $F(x, y)$ (above) to identify polynomials $F_1(x, y)$ and $F_2(x, y)$ that belong to $ker\ L_\beta$, are nonnegative on $\mathcal{V}^{(1)}$, and satisfy $\mathcal{Z}_{F_1} \bigcap \mathcal{Z}_{F_2} = \emptyset$. With the aid of linear programming, we find two such polynomials defined (approximately) by $F_1(x,y) := -2.45681(x^2y^2 - 430) - 2.72738(xy^5 - 1444) + y^6 - 4981$ (which satisfies $F_1(x_i, x_i^3) = 0$ $(1 \le i \le 4)$ and $F_1(x_i, x_i^3) > 0$ $(5 \le i \le 8)$), and $F_2(x,y) := -1.35840(x^2y^2 - 430) + 3.85396(xy^5 - 1444) - y^6 + 4981$ (which satisfies $F_2(x_i, x_i^3) > 0$ $(1 \le i \le 4)$ and $F_2(x_i, x_i^3) = 0$ $(5 \le i \le 8)$).

Finally, we consider the case $S < 1444$. Let J_8 denote the compression of J to the first 8 rows and columns, and let J_9 denote the compression of J to the first 9 rows and columns. Now $J_8 \succ 0$ and $det\ J_9 < 0$. It follows readily that there exist vectors $\widehat{a}$ and $\widehat{b}$, with $a_9 = b_9 = 0$, such that $\langle J\widehat{a}, \widehat{a}\rangle = 1$ and $\langle J\widehat{b}, \widehat{b}\rangle = -1$. Thus (2.8) implies that there exist nonzero polynomials $R(x)$ and $U(x)$, each with degree at most 8, such that (2.6) holds. Since $\mathcal{Z}_P = \mathcal{Z}_R \bigcap \mathcal{Z}_U$, it follows that $Q(x,y)$ has at most 8 distinct zeros in Γ, and since $rank\ M_3 = 9$, it follows from Corollary 2.3-i) that β has no representing measure.

This example is consistent with the solution to TMKP for $K = \Gamma$ in [25]. In [25], for $n = 2$, if $M_d \succeq 0$ and the only dependence relations in $Col\ M_d$ correspond to multiples of $y - x^3$, then there is a rational function of the moment data, $\psi(\beta)$, such that β has a representing measure if and only if $\beta_{1,2d-1} > \psi(\beta)$. In each case of the preceding example, $\psi(\beta) = 1444$. [25] depends on the Bayer–Teichmann Theorem [4] and on the Flat Extension Theorem (Theorem A.1), so the core variety provides a clearer and more direct explanation of this example. The method of this example can also be used to study $\mathcal{V}$ whenever $n = 2$ and $\mathcal{V}^{(0)} = \mathcal{Z}_p$, where $p(x,y) = y - q(x)$ for some univariate polynomial $q(x)$. $\square$

In view of Proposition 2.5, to compute $\mathcal{V}$ we must study the transition from $\mathcal{V}^{(i)}$ to $\mathcal{V}^{(i+1)}$. The next result provides an abstract characterization of this transition.

Proposition 2.7. *For each $i \ge 0$, there exists $Q_i \in ker\ L_\beta$ such that $Q_i|_{\mathcal{V}^{(i)}} \ge 0$ and $\mathcal{V}^{(i+1)} = \mathcal{V}^{(i)} \bigcap \mathcal{Z}_{Q_i}$.*

Proof. Let $\mathcal{S} := \{p \in ker\ L_\beta :\ p|_{\mathcal{V}^{(i)}} \ge 0\}$ and let J denote the ideal in $\mathbb{R}[x_1, \ldots, x_n]$ generated by $\mathcal{S}$. The Hilbert Basis Theorem [9, Theorem 4, p. 74] [29, p. 13] implies that there exists a finite collection $p_1, \ldots, p_m$ in $\mathcal{S}$ such that $J = (p_1, \ldots, p_m)$ (the ideal generated by $p_1, \ldots, p_m$). Thus, $\mathcal{V}^{(i+1)} = \bigcap\limits_{p \in \mathcal{S}} \mathcal{Z}_p = \bigcap\limits_{k=1}^{m} \mathcal{Z}_{p_k}$. Let $Q \equiv Q_i := p_1 + \cdots + p_m$; clearly, $\bigcap\limits_{k=1}^{m} \mathcal{Z}_{p_k} \subseteq \mathcal{Z}_Q$. For $1 \le k \le m$, since $p_k|_{\mathcal{V}^{(i)}} \ge 0$, if $x \in \mathcal{V}^{(i)}$ and $Q(x) = 0$, then $x \in \mathcal{Z}_{p_k} \bigcap \mathcal{V}^{(i)}$. Thus, $\mathcal{Z}_Q \bigcap \mathcal{V}^{(i)} \subseteq (\bigcap\limits_{k=1}^{m} \mathcal{Z}_{p_k}) \bigcap \mathcal{V}^{(i)} \subseteq \mathcal{Z}_Q \bigcap \mathcal{V}^{(i)}$. Now, $\mathcal{Z}_Q \bigcap \mathcal{V}^{(i)} = (\bigcap\limits_{k=1}^{m} \mathcal{Z}_{p_k}) \bigcap \mathcal{V}^{(i)}$ $= \mathcal{V}^{(i+1)} \bigcap \mathcal{V}^{(i)} = \mathcal{V}^{(i+1)}$. $\square$

In numerical examples it may be very difficult to compute Q_i as in Proposition 2.7. We next describe a procedure that can be used in certain examples to show that $\mathcal{V} = \mathcal{V}^{(i+1)} = \mathcal{V}^{(i)}$. Let $\mathcal{N} := \{p \in \mathcal{P}_d :\ \widehat{p} \in ker\ M_d\}$; clearly, $\mathcal{N}$ is a subspace of $\mathcal{P}_d$ and $V(M_d) = \bigcap\limits_{p \in \mathcal{N}} \mathcal{Z}_p$. Note that if $p \in \mathcal{N}$ and $q \in \mathcal{P}_d$, then $g \equiv pq \in ker\ L_\beta$, since $L_\beta(pq) = \langle M_d\hat{p},\ \hat{q}\rangle = 0$. Let $\widetilde{\mathcal{N}}$ denote the subspace of $ker\ L_\beta$ generated by

$$\{g \in \mathcal{P}_{2d} : g = pq\ for\ p \in \mathcal{N},\ q \in \mathcal{P}_d\}.$$

For $g \equiv pq$ (as above), $g|_{V(M_d)} = pq|_{V(M_d)} \equiv 0$, so

$$g|_{\mathcal{V}^{(i)}} \equiv 0, \quad \mathcal{V}^{(i+1)} \subseteq \mathcal{Z}_g \ \ (g \in \widetilde{\mathcal{N}},\ i \ge 0). \tag{2.10}$$

In constructing a basis $\mathcal{B}$ for $ker\ L_\beta$ to use in computing $\mathcal{V}^{(i+1)}$, it is advantageous to include a basis for $\widetilde{\mathcal{N}}$, say $\mathcal{B}_{\widetilde{\mathcal{N}}} \equiv \{g_1, \ldots, g_m\}$. Suppose $\mathcal{B} = \{f_1, \ldots,\ f_p,\ g_1, \ldots, g_m\}$ denotes a basis for $ker\ L_\beta$. Given

$Q \in ker\ L_\beta$, we have $Q \equiv F + G$, where $F := a_1 f_1 + \cdots + a_p f_p$ and $G := b_1 g_1 + \cdots + b_m g_m$ for scalars $a_1, \ldots, a_p, b_1, \ldots, b_m$. If we assume $Q|_{\mathcal{V}^{(i)}} \geq 0$, then since each $g_j|_{\mathcal{V}^{(i)}} \equiv 0$, this is equivalent to $F|_{\mathcal{V}^{(i)}} \geq 0$, i.e.,

$$a_1 f_1(w) + \cdots + a_p f_p(w) \geq 0 \qquad (w \in \mathcal{V}^{(i)}). \tag{2.11}$$

In the sequel we refer to $\{f_1, \ldots, f_p\}$ as a *reduced basis* for $ker\ L_\beta$. In the case where $\mathcal{V}^{(i)}$ is finite, $\mathcal{V}^{(i)} \equiv \{w_1, \ldots, w_s\}$, (2.11) is equivalent to the "reduced" system

$$a_1 f_1(w_j) + \cdots + a_p f_p(w_j) \geq 0 \qquad (1 \leq j \leq s). \tag{2.12}$$

By solving this system, we can gain insight into $\mathcal{Z}_Q$ (which necessarily contains $\mathcal{V}^{(i+1)}$). We may summarize the preceding discussion as follows.

Proposition 2.8. $\mathcal{V} = \mathcal{V}^{(i)} \iff \mathcal{V}^{(i+1)} = \mathcal{V}^{(i)} \iff F|\mathcal{V}^{(i)} \geq 0 \implies F|\mathcal{V}^{(i)} \equiv 0 \iff F|\mathcal{V}^{(i)} \geq 0 \implies a_1 = \cdots = a_p = 0$.

The following lemma shows that in augmenting $\{g_1, \ldots, g_m\}$ to a basis for $ker\ L_\beta$, no f_j can be a sum of squares.

Lemma 2.9. *Suppose $M_d \succeq 0$. If $f \in ker\ L_\beta$ and $f = \sum_{i=1}^{k} p_i^2$ for some $k \geq 1$ and $p_i \in \mathcal{P}_d$ $(1 \leq i \leq k)$, then $f \in \widetilde{\mathcal{N}}$.*

Proof. We have $0 = L_\beta(f) = \sum \langle M_d \widehat{p}_i, \widehat{p}_i \rangle$. Since $M_d \succeq 0$, it follows that each $M_d \widehat{p}_i = 0$, so each $p_i \in \mathcal{N}$ and thus $f \in \widetilde{\mathcal{N}}$. □

Assuming that we have computed $\mathcal{V}^{(i)}$, there are two main cases where we are able to apply Proposition 2.8 to show that $\mathcal{V} = \mathcal{V}^{(i+1)} = \mathcal{V}^{(i)}$. The first case is when $\mathcal{V}^{(i)}$ is finite. In this case, if s and p are sufficiently small, it is possible to solve (2.12) directly, using algebra (cf. Example 2.19 below). But even for slightly larger systems, this direct approach may be very difficult, and computer algebra procedures such as Mathematica's *Reduce*, which solve such systems in principle, seem to exhaust the computer's memory before achieving a solution. In such cases, (2.12) may be solved numerically, using linear programming. More generally, we can use linear programming if we can find $p \in ker\ L_\beta$ such that $p|_{\mathcal{V}^{(i)}} \geq 0$ and $card\ \mathcal{Z}_p < \infty$, for then $supp\ \mu \subseteq \mathcal{V}^{(i+1)} \subseteq \mathcal{Z}_p$ for any representing measure μ. The disadvantage of using linear programming in examples is that it may entail numerical errors which make conclusions suspect. By contrast, if (as in Example 2.6) we are able to solve (2.12) exactly, then we can clearly see why β does or does not have a representing measure geometrically, by the way zero sets of certain polynomials intersect.

The other case where we are able to apply Proposition 2.8 to show $\mathcal{V} = \mathcal{V}^{(i+1)} = \mathcal{V}^{(i)}$ is when $\mathcal{V}^{(i)}$ has the property that each polynomial F that is nonnegative on $\mathcal{V}^{(i)}$ can be expressed as a (weighted) sum of squares. In such cases we can determine algebraically whether or not the condition of Proposition 2.8 holds. We have already illustrated this technique in Example 2.6, and we further illustrate this method in the next two examples. In other cases, it may be difficult to show that $\mathcal{V}^{(i+1)} = \mathcal{V}^{(i)}$. For example if $\mathcal{V}^{(0)} = \mathbb{R}^n$, then to show $\mathcal{V}^{(1)} = \mathcal{V}^{(0)}$ it would be necessary to test the zero sets of each psd polynomial in $ker\ L_\beta$, but there is no known method for doing this. While there are algorithms for proving that a particular polynomial with numerical coefficients is psd [35] (and to then compute its zeros), we cannot do this for arbitrary (i.e., non-numerical) linear combinations of the basis elements of ker L_β. On the other hand, we illustrate in Example 3.5 and Theorem 3.8 cases where $\mathcal{V}^{(0)} = \mathbb{R}^n$ and where we can use Proposition 2.8 to compute $\mathcal{V}$ by showing that $\mathcal{V}^{(i+1)} \neq \mathcal{V}^{(i)}$. More generally, suppose $\{p_i\}_{i=0}^{k} \subset ker\ L_\beta$, with $\widehat{p_0} \in ker\ M_d$ and $p_i|\bigcap_{j=0}^{i-1} \mathcal{Z}_j \geq 0$ $(1 \leq i \leq k)$. If $card\ \bigcap_{j=0}^{k} \mathcal{Z}_j < rank\ M_d$, then $card\ \mathcal{V} < rank\ M_d$, so β has no representing measure (cf. Example 2.6 ($S = 1444$, first proof), or the first proof of Theorem 3.8).

Example 2.10. We illustrate Proposition 2.8 with $n = 2$ and M_3 given by

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 0 & 10 \end{pmatrix}.$$

M_3 is positive semidefinite, with $rank\ M_3 = 8$ and column relations $X^3 = X$ and $X^2Y = Y$. $\mathcal{V}^{(0)}$ thus consists of the lines $x = 1$ and $x = -1$, together with the origin $(0, 0)$. Consider $p(x, y) := x^3 - x$ and $q(x, y) := yx^2 - y$. A basis for $ker\ L_\beta$ consists of the 14 distinct elements of $\{x^i y^j p\}_{i,j \geq 0, i+j \leq 3} \bigcup \{x^k y^m q\}_{k,m \geq 0, k+m \leq 3}$, together with the reduced basis $\{x,\ y,\ x^2 - \frac{1}{2},\ xy,\ y^2 - 1,\ xy^2,\ y^3,\ xy^3,\ x^4 - 3,\ xy^4,\ y^5,\ xy^5,\ y^6 - 10\}$. Therefore, if $F(x, y) \in ker\ L_\beta$ satisfies $F|_{V^{(0)}} \geq 0$, then we may assume that $F(x, y) = ax + by + c(x^2 - \frac{1}{2}) + dxy + e(y^2 - 1) + fxy^2 + gy^3 + jxy^3 + m(x^4 - 3) + pxy^4 + qy^5 + sxy^5 + t(y^6 - 10)$ for scalars $a,\ b,\ c,\ \ldots$. It is not difficult to show that $F_1(y) := F(1, y)$ and $F_2(y) := F(-1, y)$ can both be represented as nonzero sums of squares, but when we impose the additional requirement $F(0, 0) \geq 0$, we find that the only solution to $F|_{\mathcal{V}^{(0)}} \geq 0$ is $F \equiv 0$. Thus $\mathcal{V} = \mathcal{V}^{(1)} = \mathcal{V}^{(0)}$. It is possible to show that although M_3 does not admit a flat extension, it does admit infinitely many positive $rank$ 9 extensions M_4, each of which admits a unique flat extension M_5; thus M_3 has infinitely many representing measures. □

We next compute the core variety for a sequence with 3 variables.

Example 2.11. We consider a degree 4 sequence β in 3 real variables, whose moment matrix M_2 has rows and columns indexed as $1, X, Y, Z, X^2, XY, XZ, Y^2, YZ, Z^2$; the entry in row $X^i Y^j Z^k$, column $X^r Y^s Z^t$ is $L_\beta(x^{i+r} y^{j+s} z^{k+t})$. M_2 is given by

$$\begin{pmatrix} 6 & 2 & 4 & 50 & 24 & -22 & -30 & 26 & 80 & 846 \\ 2 & 24 & -22 & -30 & 8 & 16 & 368 & -38 & -398 & -1342 \\ 4 & -22 & 26 & 80 & 16 & -38 & -398 & 64 & 498 & 2188 \\ 50 & -30 & 80 & 846 & 368 & -398 & -1342 & 478 & 2188 & 18074 \\ 24 & 8 & 16 & 368 & 180 & -172 & -288 & 188 & 656 & 7272 \\ -22 & 16 & -38 & -398 & -172 & 188 & 656 & -226 & -1054 & -8614 \\ -30 & 368 & -398 & -1342 & -288 & 656 & 7272 & -1054 & -8614 & -40158 \\ 26 & -38 & 64 & 478 & 188 & -226 & -1054 & 290 & 1532 & 10802 \\ 80 & -398 & 478 & 2188 & 656 & -1054 & 1-8614 & 1532 & 10802 & 58232 \\ 846 & -1342 & 2188 & 18074 & 7272 & -8614 & -40158 & 10802 & 58232 & 420438 \end{pmatrix}.$$

Let $p(x,y,z) = x + y - 1$ and $q(x,y,z) = z - (x^2 + y^2)$. M_2 is positive and recursively generated, with $rank\ M_2 = 5$, and column relations $p(X,Y,Z) = 0$, $(xp)(X,Y,Z) = 0$, $(yp)(X,Y,Z) = 0$, $(zp)(X,Y,Z) = 0$, and $q(X,Y,Z) = 0$. Thus $\mathcal{V}^{(0)} = \mathcal{Z}_p \bigcap \mathcal{Z}_q$.

To compute $\mathcal{V}^{(1)}$, we first note that the subspace $\widetilde{\mathcal{N}}$ has a basis $\mathcal{B}$ (of size 26) consisting of the multiples of p or q in $\mathcal{P}_4$ except for zp, xzp, yzp and z^2p. To form a basis for $ker\ L_\beta$, we adjoin to $\mathcal{B}$ a reduced basis for $ker\ L_\beta$ consisting of $r_1 = x - \frac{1}{3}$, $r_2 = x^2 - 4$, $r_3 = z^2 - 141$, $r_4 = yz - \frac{40}{3}$, $r_5 = z^3 - \frac{9037}{3}$, $r_6 = xz^3 + 6693$, $r_7 = y^2z^2 - \frac{5401}{3}$, and $r_8 = z^4 - 70073$. Now suppose $F(x,y,z) \in ker\ L_\beta$ satisfies $F|\mathcal{V}^{(0)} \geq 0$; we may assume $F = c_1r_1 + \cdots c_8r_8$. Since $F|\mathcal{V}^{(0)} \geq 0$, we see that $G(x) := F(x, 1-x, x^2 + (1-x)^2) \geq 0$ for all real x. Now $deg\ G = 8$, so G admits a representation as $G = A(x)^2 + B(x)^2$ for $A(x) = a_0 + \cdots + a_4x^4$ and $B(x) = b_0 + \cdots + b_4x^4$. By setting the coefficients of $S := G - (A^2 + B^2)$ to 0 (using the same method as in Example 2.6, expressing each c_i in terms of the a_k and b_k), we see that the constant term of S is 0 (and therefore all coefficients of S are 0) if and only if $\widehat{a} := (a_0, \ldots, a_4)$ and $\widehat{b} := (b_0, \ldots, b_4)$ satisfy $0 = \langle J\widehat{a}, \widehat{a}\rangle + \langle J\widehat{b}, \widehat{b}\rangle$, where

$$J := \begin{pmatrix} 12 & 4 & 48 & 16 & 360 \\ 4 & 48 & 16 & 360 & 64 \\ 48 & 16 & 360 & 64 & 3048 \\ 16 & 360 & 64 & 3048 & 256 \\ 360 & 64 & 3048 & 256 & 26760 \end{pmatrix}.$$

Since J is positive definite, we see that G is psd if and only if $\widehat{a} = \widehat{b} = 0$, which then implies that each $c_i = 0$. Thus, each polynomial in $ker\ L_\beta$ that is nonnegative on $\mathcal{V}^{(0)}$ is in $\widetilde{\mathcal{N}}$, and it follows that $\mathcal{V} = \mathcal{V}^{(1)} = \mathcal{V}^{(0)} = \{(x, 1-x, x^2 - 2x + 1)\}$ $(x \in \mathbb{R})$. Further calculations show that β has infinitely many representing measures, including the 6-atomic measure with support points $(1,0,1)$, $(0,1,1)$, $(-1,2,5)$, $(2,-1,5)$, $(-3,4,25)$, $(3,-2,13)$ and all densities equal to 1. $\square$

We next turn to the proof of the main result, Theorem 1.3, which we re-state for ease of reference.

Theorem 2.12. *Let $\beta \equiv \beta^{(2d)}$. If the core variety $\mathcal{V} \equiv \mathcal{V}(\beta)$ is nonempty, then L_β is strictly $\mathcal{V}$-positive and $\beta^{(2d-1)}$ has a $\mathcal{V}$-representing measure. Furthermore, if $\mathcal{V}$ nonempty and is either compact or a determining set for $\mathcal{P}_{2d}$, then $\beta^{(2d)}$ has a $\mathcal{V}$-representing measure.*

Remark 2.13. i) In view of [4], if $\beta^{(m)}$ has a K-representing measure, then $\beta^{(m)}$ has a finitely atomic K-representing measure. Thus, Theorem 2.12 may also be expressed in terms of finitely atomic $\mathcal{V}$-representing measures.

ii) It follows from Theorem 2.12 that if the variety $V(M_d)$ is finite, then β has a representing measure if and only if the core variety $\mathcal{V}$ is nonempty. The finite variety case is significant because, from Theorem A.2, β has a representing measure if and only if M_d admits a positive extension M_{d+k+1} whose variety $V(M_{d+k+1})$ is finite and is the support of a representing measure for β. The finite variety case arises, for example, if $n = 2$, there exist $p, q \in \mathcal{P}_d$ such that $p(X,Y) = q(X,Y) = 0$ in $Col\ M_d$, and p and q have no nontrivial common factor. In this case, it follows from Bezout's Theorem [9] that $card\ V(M_d) \leq card\ (\mathcal{Z}_p \bigcap \mathcal{Z}_q) \leq deg\ p\ deg\ q < \infty$.

Proof of Theorem 2.12. Without loss of generality in the following argument, we may assume that $L_\beta(1) \equiv \beta_0 = 1$. From Proposition 2.5 we may also assume that $\mathcal{V} = \mathcal{V}^{(j)} = \mathcal{V}^{(j+1)} = \cdots$ for some $j \geq 0$. Suppose first that L_β is not $\mathcal{V}$-positive. Thus, there exists $p \in \mathcal{P}_{2d}$ such that $p|_{\mathcal{V}} \geq 0$, but $L_\beta(p) < 0$. Let $q = p - L_\beta(p) \in \mathcal{P}_{2d}$. Then $L_\beta(q) = 0$, and for $x \in \mathcal{V}$, $q(x) = p(x) - L_\beta(p) > 0$. Since $q \in ker\ L_\beta$ and $q|_{\mathcal{V}^{(j)}} \geq 0$, then (2.1) implies $\mathcal{V}^{(j+1)} \subseteq \mathcal{Z}_q \bigcap \mathcal{V}^{(j)} = \mathcal{Z}_q \bigcap \mathcal{V} = \emptyset$, whence $\mathcal{V} = \emptyset$, a contradiction. Thus, L_β is $\mathcal{V}$-positive.

If L_β is not strictly $\mathcal{V}$-positive, then there exists $p \in \mathcal{P}_{2d}$ such that $p|_{\mathcal{V}} \geq 0$ and $p|_{\mathcal{V}} \not\equiv 0$, but $L_\beta(p) = 0$. Since $\mathcal{V} = \mathcal{V}^{(j)}$, there exists $v_0 \in \mathcal{V}^{(j)}$ such that $p(v_0) > 0$. Since $p|_{\mathcal{V}^{(j)}} \geq 0$ and $L_\beta(p) = 0$, then $v_0 \in \mathcal{V}^{(j)} = \mathcal{V}^{(j+1)} \subseteq \mathcal{V}^{(j)} \bigcap \mathcal{Z}_p \subseteq \mathcal{V}^{(j)} \setminus \{v_0\}$, a contradiction. Thus, L_β is strictly $\mathcal{V}$-positive.

Since L_β is $\mathcal{V}$-positive, Theorem 1.1 (applied with d replaced by $d-1$) implies that $\beta^{(2d-1)}$ has a $\mathcal{V}$-representing measure. Furthermore, Theorem A.3 implies that if $\mathcal{V}$ is compact, then β has a $\mathcal{V}$-representing measure. Finally, since L_β is strictly $\mathcal{V}$-positive, Theorem 1.2 implies that if $\mathcal{V}$ is a determining set for $\mathcal{P}_{2d}$, then β has a $\mathcal{V}$-representing measure. $\square$

Corollary 2.14. *If $\nu \equiv card\ \mathcal{V} < r \equiv rank\ M_d$, then $\nu = 0$ ($\mathcal{V} = \emptyset$).*

Proof. If $0 < \nu < r$, then from the compact case of Theorem 2.12, β has a representing measure. Theorem 3.8-i) thus implies $r \leq \nu$, a contradiction. $\square$

Corollary 2.15. *If $\mathcal{V} \equiv \mathcal{V}(\beta)$ is nonempty, then β is a limit of multisequences having $\mathcal{V}$-representing measures, i.e., β has a sequence of approximate representing measures.*

Proof. Since $\mathcal{V}$ is nonempty, L_β is $\mathcal{V}$-positive, so the result follows from [27]. $\square$

In the sequel, we say that β is $\mathcal{V}$-*consistent* if the conditions $p \in \mathcal{P}_{2d}$, $p|_{\mathcal{V}} \equiv 0$ together imply $L_\beta(p) = 0$; since $\mathcal{V} \subseteq V(M_d)$, $\mathcal{V}$-consistency implies consistency. The example in (A.3) (below) illustrates a case where M_d is positive and $V(M_d) \neq \emptyset$, but M_d is not recursively generated and hence β is not consistent. By contrast, we have the following result.

Corollary 2.16. *If $\mathcal{V} \equiv \mathcal{V}(\beta)$ is nonempty, then β is $\mathcal{V}$-consistent.*

Proof. Theorem 2.12 implies that L_β is $\mathcal{V}$-positive. If $p \in \mathcal{P}_{2d}$ satisfies $p|_{\mathcal{V}} \equiv 0$, then since $p|_{\mathcal{V}} \geq 0$ and $-p|_{\mathcal{V}} \geq 0$, it follows that $L_\beta(p) = 0$. $\square$

Theorem 2.12 does not provide for representing measures in the case where $\mathcal{V}$ is a proper non-compact variety. We next show how this case can arise.

Proposition 2.17. *Let $n = 2$ and let $p(x, y)$ be an irreducible polynomial with $deg\ p = d$ and $\mathcal{Z}_p$ infinite. Suppose there exists a positive finite Borel measure μ on $\mathbb{R}^2$ with $supp\ \mu = \mathcal{Z}_p$. Then $\beta \equiv \beta^{(2d)}[\mu]$ satisfies $\mathcal{V}(\beta) = \mathcal{Z}_p$.*

Proof. Let $M_d \equiv M_d(\beta)$. We first show that $V(M_d) = \mathcal{Z}_p$ ($=$ *supp* μ). Let $q(x, y) \in \mathcal{P}_d$, $q \not\equiv 0$, and suppose $q(X, Y) = 0$ in $Col\ M_d$. (A.1) implies $\mathcal{Z}_p \equiv supp\ \mu \subseteq \mathcal{Z}_q$, so $q \in I(\mathcal{Z}_p)$. Since p is irreducible and $\mathcal{Z}_p$ is infinite, it follows that $I(\mathcal{Z}_p) = (p)$, the principal ideal generated by p [29, Ch. 1.6, Cor. 1]. Thus, there is a factorization $q = ph$ for some polynomial h. Since $deg\ p + deg\ h = deg\ q$, $q \in \mathcal{P}_d$, and $deg\ p = d$, it follows that $q = \alpha p$ for some nonzero scalar α, whence $\mathcal{Z}_q = \mathcal{Z}_p$. Now $\mathcal{V}^{(0)} \equiv V(M_d) = \bigcap_{q \in \mathcal{P}_d,\ q(X,Y)=0} \mathcal{Z}_q = \mathcal{Z}_p$ ($=$ *supp* μ). Next, to compute $\mathcal{V}^{(1)}$, let $g(x, y) \in \mathcal{P}_{2d}$ satisfy $g \in ker\ L_\beta$ and $g|_{\mathcal{V}^{(0)}} \geq 0$. Since $\int g d\mu = L_\beta(g) = 0$ and $g|_{supp\ \mu} = g|_{\mathcal{V}^{(0)}} \geq 0$, we have $g|_{\mathcal{Z}_p} = g|_{supp\ \mu} \equiv 0$, i.e., $\mathcal{Z}_p \subseteq \mathcal{Z}_g$. Thus, $\mathcal{V}^{(0)} \supseteq \mathcal{V}^{(1)} = \bigcap_{g \in \mathcal{P}_{2d},\ g|_{\mathcal{V}^{(0)}} \geq 0} \mathcal{Z}_g \supseteq \mathcal{Z}_p = \mathcal{V}^{(0)}$. Now, $\mathcal{V}^{(1)} = \mathcal{V}^{(0)}$, so from (2.3) we have $\mathcal{V} = \mathcal{V}^{(1)} = \mathcal{V}^{(0)} = \mathcal{Z}_p$. $\square$

In the following result, we denote $\mathcal{V}(M_d)$ by $\mathcal{V}_d$. We further set $r_d := rank\ M_d$ and $\nu_d := card\ \mathcal{V}_d$. A consequence of the Bayer–Teichmann Theorem and of Theorem 3.8-i) is that a necessary condition for β

to have a representing measure is that M_d admit positive, recursively generated moment matrix extensions $M_{d+1},\ M_{d+2},\ \dots$ satisfying $r_{d+i} \le \nu_{d+i}$ $(i \ge 0)$. Clearly, $\{r_i\}$ is nondecreasing, and we next show that $\{\nu_i\}$ is nonincreasing. Theorem A.2 implies that β has a representing measure if and only if $r_{d+i} = r_{d+i+1}$ for some i; in this case, $r_{d+i+k} = r_{d+i+1} = \nu_{d+i+1} = \nu_{d+i+k}$ for every $k \ge 0$. More generally, for the $\mathcal{V}$-*extremal* case, when $r_i = \nu_i$ for some i, we show below (Theorem 2.20) that there exists a (unique) representing measure.

Proposition 2.18. *If $M_{d+1} \succeq 0$, then $\mathcal{V}_{d+1} \subseteq \mathcal{V}_d$.*

Proof. Let $\mathcal{V}_d^{(i)} := \mathcal{V}^{(i)}(M_d)$ and $\mathcal{V}_{d+1}^{(i)} := \mathcal{V}^{(i)}(M_{d+1})$. It suffices to prove that for $i \ge 0$, $\mathcal{V}_{d+1}^{(i)} \subseteq \mathcal{V}_d^{(i)}$. The proof is by induction on $i \ge 0$, and $\mathcal{V}_{d+1}^{(0)} \subseteq \mathcal{V}_d^{(0)}$ follows from [23]. Now suppose $\mathcal{V}_{d+1}^{(j)} \subseteq \mathcal{V}_d^{(j)}$ $(0 \le j \le i-1)$, and consider $\mathcal{V}_d^{(i)} = \bigcap_{p \in ker L_{\beta(2d)},\ p|_{\mathcal{V}_d^{(i-1)}} \ge 0} \mathcal{Z}_p$. For $p \in \mathcal{P}_{2d}$ with $L_{\beta^{(2d)}}(p) = 0$ and $p|_{\mathcal{V}_d^{(i-1)}} \ge 0$, clearly $p \in \mathcal{P}_{2d+2}$, $L_{\beta^{(2d+2)}}(p) = 0$, and since $\mathcal{V}_{d+1}^{(i-1)} \subseteq \mathcal{V}_d^{(i-1)}$ (by induction), then $p|_{\mathcal{V}_{d+1}^{(i-1)}} \ge 0$. Thus, $\mathcal{V}_{d+1}^{(i)} = \bigcap_{q \in ker L_{\beta(2d+2)},\ q|_{\mathcal{V}_{d+1}^{(i-1)}} \ge 0} \mathcal{Z}_q \subseteq \bigcap_{p \in ker L_{\beta(2d)},\ p|_{\mathcal{V}_d^{(i-1)}} \ge 0} \mathcal{Z}_p = \mathcal{V}_d^{(i)}$. $\square$

Recall that if β has a representing measure μ, then $r := rank\ M_d \le card\ supp\ \mu \le \nu := card\ \mathcal{V} \le v := card\ V(M_d)$. The results of [20] show that if β is *extremal*, i.e., $r = v$, then β has a representing measure if and only if $M_d \succeq 0$ and β is consistent. In this case, there is a unique representing measure μ, with $supp\ \mu = V(M_d)$. Thus, if β is extremal and has a representing measure, then β is also $\mathcal{V}$-*extremal*, i.e., $rank\ M_d = card\ \mathcal{V}$. We next illustrate a case where β is $\mathcal{V}$-extremal, but not extremal.

Example 2.19. For $n = 2$, $d = 3$, consider $M \equiv M_3$ given by

$$M = \begin{pmatrix} 8 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 6 \\ 6 & 0 & 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 6 \end{pmatrix}.$$

A calculation shows that $M \succeq 0$, with $rank\ M = 8$, and we claim that β is $\mathcal{V}$-extremal. $V(M)$ is determined by the column relations $X^3 = X$ and $Y^3 = Y$. Setting $p(x, y) = x^3 - x$ and $q(x, y) = y^3 - y$, we see that $V(M)$ is the 9-point grid consisting of points $w_i \equiv (x_i, y_i)$ given by $w_1 = (-1, -1)$, $w_2 = (0, -1)$, $w_3 = (1, -1)$, $w_4 = (-1, 0)$, $w_5 = (0, 0)$, $w_6 = (1, 0)$, $w_7 = (-1, 1)$, $w_8 = (0, 1)$, $w_9 = (1, 1)$. To construct a basis for $ker\ L_\beta$ so as to simplify our calculations, we seek to include as many multiples of p or q as possible. There are 20 such multiples in $\mathcal{P}_6$, as follows: $g_1 := p$, $g_2 := xp$, $g_3 := yp$, $g_4 := x^2p$, $g_5 := xyp$, $g_6 := y^2p$, $g_7 := x^3p$, $g_8 := x^2yp$, $g_9 := xy^2p$, $g_{10} := y^3p$, $g_{11} := q$, $g_{12} := xq$, $g_{13} := yq$, $g_{14} := x^2q$, $g_{15} := xyq$, $g_{16} := y^2q$, $g_{17} := x^3q$, $g_{18} := x^2yq$, $g_{19} := xy^2q$, $g_{20} := y^3q$. However, this collection is dependent, since $g_3 = g_{12} - g_{17} + g_{10}$, so we delete g_3, which results in an independent set of 19 polynomials. Since a basis for $ker\ L_\beta$ requires 27 polynomials, we augment this collection with $f_1 := x$, $f_2 := y$, $f_3 := x^2 - \frac{3}{4}$, $f_4 := xy$, $f_5 := y^2 - \frac{3}{4}$, $f_6 := x^2y$, and $f_7 := xy^2$.

The new collection provides 26 independent elements of $ker\ L_\beta$, so we require one additional polynomial for a basis. For this, we use the de-homogenized Robinson polynomial defined by

$$r(x,y) := x^6 + y^6 - x^4y^2 - x^2y^4 - x^4 - y^4 - x^2 - y^2 + 3x^2y^2 + 1$$

(cf. Section A.2). A calculation with the moments shows that $r \in ker\ L_\beta$; moreover, this polynomial is suitable for our problem because it is known to be nonnegative throughout the plane but is not a sum of squares (cf. Lemma 2.9), and has affine zeros precisely at each w_i except w_5. A further calculation shows that the new collection is indeed a basis for $ker\ L_\beta$. To compute $\mathcal{V}^{(1)}$, let $Q(x,y) \in ker\ L_\beta$ and suppose $Q|_{V(M)} \geq 0$. There exist scalars $a_1, \ldots, a_7,\ b_1,\ b_2,\ b_4, \ldots, b_{20},\ \alpha$ such that

$$Q(x,y) = a_1 f_1 + \cdots + a_7 f_7 + \alpha r + b_1 g_1 + b_2 g_2 + b_4 g_4 + \cdots + b_{20} g_{20}.$$

Since each $g_i|_{V(M)} \equiv 0$, the condition $Q|_{V(M)} \geq 0$ is equivalent to $F|_{V(M)} \geq 0$, where

$$F(x,y) := ax + by + c(x^2 - \frac{3}{4}) + dxy + e(y^2 - \frac{3}{4}) + fx^2y + gxy^2 + \alpha r(x,y),$$

for certain scalars $a,\ b,\ c,\ d,\ e,\ f,\ g,\ \alpha$. Since $F|_{V(M)} \geq 0$, we have, in particular, the system $F(x_i, y_i) \geq 0$ $(1 \leq i \leq 9,\ i \neq 5)$. It is not difficult to verify (either algebraically or using linear programming) that $a = b = c = d = e = f = g = 0$ is the unique solution to this system. We now have $F(x,y) = \alpha r(x,y)$. If we impose the condition $F(0,0) \geq 0$, we see that $\alpha \geq 0$. Thus, for $Q \in ker\ L_\beta$ with $Q|_{V(M)} \geq 0$,

$$Q(x,y) = \alpha r(x,y) + p(x,y)u(x,y) + q(x,y)v(x,y) \tag{2.13}$$

for certain $u, v \in \mathcal{P}_3$ and $\alpha \geq 0$. From (2.13), $\mathcal{Z}_Q$ includes $\mathcal{Z}_r$ for all such Q, so it follows that $\mathcal{Z}_r \subseteq \mathcal{V}^{(1)}$. Since r is psd and $r \in ker\ L_\beta$, we also have the reverse inclusion, so it follows that $\mathcal{V}^{(1)} = \mathcal{Z}_r$. To compute $\mathcal{V}^{(2)}$, we now consider $Q(x,y) \in ker\ L_\beta$ (as above) and we assume that $Q|_{\mathcal{V}^{(1)}} \geq 0$. Then, exactly as above, we have $F(x,y)|_{\mathcal{Z}_r} \geq 0$, so we again conclude that Q has the form in (2.13) for certain $u, v \in \mathcal{P}_3$ and $\alpha \in \mathbb{R}$. From this it follows that $\mathcal{Z}_Q$ contains $\mathcal{V}^{(1)}$ for all such Q, whence $\mathcal{V}^{(1)} \subseteq \mathcal{V}^{(2)}$. Now $\mathcal{V} = \mathcal{V}^{(2)} = \mathcal{V}^{(1)} = \mathcal{Z}(r)$ by (2.3). Thus, β is $\mathcal{V}$-extremal, with $r = \nu = 8 < 9 = v$. □

We next describe an analogue of [20] for the $\mathcal{V}$-extremal case. The following existence-uniqueness result for the $\mathcal{V}$-extremal case can be proved independently of the compact case of Theorem 2.12, based instead on the proof of [20, Theorem 2.8], but replacing $V(M_d)$ with $\mathcal{V}(\beta)$, and using Theorem 2.12 and Corollary 2.16 just to establish $\mathcal{V}$-consistency; we omit the details.

Theorem 2.20. *If rank $M_d = card\ \mathcal{V}$, then β admits a unique representing measure μ, and supp $\mu = \mathcal{V}$.*

We conclude this section by considering several cases in which the truncated moment problem has been completely solved, and for these cases we show that the existence of a representing measure for β is equivalent to β having a nonempty core variety. We first relate Theorem 2.20 to TMP on the real line ($n = 1$). We treat the case when M_d is singular next and the case when M_d is nonsingular in Section 3.

Theorem 2.21. *Let $n = 1$ and suppose M_d is singular. The following are equivalent for $\beta \equiv \beta^{(2d)}$ $(\beta_0 > 0)$:*
i) β has a representing measure;
ii) [10] M_d is positive and recursively generated;
iii) β is $\mathcal{V}$-extremal, i.e., $\nu = r$ (> 0);
iv) $\mathcal{V}$ is nonempty and $\mathcal{V} = V(M_d)$.
v) $\mathcal{V}$ is nonempty.

Proof. The equivalence of i) and ii) is established in [10]. Suppose iii) holds; since $\nu = r > 0$, Theorem 2.12 implies that L_β is positive, whence $M_d \succeq 0$. Similarly, Corollary 2.16 implies that β is $\mathcal{V}$-consistent, hence consistent, so M_d is recursively generated, and thus ii) holds. Now suppose ii) holds. The results of [10] show that in this case, $r = v$ and there is a representing measure, so Theorem 3.8 implies $r \le \nu \le v = r$, whence iii) holds; thus i)–iii) are equivalent. Next, if iv) holds, then so does v), and the same argument as in $iii) \Longrightarrow ii)$ shows that v) implies ii).

To complete the proof, it now suffices to show that i)–iii) imply iv). From the structure of a positive, singular, recursively generated Hankel matrix (cf. [10]), the columns $1, X, \ldots, X^{r-1}$ are independent, there is a dependence relation $g(X) = 0$ with $deg\ g = r$, and $(x^i g)(X) = 0$ $(0 \le i \le d-r)$. Further, g has r distinct real roots and $\mathcal{V}^{(0)} \equiv V(M_d) = \mathcal{Z}_g = supp\ \mu$, where μ is the unique representing measure for β. Thus, there is a basis for $ker\ L_\beta$ of the form $\{x^i - \beta_i\}_{i=1}^{r-1} \bigcup \{x^j g\}_{j=0}^{2d-r}$. Now suppose $q(x) \in ker\ L_\beta$ and $q|_{\mathcal{V}^{(0)}} \ge 0$. Then $q(x)$ is of the form $q(x) = f(x) + g(x)h(x)$, where $deg\ h(x) \le 2d - r$ and $f(x) := \sum_{i=1}^{r-1} \alpha_i (x^i - \beta_i)$ $(\alpha_i \in \mathbb{R})$. Since $q|_{\mathcal{V}^{(0)}} \ge 0$, then $f|_{\mathcal{V}^{(0)}} \ge 0$. Now $0 = L_\beta(q) = L_\beta(f) = \int_{supp\ \mu} f(x) d\mu(x)$ and $f|_{supp\ \mu} = f|_{\mathcal{V}^{(0)}} \ge 0$, so $f|_{\mathcal{V}^{(0)}} \equiv 0$. Since $card\ \mathcal{V}^{(0)} = r$ and $deg\ f \le r - 1$, it follows that $f \equiv 0$. Thus, $\mathcal{V}^{(0)} \subseteq \mathcal{Z}_q$, so $\mathcal{V}^{(1)} = \mathcal{V}^{(0)}$, and (2.3) implies $\mathcal{V} = \mathcal{V}^{(1)} = \mathcal{V}^{(0)} = \mathcal{Z}_g$. $\square$

By combining the results of [12,15,17,26], we see that if $n = 2$ and M_2 is singular, then $\beta \equiv \beta^{(2d)}$ has a representing measure if and only if M_d is positive and recursively generated, and $rank\ M_d \le card\ V(M_d)$.

Proposition 2.22. *Let $n = 2$. If $\mathcal{V} \equiv \mathcal{V}(M_d)$ is nonempty and M_2 is singular, then $\beta \equiv \beta^{(2d)}$ has a representing measure.*

Proof. If $\mathcal{V}$ is a nonempty finite set, the existence of a representing measure follows from Theorem 2.12, so we may assume that $\mathcal{V}$ is an infinite set. Since $\mathcal{V}$ is nonempty, Theorem 2.12 and Corollary 2.16 imply that L_β is positive and *consistent*; in particular, $M_d \succeq 0$ and M_d is recursively generated. Since $\mathcal{V}$ is infinite, then $rank\ M_d < +\infty = card\ \mathcal{V} \le card\ V(M_d)$, so, as described above, the existence of a representing measure follows. $\square$

Recall Hilbert's theorem on sums of squares: each psd polynomial of degree $2d$ is sos if and only if $n = 1$, $d = 1$, or $n = d = 2$. In each of these cases, TMP has been solved, and we show that in each such case the existence of a representing measure is equivalent to $\mathcal{V}(\beta) \neq \emptyset$. We treat the case when M_d is positive and singular next, and we consider the case when M_d is positive definite in Section 3.

Proposition 2.23. *In the cases of Hilbert's Theorem, if M_d is positive and singular, and if $\mathcal{V} \equiv \mathcal{V}(M_d)$ is nonempty, then β has a representing measure.*

Proof. For the case $n = 1$, the result follows from Proposition 2.21. For the case $d = 1$, since $M_d \succeq 0$, then [27, Section 4] implies that β has a measure (indeed, in this case M_d has a flat extension). For the case $n = d = 2$ and M_d singular, the result follows from Proposition 2.22. $\square$

3. The positive definite case

In this section we apply the core variety in the case when M_d is positive definite, i.e., $M_d \succ 0$. This case of TMP is largely unsolved, due to the lack of dependence relations in $Col\ M_d$. (Such relations, when present and combined with recursiveness, are useful in constructing positive extensions leading to flat extensions and representing measures (cf. Theorem A.1 and A.2).) Recall that Hilbert's theorem shows each psd polynomial is sos if and only if $n = 1$, $d = 1$, or $n = d = 2$ (cf. Section A.2). In the cases of Hilbert's theorem, it is known that a positive definite moment matrix has a representing measure (cf. [10] for $n = 1$ and [27] for $d = 1$ and $n = d = 2$). We begin with a new proof of this result based on the core variety.

Proposition 3.1. *In the cases of Hilbert's Theorem, if $M_d(\beta) \succ 0$, then $\mathcal{V}(\beta) = \mathbb{R}^n$ and β has a representing measure.*

Proof. Since $M_d \succ 0$, then $V(M_d) = \mathbb{R}^n$. From (2.3), to show $\mathcal{V} = \mathbb{R}^n$, it suffices to verify that $\mathcal{V}^{(1)} = \mathbb{R}^n$. Suppose $q \in ker\ L_\beta$ and q is psd. Then q is of the form $q = \sum q_i^2$ for certain $q_i \in \mathcal{P}_d$, and thus $0 = L(q) = \sum L(q_i^2) = \sum \langle M_d \hat{q}_i, \widehat{q_i} \rangle$. Since $M_d \succ 0$, it follows that each $\widehat{q_i} = 0$, whence $q = 0$ and $\mathcal{Z}_q = \mathbb{R}^n$. We thus have $\mathcal{V}^{(1)} = \bigcap_{q \in ker\ L_\beta, q|_{\mathbb{R}^n} \geq 0} \mathcal{Z}_q = \mathbb{R}^n = \mathcal{V}^{(0)}$, whence (2.3) implies $\mathcal{V} = \mathcal{V}^{(1)} = \mathcal{V}^{(0)} = \mathbb{R}^n$. Since $\mathbb{R}^n$ is a determining set, Theorem 1.3 implies that β has a representing measure. $\square$

Remark 3.2. For the cases $d = 1$ or $n = 1$, it is known that if $M_d \succ 0$, then M_d admits a flat extension M_{d+1} (cf. [11,27]). For the case $n = d = 2$, it was an open question for several years as to whether a positive definite M_2 admits a flat extension M_3. This has recently been answered affirmatively in [21], using the *proof* of [17] and a "reduction of rank" technique.

Relatively little is known concerning the positive definite case beyond the scope of Hilbert's Theorem. However, we do have the following general result.

Proposition 3.3. *The following are equivalent:*
i) L_β is strictly positive;
ii) $M_d \succ 0$ and $\mathcal{V} = \mathbb{R}^n$;
iii) $\mathcal{V} = \mathbb{R}^n$.

Proof. We begin with i) $\Longrightarrow$ ii). Suppose L_β is strictly positive. For $p \in \mathcal{P}_d$ with $p \not\equiv 0$, we have $\langle M_d \widehat{p}, \widehat{p} \rangle = L_\beta(p^2) > 0$, so $M_d \succ 0$. It follows that $\mathcal{V}^{(0)} \equiv V(M_d) = \mathbb{R}^n$. From (2.3), to show $\mathcal{V} = \mathbb{R}^n$, it suffices to prove that $\mathcal{V}^{(1)} = \mathbb{R}^n$. Suppose $p \in ker\ L_\beta$ and $p|_{\mathcal{V}^{(0)}} \geq 0$. Then since p is psd and L_β is strictly positive, we have $p \equiv 0$, whence $\mathcal{Z}_p = \mathbb{R}^n$. It follows that $\mathbb{R}^n = \mathcal{V}^{(1)} \subseteq \mathcal{V}^{(0)} = \mathbb{R}^n$, so the conclusion $\mathcal{V} = \mathbb{R}^n$ follows. Clearly ii) implies iii), and the implication iii) $\Longrightarrow$ i) follows immediately from Theorem 1.3. $\square$

For the case $n = 2$, $d = 3$, and $M_3 \succ 0$, we next characterize the existence of representing measures in terms of the core variety. In the sequel we will denote by $\Delta_{n,2d}$ the polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ of degree at most $2d$ that are nonnegative on $\mathbb{R}^n$, but cannot be expressed as sums of squares of polynomials (cf. Section A.2).

Theorem 3.4. *For $n = 2$ and $M_3 \succ 0$, exactly one of the following holds:*
i) $\mathcal{V} = \mathbb{R}^2$ and there is a representing measure;
ii) $\nu = 10$ and there is a unique representing measure, whose support is $\mathcal{V}$;
iii) $\nu = 0$ and there is no representing measure.

Proof. i) If $\mathcal{V} = \mathbb{R}^2$ (as described in Proposition 3.3), then since $\mathbb{R}^2$ is a determining set, Theorem 1.3 implies the existence of a representing measure.

ii) If $\nu = 10$, then since $M_3 \succ 0$, we have $r = \nu$. Thus β is $\mathcal{V}$-extremal, so the existence of a unique representing measure, with support $\mathcal{V}$, follows from Theorem 2.20.

iii) Since $M_3 \succ 0$, we have $\mathcal{V}^{(0)} = \mathbb{R}^2$, and since $ker\ M_d = \{0\}$, then $\widetilde{\mathcal{N}} = \{0\}$. To compute $\mathcal{V}^{(1)}$, suppose $p \in ker\ L_\beta$ and p is psd. If $card\ \mathcal{Z}_p > 10$, then it follows from [40, Sec. 7] (see also [8, Thm. 3.7]) that p is sos, and thus Lemma 2.9 implies $p \in \widetilde{\mathcal{N}}$, whence $p = 0$. Thus, if $\mathcal{V} \neq \mathbb{R}^2$, then there exists $p \in \ker L_\beta \bigcap \Delta_{2,6}$ such that $card\ \mathcal{Z}_p \leq 10$, so $\nu \leq 10$. If we further assume that $\nu \neq 10$, then $\nu < 10 = rank\ M_3$, so Corollary 2.14 implies that $\nu = 0$ and that there is no representing measure. □

Example 3.5. We may illustrate the three cases of Theorem 3.4 as follows. For case i), let μ denote a positive finite Borel measure on the plane with $supp\ \mu = \mathbb{R}^2$, e.g., $d\mu = e^{-x^2-y^2}dxdy$. Let $M \equiv M_3[\mu]$. For $p \in \mathcal{P}_3$, $p \not\equiv 0$, $\langle M\widehat{p}, \widehat{p}\rangle = L_\beta(p^2) = \int p^2 d\mu > 0$, so $M \succ 0$. Similarly, if $p \in \ker L_\beta$ is psd, then since $\int p d\mu = L_\beta(p) = 0$, we have $p \equiv 0$. Thus $\mathcal{V} = \mathcal{V}^{(1)} = \mathcal{V}^{0)} = \mathbb{R}^2$.

To illustrate case ii), recall the Robinson polynomial $r(x, y)$, and let $R(x, y, z)$ denote its homogenization, defined by

$$R(x, y, z) := x^6 + y^6 + z^6 - x^4y^2 - x^2y^4 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2.$$

It is known that $r(x, y)$ has precisely eight affine zeros $w_i \equiv (x_i, y_i)$ $(1 \leq i \leq 8)$, as follows: $w_1 = (-1, -1)$, $w_2 = (0, -1)$, $w_3 = (1, -1)$, $w_4 = (-1, 0)$, $w_5 = (1, 0)$, $w_6 = (-1, 1)$, $w_7 = (0, 1)$, $w_8 = (1, 1)$. Corresponding to these are eight projective zeros of the Robinson form R: $\widetilde{w}_i := (x_i, y_i, 1)$ $(1 \leq i \leq 8)$. The Robinson form has two additional projective zeros, $\widetilde{w}_9 \equiv (x_9, y_9, z_9) := (1, 1, 0)$ and $\widetilde{w}_{10} \equiv (x_{10}, y_{10}, z_{10}) := (1, -1, 0)$. We now define $T(x, y, z)$ to be the Robinson form composed with a linear change of variables in $\mathbb{R}^3$, as follows:

$$T(x, y, z) := R(3x - 3y + z, -3x + 5y - 2z, x - 2y + x).$$

Since $R \in \Delta_{3,6}$, so is T. Then the dehomogenization of T, defined by $t(x, y) := T(x, y, 1)$, is in $\Delta_{2,6}$. For each projective zero (x, y, z) of R, $(u, v, w) := (x + y + z, x + 2y + 3z, x + 3y + 6z)$ is a projective zero of T, and if $w \neq 0$, then $(\frac{u}{w}, \frac{v}{w})$ is an affine zero of t. A calculation now shows that t has 10 distinct affine zeros, $u_i \equiv (a_i, b_i)$ $(1 \leq i \leq 10)$, as follows: $u_1 = (-\frac{1}{2}, 0)$, $u_2 = (0, \frac{1}{3})$, $u_3 = (\frac{1}{4}, \frac{1}{2})$, $u_4 = (0, \frac{2}{5})$, $u_5 = (\frac{2}{7}, \frac{4}{7})$, $u_6 = (\frac{1}{8}, \frac{1}{2})$, $u_7 = (\frac{2}{9}, \frac{5}{9})$, $u_8 = (\frac{3}{10}, \frac{3}{5})$, $u_9 = (\frac{1}{2}, \frac{3}{4})$, $u_{10} = (0, \frac{1}{2})$. Setting $\mu := \sum_{i=1}^{10} \delta_{(a_i,b_i)}$, a straightforward calculation with nested determinants shows that $M_3[\mu] \succ 0$, and a calculation with the moments $\beta \equiv \beta^{(6)}[\mu]$ shows that $L_\beta(t) = 0$. Since t is psd, we have $\mathcal{V}^{(1)} \subseteq \mathcal{Z}_t$, whence $\nu \leq 10$. Since β has the representing measure μ, we also have $10 = card\ supp\ \mu \leq \nu$, so we see that $\nu = 10$ and $\mathcal{V} = \mathcal{Z}_t$.

Concerning iii), the first example of a positive definite moment matrix not having a representing measure appears in [12, Section 4], for the case $n = 2$ and $d = 3$. This example is based on Schmüdgen's construction in [42] of a polynomial in $\Delta_{2,6}$ given by

$$\psi(x, y) := 200(x^3 - 4x)^2 + 200(y^3 - 4y)^2 + (y^2 - x^2)x(x + 2)[x(x - 2) + 2(y^2 - 4)].$$

This polynomial is used in [42] to explicitly construct a linear functional L on $\mathcal{P}_6$ that is positive on $\Sigma_{2,6}$, but is not positive on $\mathcal{P}_6$. The functional is defined by

$$L(p) := 32 \sum_{i=1}^{8} p(A_i) + p(B_1) + p(B_2) - p(A_0) \qquad (p \in \mathcal{P}_6),$$

where

$$\begin{array}{llll} A_1 = (-2, -2), & A_2 = (0, -2), & A_3 = (2, -2), & A_4 = (-2, 0), \\ A_5 = (0, 0), & A_6 = (-2, 2), & A_7 = (0, 2), & A_8 = (2, 2), \\ A_9 = (2, 0), & B_1 = (\frac{1}{100}, 0), & B_2 = (0, \frac{1}{100}). & \end{array}$$

The moment matrix $M_3(\beta)$ corresponding to the moment sequence $\beta \equiv \beta^{(6)}$, where $\beta_{ij} = L(x^i y^j)$ $(i, j \geq 0,$ $i+j \leq 6)$, illustrates case iii) in Theorem 3.4. The fact that M_3 is positive definite was asserted in [12], but in [12] there are errors in recording the moments; The correct values are as follows:

$\beta_{00} = 257$, $\beta_{10} = -\frac{6599}{100}$, $\beta_{01} = \frac{1}{100}$, $\beta_{20} = \frac{6360001}{10000}$, $\beta_{11} = 0$, $\beta_{02} = \frac{7680001}{10000}$, $\beta_{30} = -\frac{263999999}{1000000}$, $\beta_{21} = \beta_{12} = 0$, $\beta_{03} = \frac{1}{1000000}$, $\beta_{40} = \frac{254400000001}{100000000}$, $\beta_{31} = \beta_{13} = 0$, $\beta_{22} = 2048$, $\beta_{04} = \frac{307200000001}{100000000}$, $\beta_{50} = -\frac{105599999999999}{10000000000}$, $\beta_{41} = \beta_{32} = \beta_{23} = \beta_{14} = 0$, $\beta_{05} = \frac{1}{10000000000}$, $\beta_{60} = \frac{10176000000000001}{1000000000000}$, $\beta_{51} = \beta_{33} = \beta_{15} = 0$, $\beta_{42} = \beta_{24} = 8192$, $\beta_{06} = \frac{12288000000000001}{1000000000000}$.

Using nested determinants, it is straightforward to verify that $M_3 \succ 0$. Now ψ is psd and $L(\psi) = -\frac{255360015879601}{1000000000000} < 0$, so L is not positive, and therefore β has no representing measure. It now follows from Theorem 3.4 that $\nu = 0$. $\square$

We next use the core variety to prove the existence of a large family of positive definite moment matrices not having representing measures.

Proposition 3.6. *Let $p \in \Delta_{n,2d}$ with card $\mathcal{Z}_p <$ dim $\mathcal{P}_d$. Then there exists $M_d \equiv M_d(y_p)$ such that $M_d \succ 0$, $L_{y_p}(p) = 0$, and $\mathcal{V}(M_d) = \emptyset$, whence y_p has no representing measure.*

Proof. Since Σ_{2d} is a closed convex set in $\mathcal{P}_{2d}$ [36, Ch. 3.8, Cor. 3.50, p. 50], and $p \notin \Sigma_{2d}$, the Separation Theorem [5, Thm. 34.1, p. 134] implies that there exists a linear functional $L : \mathcal{P}_{2d} \mapsto \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $L|_{\Sigma_{2d}} > \alpha$ and $\delta := L(p) < \alpha$. Let $q \in \Sigma_{2d}$, $q \not\equiv 0$. Since for each $n > 0$, $L(\frac{1}{n}q) > \alpha$, then $L(q) > n\alpha$, whence $\alpha \leq 0$ and $\delta < 0$. Moreover, $nL(q) = L(nq) > \alpha$, so $L(q) > \frac{\alpha}{n}$, and thus $L(q) \geq 0$. Now we have $L|_{\Sigma_{2d}} \geq 0$ and $L(p) = \delta < 0$.

Let μ denote a finite positive Borel measure such that *supp* $\mu = \mathbb{R}^n$. Then for $\beta \equiv \beta^{(2d)}[\mu]$, L_β is strictly positive. Let $\epsilon := L_\beta(p)$ (> 0) and set $\gamma = -\frac{\epsilon}{\delta}$ (> 0). Let $y \equiv y_p$ denote the sequence associated with $\mathcal{L} := \gamma L + L_\beta$, i.e., $y_i = \mathcal{L}(x^i)$ $(|i| \leq 2d)$. For $q \in \mathcal{P}_d$, $q \not\equiv 0$, $L_y(q^2) \equiv \mathcal{L}(q^2) = \gamma L(q^2) + L_\beta(q^2) \geq L_\beta(q^2) > 0$, whence $M_d(y) \succ 0$ and $\mathcal{V}^{(0)} \equiv \mathcal{V}^{(0)}(M_d(y)) = \mathbb{R}^n$. Now, $L_y(p) = \gamma L(p) + L_\beta(p) = -\frac{\epsilon}{\delta}\delta + \epsilon = 0$. Since $p|_{\mathcal{V}^{(0)}(M_d(y))} = p|_{\mathbb{R}^n} \geq 0$, it follows that $\mathcal{V}(y) \subseteq \mathcal{V}^{(1)}(M_d(y)) \subseteq \mathcal{Z}_p$, and since *card* $\mathcal{Z}_p <$ *dim* $\mathcal{P}_d =$ *rank* $M_d(y)$, Corollary 2.14 implies that $\mathcal{V}(M_d(y)) = \emptyset$, so y has no representing measure. $\square$

Remark 3.7. The Robinson polynomial $r(x, y)$ satisfies the hypothesis of Proposition 3.6, since $r \in \Delta_{2,6}$ and *card* $\mathcal{Z}_r = 8 < 10 =$ *dim* $\mathcal{P}_3$. Similarly, the Motzkin polynomial $m(x, y)$ satisfies *card* $\mathcal{Z}_m = 4$, and Schmüdgen's polynomial $\psi(x, y)$ satisfies *card* $\mathcal{Z}_\psi = 8$. Given $p \in \Delta_{n,2d}$ as in Proposition 3.6, it is generally very difficult to express a separating functional L explicitly, because the proof of the Separation Theorem is non-constructive. For this reason, it is generally difficult to describe L_{y_p} numerically. However, as discussed above in Example 3.5-iii), in [42], Schmüdgen explicitly constructed a functional L separating ψ from $\Sigma_{2,6}$. Now let μ be any positive Borel measure such that *supp* $\mu = \mathbb{R}^n$ and such that μ has convergent moments up to degree $2d$ that can be explicitly computed, e.g., $d\mu = e^{-x^2-y^2}dxdy$. For $\beta \equiv \beta^{(2d)}[\mu]$, let $\mathcal{L} = -\frac{\epsilon}{\delta}L + L_\beta$, where $\delta = L(\psi)$ (< 0) and $\epsilon = L_\beta(\psi)$ (> 0). Then y_ψ, the moment sequence of degree $2d$ corresponding to $\mathcal{L}$, is a numerically computable sequence that satisfies the requirements of Proposition 3.6.

The analysis of the relationship between $\Sigma_{n,2d}$ and $\mathcal{P}^+_{n,2d}$, initiated by Hilbert, continues to be much-studied, notably in the work of B. Reznick [39,40] and in recent work of G. Blekherman concerning Cayley–Bacharach theory (e.g. [6]). In [6] Blekherman showed that for $p \in \Delta_{3,6}$ there exists a 9-atomic separating functional; one of the parameters used to construct this functional is computed via semidefinite optimization (cf. [36]). Building on Blekherman's work, S. Iliman and T. de Wolff [33] have developed an exact (algebraic) method for constructing separating functionals L for certain classes of polynomials p in $\Delta_{3,6}$. In particular, [33, Section 5] has a concrete (numerical) formula for a 9-atomic separating functional $\widetilde{L}$ for the homogeneous Motzkin form $M(x, y, z)$. By de-homogenizing $\widetilde{L}$, one has a separating functional L for $m(x, y)$. Of course, once L is computed and μ is chosen as above, it is straightforward to construct y_p exactly (as described above).

If L_β is strictly positive, then $M_d \succ 0$, and Theorem 1.2 implies that β has a representing measure. In [27, Question 1.2] we asked whether the same conclusion holds if $M_d \succ 0$ and L_β is merely positive (cf. [18, Question 2.9]). The following result uses the core variety to provide a negative answer.

Theorem 3.8. *For $n = 2$, there exists $\beta \equiv \beta^{(6)}$ with $M_3 \equiv M_3(\beta) \succ 0$ and L_β positive, but with $\mathcal{V} = \emptyset$, so β does not have a representing measure.*

Proof. As noted in Example 3.5, the de-homogenized Robinson polynomial $r(x, y)$ has the following eight affine zeros $w_i \equiv (x_i, y_i)$ $(1 \le i \le 8)$: $w_1 = (-1, -1)$, $w_2 = (0, -1)$, $w_3 = (1, -1)$, $w_4 = (-1, 0)$, $w_5 = (1, 0)$, $w_6 = (-1, 1)$, $w_7 = (0, 1)$, $w_8 = (1, 1)$. Corresponding to these are eight projective zeros of the Robinson form R, $\widetilde{w}_i := (x_i, y_i, 1)$ $(1 \le i \le 8)$, and there are two additional projective zeros for R, $\widetilde{w}_9 \equiv (x_9, y_9, z_9) := (1, 1, 0)$ and $\widetilde{w}_{10} \equiv (x_{10}, y_{10}, z_{10}) := (1, -1, 0)$. Define the measure ω on $\mathbb{R}^3$ by $\omega := \sum_{i=1}^{10} \delta_{\widetilde{w}_i}$. Let $\widetilde{\beta} \equiv \widetilde{\beta}^{(=6)}$ denote the ω-moments of degree exactly 6, and let $L_{\widetilde{\beta}}$ denote the corresponding functional on homogeneous forms of degree 6 in $\mathbb{R}[x, y, z]$. Now define $L : \mathbb{R}[x, y]_6 \mapsto \mathbb{R}$ by $L(p(x, y)) := L_{\widetilde{\beta}}(\widetilde{p}(x, y, z))$ (where $\widetilde{p}$ denotes the homogenization of p, i.e., $\widetilde{p}(x, y, z) = z^6 p(\frac{x}{z}, \frac{y}{z})$). Let $\beta_{ij} := L(x^i y^j)$ $(i, j \ge 0,\ i+j \le 6)$, so that $L_\beta = L$. (Note that $\beta_{ij} = \widetilde{\beta}_{i,j,6-i-j}$, so $\widetilde{\beta}$ is the *homogenization* of β in the language of [28], cf. Section A.2.) The moment matrix corresponding to β, $M \equiv M_3(\beta)$, is given by

$$M = \begin{pmatrix} 8 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 6 \\ 6 & 0 & 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 8 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 8 \end{pmatrix}.$$

Using nested determinants, it is easy to check that $M \succ 0$. Since $\widetilde{\beta}$ has the representing measure ω in HTMP, it follows from Theorem A.4 that L_β is positive.

We next compute $\mathcal{V}(M)$. Clearly, $\mathcal{V}^{(0)} = \mathbb{R}^2$. A calculation with the moments of M shows that $L_\beta(r) = 0$, and since r is psd, it follows that $\mathcal{V}^{(1)} \subseteq \mathcal{Z}_r$. Thus, $\nu \le card\ \mathcal{Z}_r = 8 < 10 = rank\ M$, so Corollary 2.14 already implies that $\nu = 0$ and that β has no representing measure. We will also verify this conclusion explicitly. If $t(x, y) \in \mathcal{P}_6$ is psd and $L_\beta(t) = 0$, then $\int \widetilde{t}(x, y, z) d\omega = L_{\widetilde{\beta}}(\widetilde{t}) = L_\beta(t) = 0$. Since t is psd, so is $\widetilde{t}$, and thus $\widetilde{t}|_{supp\ \omega} \equiv 0$. It follows that $t|_{\mathcal{Z}_r} \equiv 0$, which implies $\mathcal{Z}_r \subseteq \mathcal{V}^{(1)}$. Since, from above, we also have $\mathcal{V}^{(1)} \subseteq \mathcal{Z}_r$, then $\mathcal{V}^{(1)} = \mathcal{Z}_r$. Now let $f(x, y) := 2 - x^2 - y^2$ and $g(x, y) := \frac{3}{2}x^2y^2 - x^2y^4$. Then $L_\beta(f) = L_\beta(g) = 0$ and f and g are nonnegative on $\mathcal{Z}_r$. Moreover, for $1 \le i \le 8$, either $f(w_i) = 0$ and $g(w_i) = \frac{1}{2}$, or $f(w_i) = 1$ and $g(w_i) = 0$, so $\mathcal{V} \subseteq \mathcal{V}^{(2)} \subseteq \mathcal{Z}_r \bigcap \mathcal{Z}_f \bigcap \mathcal{Z}_g = \emptyset$. $\square$

4. Open questions

In the examples in this paper, whenever there is no representing measure we have $\mathcal{V} = \emptyset$. This observation, Theorems 1.3, 2.21, 3.4, and Propositions 2.22–2.23 suggest the following basic question.

Question 4.1. If $\mathcal{V}(\beta)$ is nonempty, does β have a representing measure?

Theorem 2.21 and Proposition 3.1 show that for $n = 1$, we either have $\mathcal{V}^{(0)} = \mathcal{V}$ ($\neq \emptyset$) or $\mathcal{V}^{(0)} \supset \mathcal{V}^{(1)} = \mathcal{V} = \emptyset$. For $n = 2$, in all of the examples of this paper we have either $\mathcal{V}^{(0)} = \mathcal{V}$, $\mathcal{V}^{(0)} \supset \mathcal{V}^{(1)} = \mathcal{V}$, or $\mathcal{V}^{(0)} \supset \mathcal{V}^{(1)} \supset \mathcal{V}^{(2)} = \mathcal{V}$ ($= \emptyset$).

Question 4.2. For fixed n and d, what is the maximal length k of a sequence of proper inclusions $\mathcal{V}^{(0)} \supset \cdots \supset \mathcal{V}^{(k)} = \mathcal{V}$?

For $n = 1$, if M_d has a measure, then it has a flat extension M_{d+1} and *card* $V_d \geq$ *card* V_{d+1}, with strict inequality if and only if M_d is positive definite. For $n = 2$, in the examples of [10–19] and [23–26] concerning Theorem A.2, for a sequence of positive extensions $M_d, \ldots, M_{d+k}, M_{d+k+1}$, where M_{d+k+1} is a flat extension of M_{d+k}, so that *card* $V(M_d) \geq \cdots \geq$ *card* $V(M_{d+k+1})$, we have never observed more than 2 strict inequalities before the varieties stabilize in the support of a representing measure for M_d. This motivates the following question which seems to be related to Question 4.2.

Question 4.3. For fixed n and d, what is the largest number of strict inequalities possible in the sequence of inequalities *card* $V(M_d) \geq \cdots \geq$ *card* $V(M_{d+k+1})$ (where M_{d+k+1} is a flat extension of $M_{d+k} \succeq 0$)?

If $\mathcal{V} = \mathbb{R}^n$ (a determining set), then there is a representing measure by Theorem 1.3. The Bayer–Teichmann Theorem implies that there is then a finitely atomic representing measure μ, so in this case *supp* μ is a proper subset of the core variety. In all of the examples we have seen in which $\mathcal{V}$ is a nonempty finite set, there is a unique representing measure, whose support coincides with the core variety.

Question 4.4. Suppose $\mathcal{V}$ is nonempty and finite. Is there a unique representing measure? Does the support of a representing measure necessarily coincide with $\mathcal{V}$?

Note added in proof

In a forthcoming paper by Grigoriy Blekherman and the author, Question 4.1 is answered affirmatively; partial solutions to Questions 4.2–4.4 are also presented.

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Appendix A

A.1. Positive moment matrices and extensions

Unless otherwise stated, we are in the general case, i.e., $n \geq 1$. For $p \equiv \sum_{i \in \mathbb{Z}_+^n, |i| \leq d} a_i x^i \in \mathcal{P}_d$, let $\widehat{p} \equiv (a_i)$ denote the coefficient vector of p relative to the basis $\mathcal{B}_d$ of monomials in $\mathcal{P}_d$ in degree-lexicographic order. Following [11,16], we associate to $\beta \equiv \beta^{(2d)}$ the *moment matrix* $M_d \equiv M_d(\beta)$, with rows and columns X^i indexed by the elements of $\mathcal{B}_d$. The entry in row X^i, column X^j of M_d is β_{i+j} ($i, j \in \mathcal{Z}_+^n$, $|i|, |j| \leq d$), so M_d is a real symmetric matrix characterized by $\langle M_d \widehat{p}, \widehat{q} \rangle = L_\beta(pq)$ ($p, q \in \mathcal{P}_d$). If L_β is positive (in particular, if β has a representing measure), then $\langle M_d \widehat{p}, \widehat{p} \rangle = L_\beta(p^2) \geq 0$, and since M_d is real symmetric, it follows that M_d is positive semidefinite ($M_d \succeq 0$).

For $p(x) \equiv \sum a_i x^i \in \mathcal{P}_d$, we have the column space element $p(X) \equiv \sum a_i X^i$, and $p(X) = M_d \widehat{p}$. If β admits a representing measure μ, then

$$for\ p \in \mathcal{P}_d,\ supp\ \mu \subseteq \mathcal{Z}_p \Longleftrightarrow p(X) = 0 \tag{A.1}$$

(cf. [11, Prop. 3.1]). It follows from (A.1) that $supp\ \mu \subseteq V(M_d)$, whence

$$r \equiv rank\ M_d \leq card\ supp\ \mu \leq v \equiv card\ V(M_d) \tag{A.2}$$

(cf. [11, Cor. 3.7]). We will cite the following basic existence theorem of [11,16] for a "minimal" representing measure μ satisfying $card\ supp\ \mu = rank\ M_d$.

Theorem A.1 *(Flat Extension Theorem, cf. [16, Thm. 1.1–1.2], [46]). $\beta \equiv \beta^{(2d)}$ has a rank M_d-atomic representing measure if and only if $M_d \succeq 0$ and M_d admits a flat moment matrix extension, i.e., a moment matrix extension M_{d+1} satisfying rank $M_{d+1} = rank\ M_d$. In this case, $\beta^{(2d+2)}$ admits a unique representing measure, $\mu \equiv \mu_{M_{d+1}}$, satisfying $supp\ \mu = V(M_{d+1})$ and $card\ supp\ \mu = rank\ M_d$.*

For the case of *flat data* ($M_d \succeq 0$ and $rank\ M_d = rank\ M_{d-1}$), Theorem A.1 (applied to M_{d-1}) implies a unique ($rank\ M_d$-atomic) representing measure for $\beta^{(2d)}$.

By combining Theorem A.1 with [4], we have the following solution to the Truncated Moment Problem, expressed in terms of moment matrix extensions.

Theorem A.2 *(Moment Matrix Extension Theorem, cf. [16, Corollary 1.4]). $\beta^{(2d)}$ has a representing measure if and only if there is an integer $k \geq 0$ such that M_d admits a positive moment matrix extension M_{d+k} which in turn admits a flat extension M_{d+k+1}.*

Theorem A.2 is not, by itself, a concrete solution to TMP, but it does provide a framework for obtaining concrete solutions in certain cases (cf. [15,17]).

A.2. Positive Riesz functionals

In [49], V. Tchakaloff established the fundamental existence theorem in cubature theory. Let K denote a compact subset of $\mathbb{R}^n$ with positive n-dimensional Lebesgue measure. Let μ denote the restriction of Lebesgue measure on $\mathbb{R}^n$ to K, and let m be a positive integer. Tchakaloff proved that there exist finitely many points in K, $w_1, \ldots, w_N$ ($N \leq dim\ \mathcal{P}_m$), and positive weights $\alpha_1, \ldots, \alpha_N$, such that for each $p \in \mathcal{P}_m$, $L(p) := \int_K p(x)d\mu(x) = \sum_{i=1}^{N} \alpha_i p(w_i)$. A careful examination of [49] reveals that the role of μ is simply to establish that $L : \mathcal{P}_m \mapsto \mathbb{R}$ is K-positive. Thus we may paraphrase Tchakaloff's Theorem as the analogue of Riesz–Haviland for TKMP in the compact case, as follows.

Theorem A.3. *(cf. Tchakaloff [49], [28, Theorem 2.2]) Let $\beta \equiv \beta^{(m)}$, $\beta_0 > 0$, and let K be a compact subset of $\mathbb{R}^n$. β has a K-representing measure if and only if $L_\beta : \mathcal{P}_m \mapsto \mathbb{R}$ is K-positive, in which case β admits a K-representing measure μ with card $supp\ \mu \leq dim\ \mathcal{P}_m$.*

Theorem A.3 plays a role in the proof of Theorem 1.3, but its import extends beyond the case when K is compact. There are examples where K is noncompact, but $\mathcal{V}$ is finite (e.g., Example 3.5-ii), where $K \equiv V(M_d) = \mathbb{R}^2$ but $card\ \mathcal{V} = 10$), so Theorem 1.3 shows that in such examples representing measures exist.

Considerations with Taylor series imply that if $K \subseteq \mathbb{R}^n$ has nonempty interior, then K is a determining set for $\mathcal{P}_m$. However, it is also possible for a finite set to be determining. Suppose μ is a finite positive Borel measure and let $\mathcal{K} := supp\ \mu$, where $int(\mathcal{K}) \neq \emptyset$. Suppose μ has convergent moments up to degree $2m$. The Bayer–Teichmann Theorem [4] implies that there exist points $w_1, \ldots, w_k$ in $\mathcal{K}$ and positive weights

$\alpha_1, \ldots, \alpha_k$ such that $\int_{\mathcal{K}} p(x)d\mu(x) = \sum_{i=1}^{k} \alpha_i p(w_i)$ for all $p \in \mathcal{P}_{2m}$. We claim that $W := \{w_1, \ldots, w_k\}$ is a determining set for $\mathcal{P}_m$. Indeed, if $p \in \mathcal{P}_m$ and $p|_W \equiv 0$, then $q := p^2 \in \mathcal{P}_{2m}$ satisfies $q|_W \equiv 0$. Now $\int_{\mathcal{K}} q \, d\mu = \sum \alpha_i q(w_i) = 0$ and $q|_{\mathcal{K}} \geq 0$, so $q|_{\mathcal{K}} \equiv 0$. Since $int\ \mathcal{K} \neq \emptyset$, then $q \equiv 0$, whence $p \equiv 0$.

Following [40], we refer to $p \in \mathcal{P}_{2d}$ as *positive semidefinite* (psd) if $p|_{\mathbb{R}^n} \geq 0$, and as a *sum of squares* (sos) if there exist $p_1, \ldots, p_k \in \mathcal{P}_d$ such that $p = \sum_{i=1}^{k} p_i^2$. For $\beta \equiv \beta^{(2d)}$, positivity of L_β is easily established if each psd polynomial in $\mathcal{P}_{2d}$ is sos, since then L_β is positive if and only if $M_d \succeq 0$; indeed, in this case, if p is psd, then $p = \sum_{i=1}^{k} p_i^2$, so $L_\beta(p) = \sum L_\beta(p_i^2) = \sum \langle M_d \widehat{p}_i, \widehat{p}_i \rangle \geq 0$. A well-known theorem of Hilbert shows that each psd polynomial is sos if and only if $n = 1$, $d = 1$, or $n = d = 2$ (cf. [40]). As discussed above, if K is compact and L_β is K-positive, then β has a finitely atomic K-representing measure. Perhaps the simplest example of β for which L_β is K-positive but β has no representing measure occurs with $n = 1$, $K = \mathbb{R}$, $d = 2$, and $M_2(\beta)$ given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{A.3}$$

[18, Example 2.1]. In (A.3), since $n = 1$ and $M_2 \succeq 0$, it follows from the preceding discussion that L_β is positive. However, since M_2 is not recursively generated ($X = 1$, but $X^2 \neq X$), [10] implies that there is no representing measure. In this example, $M_2 \succeq 0$ and $V(M_2) \equiv \{1\}$ is nonempty and compact, so the nonexistence of a representing measure contrasts with Theorem 1.3.

Let $\mathcal{P}^+_{n,2d}$ and $\Sigma_{n,2d}$ denote, respectively, the positive cones in $\mathcal{P}_{2d}$ consisting of the psd and sos polynomials, and let $\Delta \equiv \Delta_{n,2d} := \mathcal{P}^+_{n,2d} \setminus \Sigma_{n,2d}$. Concrete examples of polynomials in Δ were discovered beginning some 60 years after Hilbert's work. We will refer to several such examples from $\Delta_{3,6}$ that are discussed by Reznick [39,40], including the Motzkin form

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2;$$

and the Robinson form

$$R(x, y, z) = x^6 + y^6 + z^6 - x^4y^2 - x^2y^4 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2.$$

It is well-known that a homogeneous form $F(x, y, z)$ is psd (respectively, sos) if and only if its dehomogenization $f(x, y) := F(x, y, 1)$ is psd (respectively, sos). In the sequel we will denote the dehomogenizations of M and R by m and r. In [28] we studied the connection between TMP for an n-dimensional sequence $\beta \equiv \beta^{(2d)}$ and the moment problem with respect to homogeneous polynomials of degree m in $n + 1$ variables $x_0, x_1, \ldots, x_n$, with moment data $\widetilde{\beta} \equiv \widetilde{\beta}^{(=2d)}$ defined by

$$\widetilde{\beta}_{(2d-|\alpha|,\alpha)} := \beta_\alpha$$

for every $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq 2d$. We denote this problem by HTMP. The moment problem for β and the homogeneous moment problem for $\widetilde{\beta}$ are not equivalent, but are closely related.

Theorem A.4. *([28, Theorem 3.1]) If $\beta \equiv \beta^{(2d)}$ has a representing measure in TMP, then $\widetilde{\beta}$ has a representing measure in HTMP. Moreover, L_β is positive if and only if $\widetilde{\beta}$ has a representing measure in HTMP.*

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