Operators and Matrices
[0263] Second Galley Proofs

POSITIVE LINEAR FUNCTIONALS WITHOUT REPRESENTING MEASURES

CHIRAKKAL EASWARAN AND LAWRENCE FIALKOW

(*Communicated by J. William Helton*)

Abstract. For k even, let $\mathscr{P}_k$ denote the vector space of polynomials in 2 real variables of degree at most k. A linear functional $L : \mathscr{P}_k \longrightarrow \mathbb{R}$ is *positive* if $p \in \mathscr{P}_k$, $p|\mathbb{R}^2 \geqslant 0 \Longrightarrow L(p) \geqslant 0$. Hilbert's theorem on sums of squares (cf. [15]) implies that $L : \mathscr{P}_4 \longrightarrow \mathbb{R}$ is positive if and only if the moment matrix associated to L is positive semidefinite. In this note, using $k = 6$, we exhibit the first family of positive linear functionals $L : \mathscr{P}_k \to \mathbb{R}$ whose positivity cannot be derived from the positive semidefiniteness of the associated moment matrices, and which do not correspond to integration with respect to positive measures.

1. Introduction

For $n \geqslant 1$, $k \geqslant 0$, let $\beta \equiv \beta^{(k)} := \{\beta_i : i \in \mathbb{Z}_+^n, |i| \leqslant k\}$ denote an n-dimensional real multisequence of degree k (where $i \equiv (i_1, \dots, i_n)$ and $|i| = i_1 + \cdots + i_n$). Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set. The *truncated K-moment problem* asks for conditions on β so that there exists a positive Borel measure μ on $\mathbb{R}^n$ satisfying

$$\operatorname{supp} \mu \subseteq K$$

and

$$\beta_i = \int_K x^i \, d\mu \quad (|i| \leqslant k)$$

(where $x \equiv (x_1, \dots, x_n)$ and $x^i = x_1^{i_1} \cdots x_n^{i_n}$); we refer to such a measure as a K-*representing measure* for β.

Let $\mathscr{P} \equiv \mathbb{R}[x_1, \dots, x_n]$ and let $\mathscr{P}_k$ denote the subspace consisting of all polynomials p with *deg* $p \leqslant k$. We associate to β the *Riesz functional* $L_\beta : \mathscr{P}_k \longrightarrow \mathbb{R}$ defined by $L_\beta(\sum a_i x^i) = \sum a_i \beta_i$. Note that if β admits a K-representing measure μ, then L_β is K-*positive* in the sense that $p \in \mathscr{P}_k$, $p|K \geqslant 0 \Longrightarrow L_\beta(p) \geqslant 0$; indeed, in this case, $L_\beta(p) = \int_K p \, d\mu \geqslant 0$ (since μ is a positive measure supported in K). For $K = \mathbb{R}^n$, we refer to μ simply as a *representing measure* and to a K-positive functional L_β as *positive*. For K compact, the proof of Tchakaloff's Theorem [20] shows that if L_β is K-positive, then β admits a K-representing measure. Even for $n = 1$, this implication does not always hold for non-compact K [9, Example 2.1], but the following result of [9] characterizes the existence of K-representing measures in terms of K-positivity.

Mathematics subject classification (2010): 47A57, 44A60, 47A20.

Keywords and phrases: Truncated moment sequence, Riesz functional, K-positivity, moment matrix, representing measure.

Research partially supported by NSF Grant DMS-0758378.

THEOREM 1.1. ([9, Theorem 1.2]) *Let $k=2m$ or $k=2m+1$. $\beta \equiv \beta^{(k)}$ admits a K-representing measure if and only if L_β admits a K-positive extension $L_{\tilde{\beta}} : \mathscr{P}_{2m+2} \longrightarrow \mathbb{R}$.*

Theorem 1.1 is the analogue for the truncated moment problem of the classical theorem of M. Riesz ($n=1$) [17] and E.K. Haviland ($n>1$) [13], which shows that a *full* multisequence $\beta \equiv \beta^{(\infty)}$ admits a K-representing measure if and only if the corresponding Riesz functional $L_\beta : \mathscr{P} \longrightarrow \mathbb{R}$ is K-positive. (Theorem 1.1 actually implies the Riesz-Haviland Theorem (see [9]).) In view of Theorem 1.1 and the Riesz-Haviland Theorem, it is important to be able to recognize when a functional L_β is K-positive. A celebrated theorem of K. Schmüdgen [19] shows that if K is a compact semialgebraic set, then a strictly positive polynomial on K can be expressed as a weighted sum of squares, and this permits one to establish K-positivity for $L_{\beta^{(\infty)}}$ simply by verifying the positive semidefiniteness of a finite number of *localizing matrices* associated to $\beta^{(\infty)}$ (cf. [9] [14] [19]). Nevertheless, a basic difficulty is that for a general closed (even semialgebraic) set K, there is no concrete description of the polynomials that are nonnegative on K, so there may be no direct test for K-positivity of $L_{\tilde{\beta}}$ or $L_{\beta^{(\infty)}}$. [12, Theorem 2.2] shows that $L_{\beta^{(k)}}$ is K-positive if and only if $\beta^{(k)}$ is in the closure of the multisequences having K-representing measures. Motivated by this result, in the sequel we will use an approximation technique to provide the first examples of K-positive functionals $L_{\beta^{(k)}}$ in cases where $\beta^{(k)}$ has no K-representing measure and where the strictly positive polynomials on K cannot be represented as weighted sums of squares.

To explain our results, let us first recall the scope of the sums of squares approach when $K=\mathbb{R}^n$. Following [7], for $k=2d$, we may associate to $\beta \equiv \beta^{(k)}$ the *moment matrix* $M \equiv M_d(\beta)$, of size $\dim \mathscr{P}_d$, defined by

$$\langle M\hat{p}, \hat{q}\rangle = L_\beta(pq) \quad (p,q \in \mathscr{P}_d) \tag{1.1}$$

(where $\hat{p}$ is the vector of coefficients of p relative to the basis of monomials of $\mathscr{P}_d$ in degree-lexicographic order). If β admits a representing measure μ, then for $q \in \mathscr{P}_d$, $\langle M\hat{q}, \hat{q}\rangle = L_\beta(q^2) = \int q^2 d\mu \geqslant 0$, so M is positive semidefinite ($M \succeq 0$). In general, even for $n=1$, positive semidefiniteness of M is *not* sufficient for β to admit a representing measure [3]. However, if $M \succeq 0$ and each polynomial p in $\mathscr{P}_{2d}$ that is strictly positive on $\mathbb{R}^n$ admits a sum of squares decomposition, i.e., $p = \sum q_j^2$ ($q_j \in \mathscr{P}_d$), then L_β is positive. Indeed, in this case,

$$L_\beta(p) = \sum L_\beta(q_j^2) = \sum \langle M\hat{q}_j, \hat{q}_j\rangle \geqslant 0; \tag{1.2}$$

now, if $q|\mathbb{R}^2 \geqslant 0$, then for every $\varepsilon > 0$, $q+\varepsilon$ is strictly positive, so (1.2) implies $L_\beta(q) \geqslant -\varepsilon L_\beta(1)$, and thus $L_\beta(q) \geqslant 0$.

A well-known theorem of Hilbert (cf. [15] [16]) shows that for k even, every polynomial in $\mathscr{P}_k$ that is nonnegative on $\mathbb{R}^n$ may be expressed as a sum of squares of polynomials if and only if $n=1$; or $n=2$ and $k=4$; or $n \geqslant 1$ and $k=2$. We note that these are precisely the cases in which each polynomial that is *strictly* positive on $\mathbb{R}^n$ is a sum of squares, so that precisely in these cases can positivity of L_β be established through the positive semidefiniteness of $M_k(\beta)$ (as in the argument following (1.2)). To see this, suppose $q \in \mathscr{P}_k$ is nonnegative on $\mathbb{R}^n$ but is not a sum of squares. It suffices to show that for all sufficiently small $\varepsilon > 0$, the strictly positive polynomial $q+\varepsilon$ is not a sum of squares. Let Σ_k denote the convex cone in $\mathscr{P}_k$ consisting of sums of squares and recall from [14, Section 3.8] that Σ_k is closed in $\mathscr{P}_k$ (relative to the norm $||\Sigma a_i x^i|| = \max|a_i|$). It follows from the Minkowski separation theorem [2, (34.2)] that there exists a linear functional $L : \mathscr{P}_k \longrightarrow \mathbb{R}$ for which $L|\Sigma_k \geqslant 0$ and $L(q) < 0$. By continuity, $L(q+\varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$, so $q+\varepsilon$ is not a sum of squares.

Let us apply the preceding observations when $n = 2$. For $n = 2$ and $k = 4$, Hilbert's theorem and (1.2) imply that if $\beta \equiv \beta^{(4)}$ satisfies $M_2(\beta) \succeq 0$, then L_β is positive. However, the above discussion implies that there exists $\beta \equiv \beta^{(6)}$ such that $M_3(\beta)$ is positive semidefinite but L_β is not positive; moreover, an example of Schmüdgen [18] illustrates a case where $M_3(\beta)$ is positive definite but L_β is not positive. In view of these examples, for $\beta \equiv \beta^{(6)}$ we cannot use sums of squares, as in (1.2), to promote positive semidefiniteness of $M_3(\beta)$ into positivity of L_β. In cases where $\beta^{(6)}$ fails to have a representing measure, we require a new technique, beyond sums of squares, to establish that L_β is positive, and the goal of this note is to illustrate such a technique.

In the sequel, for $n = 2$, we denote the successive rows and columns of the moment matrix $M \equiv M_3(\beta)$ by $1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3$. We denote the elements of $\beta^{(6)}$ by β_{ij} $(i, j \geqslant 0,\ i+j \leqslant 6)$, where β_{ij} corresponds to the monomial $x^i y^j$. Let $Col\ M$ denote the column space of M in $\mathbb{R}^{10}$. Under the conditions

$$M \equiv M_3(\beta) \succeq 0,\ \ Y = X^3 \ in\ Col\ M,\ \ rank(M) = 9, \tag{1.3}$$

we will associate to β an expression $\psi(\beta)$, a certain rational function of the moment data in β (see Section 2). [11, Theorem 1.1] implies that under the conditions of (1.3), β has a representing measure (necessarily supported in the curve $y = x^3$) if and only if $\beta_{15} > \psi(\beta)$. Our main result, which follows, displays a family of positive functionals $L_{\beta^{(6)}}$ whose positivity does not arise from the existence of representing measures or from sums of squares as in (1.2).

THEOREM 1.2. *For $\beta \equiv \beta^{(6)}$, suppose $M \equiv M_3(\beta) \succeq 0$, $Y = X^3$, and rank $M =$ 9. If $\beta_{15} = \psi(\beta)$, then β has no representing measure, but L_β is positive.*

REMARK 1.3. i) The positivity of L_β was conjectured in [12, Example 2.5], where it was established for particular numerical instances of M satisfying (1.3); the key new ingredient for the proof of Theorem 1.2 is Proposition 2.2, which is proved by means of a highly intensive symbolic algebra calculation. The functionals of Theorem 1.2 (including those of [12, Example 2.5]) seem to be the first concrete examples of L_β where positivity cannot be established through the existence of representing measures or via sums of squares.

ii) We may view a multisequence $\beta \equiv \beta^{(k)}$ as an element of $\mathbb{R}^\eta$, where $\eta =$ dim $\mathscr{P}_k$. Let $\mathscr{C}$ denote the convex cone in $\mathbb{R}^\eta$ consisting of those multisequences β having K-representing measures. [12, Theorem 2.2] shows that L_β is K-positive if and only if $\beta \in \bar{\mathscr{C}}$. From this viewpoint, with $k = 6$ and $K = \mathbb{R}^2$, the sequences β in Theorem 1.2 belong to $bdry\, \mathscr{C} \setminus \mathscr{C}$.

The case of the bivariate moment problem for $\beta^{(4)}$ with $M_2(\beta^{(4)})$ *singular* was solved in [6] [8]: concrete necessary and sufficient conditions for representing measures are known, and finitely atomic representing measures with the fewest atoms can be explicitly computed. Note that in Theorem 1.2, $M_2(\beta^{(4)})$ is *positive definite*. In [12], J. Nie and the second-named author proved that for a bivariate $\beta \equiv \beta^{(4)}$, if $M_2(\beta) \succ 0$, then β does have a representing measure, but at present it is not known how to construct such a measure. It follows from [1, Theorem 2] that since there is a representing measure, then there exists a cubature rule ν (a finitely atomic representing measure) with card supp $\nu \leqslant \dim P_4 = 15$, but there is no method known for computing ν. Further, any representing measure μ necessarily satisfies card supp $\mu \geqslant rank\, M_2(\beta) = 6$ (cf. [4] [7]), but it remains unknown whether 6-atomic representing measures always exist. In general, if a positive semidefinite M_d admits a *flat* (i.e., rank-preserving) extension M_{d+1}, then $\beta^{(2d)}$ admits a *rank* M_d-atomic representing measure that can be explicitly computed [7], but it is unknown whether a positive definite M_2 always admits a flat extension M_3 (and a corresponding 6-atomic measure). We will show that for the sequences $\beta^{(4)}$ which appear in Theorem 1.2, there always exist computable 9-atomic representing measures:

COROLLARY 1.4. *If $\beta^{(4)}$ admits an extension to a sequence $\beta^{(6)}$ satisfying (1.3) and $\beta_{15} \geqslant \psi(\beta^{(6)})$, then $\beta^{(5)}$ admits a computable 9-atomic representing measure.*

2. A family of positive functionals $L_{\beta^{(6)}}$

In this section we prove Theorem 1.2, and to this end we require a preliminary result and some notation. Recall that a real symmetric 2×2 block matrix $M \equiv \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is positive semidefinite if and only if $A \succeq 0$, $B = AW$ for some matrix W (equivalently, $Ran\ B \subseteq Ran\ A$), and $C \succeq W^T AW$ (cf. [7]). For a bivariate moment matrix $M \equiv M_d(\beta)$, we denote the columns by 1, X, Y, $\ldots$, X^d, $X^{d-1}Y$, $\ldots$, XY^{d-1}, Y^d (following the degree-lexicographic ordering of the monomials in $\mathscr{P}_d$). Each linear dependence relation in the columns of M may thus be expressed as $p(X,Y) = 0$, where $p(x,y) \equiv \sum a_{ij} x^i y^j \in \mathscr{P}_d$ and $p(X,Y) := \sum a_{ij} X^i Y^j$. Recall from [7] that M is *recursively generated* if p, q, $pq \in \mathscr{P}_d$, $p(X,Y) = 0 \Longrightarrow (pq)(X,Y) = 0$. Recursiveness is a necessary condition for the existence of a representing measure for β. The following result is implicit in [11].

PROPOSITION 2.1. *If $\beta \equiv \beta^{(2d)}$ has a representing measure, then $M \equiv M_d(\beta)$ admits a positive, recursively generated moment matrix extension*

$$M_{d+1}(\tilde{\beta}) \equiv \begin{pmatrix} M & B(d+1) \\ B(d+1)^T & C(d+1) \end{pmatrix};$$

in particular, $Ran\, B(d+1) \subseteq Ran\, M$.

Proof. Since β has a representing measure, say μ, it follows from [1] that μ admits a *cubature rule* of degree $2d$, i.e., there exists a *finitely atomic* positive measure ν that has the same moments as μ up to (at least) degree $2d$. Since ν is finitely atomic, it has finite moments of degrees $2d+1$ and $2d+2$, and a corresponding moment sequence $\tilde{\beta} \equiv \tilde{\beta}^{(2d+2)}$. Since ν is obviously a representing measure for $\tilde{\beta}$, it follows that $M_{d+1}(\tilde{\beta})$ is a positive, recursively generated extension of M, and, in particular, $Ran\, B(d+1) \subseteq Ran\, M$. $\square$

We note that the preceding result holds for general $n \geqslant 1$.

We next describe how to construct $M_3(\beta)$ satisfying the hypotheses of Theorem 1.2. Let $n=2$ and $d=3$. We consider the general form of a moment matrix $M_3(\beta)$ with a column relation $Y = X^3$ (normalized with $\beta_{00}=1$):

$$M \equiv M_3(\beta) = \begin{pmatrix} 1 & a & b & c & e & d & b & f & g & x \\ a & c & e & b & f & g & e & d & h & j \\ b & e & d & f & g & x & d & h & j & k \\ c & b & f & e & d & h & f & g & x & u \\ e & f & g & d & h & j & g & x & u & v \\ d & g & x & h & j & k & x & u & v & w \\ b & e & d & f & g & x & d & h & j & k \\ f & d & h & g & x & u & h & j & k & r \\ g & h & j & x & u & v & j & k & r & s \\ x & j & k & u & v & w & k & r & s & t \end{pmatrix}. \tag{2.1}$$

For suitable values of the moment data, M satisfies the following properties:

$$M \succeq 0, \quad Y = X^3, \quad rank\, M = 9; \tag{2.2}$$

this is the case, for example, with

$$\begin{gathered} a=b=f=g=u=v=w=x=0,\ c=1,\ e=2,\ d=5,\ h=14, \\ j=42,\ k=132,\ r=429,\ s=1429,\ t=4847, \end{gathered} \tag{2.3}$$

$$M \equiv M_3(\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & 1429 \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & 1429 & 4847 \end{pmatrix}. \tag{2.4}$$

In [11] we solved the truncated K-moment problem for $K = \{(x,y) \in \mathbb{R}^2 : y = x^3\}$. In particular, [11] provides a numerical test, that we next describe, for the existence of K-representing measures whenever M as in (2.1) satisfies (2.2). From Proposition 2.1, we know that if β admits a representing measure, then M admits a positive, recursively generated extension $M_4(\tilde{\beta})$. In any such extension, the moments must be consistent with the relation $y = x^3$, so in particular, we must have $\beta_{44} = \beta_{15} (\equiv s)$. To insure positivity, $M_4(\tilde{\beta})$ must satisfy a lower bound for the diagonal element β_{44} (in row X^2Y^2, column X^2Y^2), which we may derive as in [11]. Let J denote the compression of M obtained by deleting row X^3 and column X^3; thus, $J \succ 0$. Let us write

$$J = \begin{bmatrix} N & U \\ U^T & \Delta \end{bmatrix},$$

where N is the compression of J to its first 8 rows and columns, U is a column vector, and $\Delta \equiv \beta_{06} (\equiv t) > 0$. Consider the corresponding block decomposition of J^{-1}, which is of the form

$$J^{-1} = \begin{bmatrix} P & V \\ V^T & \varepsilon \end{bmatrix},$$

where $P \succ 0$ and $\varepsilon > 0$. Since $M_4(\tilde{\beta})$ is recursively generated, the relation $Y = X^3$ in $Col\ M$ propagates to the column relations $X^4 = XY$ and $X^3Y = Y^2$ in $Col\ M_4(\tilde{\beta})$, so by moment matrix structure, after deleting the element in row X^3, the first 8 remaining elements of column X^2Y^2 must be $W \equiv (h,\ x,\ u,\ j,\ k,\ r,\ v,\ w)^T$. Let $\omega = \langle PW, W \rangle$ and define

$$\psi(\beta) := \frac{\omega\varepsilon - \langle V, W \rangle^2}{\varepsilon}. \tag{2.5}$$

In [11] we showed that in $M_4(\tilde{\beta})$ we must have $\beta_{44} \geqslant \psi(\beta)$, so in M we require $\beta_{15} \geqslant \psi(\beta)$, and [11, Theorem 1.1] implies that β has a representing measure if and only $s \equiv \beta_{15} > \psi(\beta)$; since $Y = X^3$ in $Col\ M$, such a measure is necessarily supported in K (cf. [7]).

Although $\psi(\beta)$ is formally defined in terms of all of the moments in β, we will show below (Proposition 2.2) that for M as in (2.1)-(2.2), the value of $\psi(\beta)$ is actually *independent* of s and t. For any M satisfying (2.1) and (2.2), so that $\psi(\beta)$ is independent of s and t, we now specify $s \equiv \beta_{15} = \psi(\beta)$, and we adjust t (if necessary) so that M continues to be positive with $rank\ M = 9$. Thus, M satisfies all of the hypotheses of Theorem 1.2; for a numerical example, consider (2.3), where a calculation shows that $\beta_{15} \equiv 1429 = \psi(\beta)$.

Proof of Theorem 1.2. With β as in the hypothesis, we claim that L_β is positive. Since $\beta_{15} = \psi(\beta)$, positivity for L_β cannot be derived from the existence of a representing measure, since [11, Theorem 1.1] shows that β has no representing measure. Moreover, as we discussed in Section 1, positivity for L_β cannot be derived from the positivity of M via sums of squares arguments as in (1.2) because, by Hilbert's theorem, there exist degree 6 bivariate polynomials that are everywhere nonnegative but are not sums of squares. To prove that L_β is positive, we employ a sequence of approximate representing measures. Since $J \succ 0$, then $t \equiv \Delta > U^T N^{-1} U$. Thus, there exists $\delta > 0$ such that if we replace s $(= \psi(\beta))$ by $s + \frac{1}{m}$ (with $\frac{1}{m} < \delta$), then the resulting moment matrix, $M_3(\beta^{[m]})$, remains positive, with $rank\ M_3(\beta^{[m]}) = 9$ and $Y = X^3$ in

Col $M_3(\beta^{[m]})$. Since, from Proposition 2.2 (below), the value of $\psi(\beta^{[m]})$ is independent of $\beta_{15}[\beta^{[m]}]$ and $\beta_{06}[\beta^{[m]}]$, we have $\psi(\beta^{[m]}) = \psi(\beta) = s < s + \frac{1}{m} = \beta_{15}[\beta^{[m]}]$. It now follows from [11, Theorem 1.1] that $\beta^{[m]}$ has a representing measure, whence $L_{\beta^{[m]}}$ is positive. Note that the convex cone $\{\beta \equiv \beta^{(6)} \in \mathbb{R}^{10} : L_\beta \textit{ is positive}\}$ is closed; since $\|\beta^{[m]} - \beta\| = \frac{1}{m} \longrightarrow 0$, we conclude that L_β is positive. $\square$

To complete the proof of Theorem 1.2 it now suffices to prove the following result.

PROPOSITION 2.2. *For $M \equiv M_3(\beta)$ as in (2.1)-(2.2), the value of $\psi \equiv \psi(\beta)$ is independent of s and t, i.e., if $M_3(\beta')$ is as in (2.1)-(2.2) and β' agrees with β except possibly in the values of s and t, then $\psi(\beta') = \psi(\beta)$.*

Proof. Our proof that the value of ψ is independent of s and t is computational. We represent J^{-1} in the form

$$J^{-1} = \begin{bmatrix} P & V \\ V^T & \varepsilon \end{bmatrix} \equiv \frac{1}{D} \begin{bmatrix} P' & V' \\ V'^T & \varepsilon' \end{bmatrix},$$

where $D = \det J$. Then we have

$$\psi = \frac{\langle P'W, W\rangle - \frac{1}{\varepsilon'} \langle V', W\rangle^2}{D}. \tag{2.6}$$

One can verify, either computationally or by examining the relevant terms, that:

i) $\langle P'W, W\rangle$ is a polynomial in s and t of the form $A_1 + A_2 s + A_3 s^2 + A_4 t$, with coefficients A_i that are polynomials in the remaining moment variables that define M.

ii) $\langle V'^T, W\rangle$ is a linear polynomial in s, with coefficients that are polynomials in the remaining moment variables, excluding t.

iii) D is quadratic in s and linear in t (as in i)).

iv) ε' is a polynomial in the moment variables, but with no terms including s or t.

ψ can then be represented as ratio of two polynomials in s and t:

$$\psi = \frac{n_0 + n_1 s + n_2 s^2 + n_3 t}{d_0 + d_1 s + d_2 s^2 + d_3 t}, \tag{2.7}$$

where $\{n_i\}_{i=0}^3$ and $\{d_i\}_{i=0}^3$ are sequences of polynomial functions in the variables a, b, f, g, u, v, w, c, e, d, h, j, k, r, and x (and which contain no terms with s or t). Then the value of ψ is independent of s and t if and only if

$$\frac{n_0}{d_0} = \frac{n_1}{d_1} = \frac{n_2}{d_2} = \frac{n_3}{d_3}. \tag{2.8}$$

Our proof of (2.8) consists of computing $f_0 = \frac{n_0}{d_0}$, $f_1 = \frac{n_1}{d_1}$, $f_2 = \frac{n_2}{d_2}$, $f_3 = \frac{n_3}{d_3}$ and showing that

$$\frac{f_0}{f_3} = \frac{f_1}{f_3} = \frac{f_2}{f_3} = 1. \tag{2.9}$$

By explicit computation using the computer algebra system *Maxima*, we were able to verify (2.9), thus completing the proof. However, the calculations of the terms involved in verifying (2.9) present particular computational challenges that we discuss in the next section. $\square$

Proof of Corollary 1.4. Suppose $M_2 \succ 0$ and that M_2 can be extended to $M_3(\beta)$ satisfying (1.3) and $\beta_{15} \geqslant \psi(\beta)$. If $\beta_{15} > \psi(\beta)$, then [11, Theorem 1.1] implies that M_3 admits a flat extension M_4, so $\beta^{(6)}$ admits a 9-atomic representing measure that can be explicitly constructed using the method of [7]. If $\beta_{15} = \psi(\beta)$, then Theorem 1.2 implies that $L_{\beta^{(6)}}$ is positive, so Theorem 1.1 implies that $\beta^{(5)}$ has a representing measure. For a more constructive approach, we recall that in the proof of Theorem 1.2., $\beta^{[m]}$ satisfies the conditions of [11, Theorem 1.1], which implies that $\beta^{[m]}$ has a 9-atomic representing measure that can be explicitly constructed using the method of [7]; clearly, such a measure is also a representing measure for $\beta^{(5)}$. $\square$

We conclude with a question.

QUESTION 2.3. If $M_3(\beta)$ satisfies (2.2) and $\beta_{15} < \psi(\beta)$, can L_β be positive?

If, in Question 2.3, L_β were positive, then, from [12, Theorem 2.2], $\beta = lim\, \beta^{[n]}$, where each $\beta^{[n]}$ has a representing measure. Such $\beta^{[n]}$ cannot all satisfy (2.2), for otherwise $\beta^{[n]}_{15} > \psi(\beta^{[n]})$ from [11, Theorem 1.1], implying $\beta_{15} \geqslant \psi(\beta)$. However, there may be sequences having representing measures which approximate β and which do not satisfy (2.2).

3. Appendix

In order to establish (2.9), we first need to compute J^{-1}, and to then represent ψ in the form (2.6). We can then extract from (2.6) the coefficients of various powers of s and t in the numerator and denominator, as well as terms independent of s and t, so as to calculate the quantities $\{n_i\}_{i=0}^{3}$ and $\{d_i\}_{i=0}^{3}$ in (2.7). This then allows us to compute $\{f_i\}_{i=0}^{3}$ for use in verifying (2.9).

We first attempted to perform these computations using *Mathematica* software. On a dual-core x86_64 computer with 10GB RAM (memory for computation) and 8GB swap space (auxiliary storage), and using the Linux operating system, the program ran for several days without producing even J^{-1}. We noted that as the computation progressed, available RAM and swap space became almost wholly consumed, eventually shutting down the operating system. We repeated the above calculations with *Mathematica* on several different machines and operating systems, all of which failed to yield results. We then tried the same computations using *Matlab*, which employs the *Maple* symbolic algebra system, but *Matlab* also failed to yield any result, eventually shutting down the operating system.

We then turned to *Maxima*, an open source computational software system descended from the now-extinct commercial *Macsyma* software. On the above-mentioned machine, *Maxima* was able to compute J^{-1} and the representation of ψ as in (2.7), and store the results on disk, in approximately 100 minutes. The intermediate expressions leading up to each f_i $(0 \leqslant i \leqslant 3)$ are very large, taking several hundred megabytes of storage for each. Nevertheless, *Maxima* was able to retrieve the components of ψ from disk, and compute and simplify all of the expressions f_1, f_2 and f_3, in approximately 1 hour. Although *Maxima* also computed f_0, its simplification via the *ratsimp* function in *Maxima* failed, with *Maxima* reporting that the computation was aborted due to heap space exhaustion. Now *Maxima* is written in the Lisp programming language, and relies on a Lisp engine to do its computations. On our Linux operating system, *Maxima* was using the SBCL Lisp engine, Steel Box Common Lisp, another open source software, to perform the calculations. Because the source code was accessible to us, we recompiled SBCL so that it would be able to utilize up to 25GB of heap space. On this recompiled Lisp engine, *Maxima* loaded the components of ψ from disk and computed and simplified f_0 in approximately 4200 seconds of cpu time (and an amount of real time that varied, on several trials, between 2.2 hours and 8.5 hours, depending on overall system conditions). The final outcome of these calculations is that each of the quantities $\{f_i\}_{i=0}^3$ in (2.9) reduces to the same expression (requiring 2,343,199 bytes of memory and about 50 pages to print) for arbitrary values of the moment variables (not including s and t). The entire computation described above thus establishes (2.9) and proves Proposition 2.2.

The details of the calculations that eventually proved successful, including annotated *Maxima* code, are posted online at `http://cs.newpaltz.edu/~easwaran/PLF`. It is not clear to us why *Maxima* was successful, as compared to other software. This seems partly due to the way the computations are organized. For example, the *Mathematica* command $Inverse[J]$ is carrried out with the enormous term $Det[J]$ present in the denominator of each entry of the inverse, whereas *Maxima* provides an option for carrying $1/(\det\ J)$ outside, thus reducing storage requirements on swap space. We suspect that the use of Lisp as its underlying processing mechanism also contributes to the success of *Maxima*. In particular, the open source nature of *Maxima* and of the *SBCL* Lisp processor allowed us to recompile source code, which was crucial to the success of the computation (as described above). (More information about *Maxima* and SBCL can be found at `http://maxima.sourceforge.net/` and `http://sbcl.sourceforge.net/`.)

REFERENCES

[1] C. BAYER AND J. TEICHMANN, *The proof of Tchakaloff's Theorem*, Proc. Amer. Math. Soc., **134** (2006), 3035–3040.

[2] S. BERBERIAN, *Lectures in Functional Analysis and Operator Theory*, Springer-Verlag, 1973.

[3] R. CURTO AND L. FIALKOW, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991) 603–635.

[4] R. CURTO AND L. FIALKOW, *Solution of the truncated complex moment problem for flat data*, Memoirs of the American Mathematical Society, **119** (1996), No. 568, Amer. Math. Soc. Providence, RI, 1996.

[5] R. CURTO AND L. FIALKOW, *Flat extensions of positive moment matrices: Relations in analytic or conjugate terms*, Operator Th.: Adv. Appl. **104** (1998), 59–82.

[6] R. CURTO AND L. FIALKOW, *Solution of the singular quartic moment problem*, Journal of Operator Theory, **48** (2002), 315–354.

[7] R. CURTO AND L. FIALKOW, *Truncated K-moment problems in several variables*, Journal of Operator Theory, **54** (2005), 189–226.
[8] R. CURTO AND L. FIALKOW, *Solution of the truncated hyperbolic moment problem*, Integral Equations and Operator Theory, **52** (2005), 181–219.
[9] R. CURTO AND L. FIALKOW, *An analogue of the Riesz-Haviland Theorem for the truncated moment problem*, J. Functional Analysis, **225** (2008), 2709–2731.
[10] L. FIALKOW, *Truncated multivariable moment problems with finite variety*, J. Operator Theory, **60** (2008), 343–377.
[11] L. FIALKOW, *Solution of the truncated moment problem with variety* $y = x^3$, Trans. Amer. Math. Soc., to appear.
[12] L. FIALKOW AND J. NIE, *Positivity of Riesz functionals and solutions of quadratic and quartic moment problems*, J. Functional Analysis, **258**, 1 (2010), 328–356.
[13] E. K. HAVILAND, *On the momentum problem for distributions in more than one dimension II*, Amer. J. Math., **58** (1936) 164–168.
[14] M. LAURENT, *Sums of squares, moment matrices and optimization over polynomials, Emerging Applications of Algebraic Geometry*, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds), Springer, pages 157–270, 2009.
[15] B. REZNICK, *Some concrete aspects of Hilbert's* 17^{th} *problem*, in Contemp. Math., volume 253, pages 251–272, American Mathematical Society, 2000.
[16] B. REZNICK, *On Hilbert's construction of positive polynomials*, preprint, 2007.
[17] M. RIESZ, *Sur le probleme des moments, troisieme note*, Ark. Mat., **17** (1923), 1–52.
[18] K. SCHMÜDGEN, *An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional*, Math. Nachr., **88** (1979), 385–390.
[19] K. SCHMÜDGEN, *The* K*-moment problem for compact semi-algebraic sets*, Math. Ann. **289** (1991), 203–206.
[20] V. TCHAKALOFF, *Formules de cubatures mécanique à coefficients non négatifs*, Bull. Sci. Math., **(2)82** (1957), 123–134.

(Received November 6, 2009)

Chirakkal Easwaran
Department of Computer Science
State University of New York
New Paltz, New York 12561
e-mail: `easwaran@newpaltz.edu`

Lawrence Fialkow
Department of Computer Science and Department of Mathematics
State University of New York
New Paltz, New York 12561
e-mail: `fialkowl@newpaltz.edu`